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Extension of a result of Haynsworth and Hartfiel

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Abstract. Haynsworth (Proc Am Math Soc 24:512–516, 1970) used a result of the Schur complement to refine a determinant inequality for positive definite matrices. Haynsworth's result was improved by Hartfiel (Proc Am Math Soc 41:463–465, 1973). We extend their results to a larger class of matrices, namely, matrices whose numerical range is contained in a sector. Our proof relies on a number of new relations for the Schur complement of this class of matrices.

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1. Introduction. Let $\mathbb{M}_n$ be the set of all $n \times n$ complex matrices. For $A \in \mathbb{M}_n$, the conjugate transpose of A is denoted by A^*, the real and imaginary parts of A are in the sense of the Cartesian decomposition and they are denoted by $\Re A = \frac{1}{2}(A + A^*)$ and $\Im A = \frac{1}{2i}(A - A^*)$, respectively. For two Hermitian matrices $A, B \in \mathbb{M}_n$, we write $A \geq B$ (or $B \leq A$) to mean that $A - B$ is positive semidefinite. We also consider $A \in \mathbb{M}_n$ to be partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \tag{1.1}$$

where diagonal blocks are square matrices. If A is nonsingular, then we partition A^{-1} conformally as A. If A_{11} is nonsingular, then the Schur complement of A_{11} in A is defined by $A/A_{11} = A_{22} - A_{21}A_{11}^{-1}A_{12}$. The term "Schur complement" and the notation were first brought in by Haynsworth. We refer the readers to [14] for a survey of this important notion and its far reaching applications in various branches of mathematics.

Recall that the numerical range (also known as the field of values) of $A \in \mathbb{M}_n$ is defined by

$$W(A) = \{x^*Ax : x \in \mathbb{C}^n, x^*x = 1\}.$$

Also, we define a sector on the complex plane

$$S_\alpha = \{z \in \mathbb{C} : \Re z > 0, |\Im z| \leq (\Re z) \tan \alpha\}, \qquad \alpha \in [0, \pi/2).$$

Clearly, if A is positive definite, then $W(A) \subset S_0$.

For fundamentals of numerical range, see [4,8]. As $0 \notin S_\alpha$, if $W(A) \subset S_\alpha$, then A is necessarily nonsingular.

The main object of this paper is a class of matrices whose numerical range is contained in S_α. Part of the motivation for investigating this class of matrices comes from the search for the optimal growth factor in Gaussian elimination; see, for example, [1,2,7,10,12].

Let $A, B \in \mathbb{M}_n$ be positive definite. It is well known that (e.g., [9, p. 511])

$$\det(A + B) \geq \det A + \det B. \tag{1.2}$$

Haynsworth proved the following refinement of (1.2).

Theorem 1.1. ([6, Theorem 3]) *Suppose $A, B \in \mathbb{M}_n$ are positive definite. Let A_k and B_k, $k = 1, \ldots, n-1$, denote the k-th principal submatrices of A and B respectively. Then*

$$\det(A + B) \geq \left(1 + \sum_{k=1}^{n-1} \frac{\det B_k}{\det A_k}\right) \det A + \left(1 + \sum_{k=1}^{n-1} \frac{\det A_k}{\det B_k}\right) \det B. \tag{1.3}$$

Hartfiel [5] obtained an improvement of (1.3): under the same condition as in Theorem 1.1,

$$\det(A + B) \geq \left(1 + \sum_{k=1}^{n-1} \frac{\det B_k}{\det A_k}\right) \det A + \left(1 + \sum_{k=1}^{n-1} \frac{\det A_k}{\det B_k}\right) \det B + (2^n - 2n)\sqrt{\det AB}. \tag{1.4}$$

Haynsworth's proof of (1.3) relies on an inequality for the Schur complement [6, Theorem 2]: Let $A, B \in \mathbb{M}_n$ be positive definite and be comformally partitioned as in (1.1). Then

$$(A + B)/(A_{11} + B_{11}) \geq A/A_{11} + B/B_{11}. \tag{1.5}$$

In this paper, we first extend (1.5), then as an application, we obtain a generalization of (1.4) and so (1.3).

2. Preliminaries. We first show a closure property of numerical range under the Schur complement.

Proposition 2.1. *Let $A \in \mathbb{M}_n$ be partitioned as in (1.1). If $W(A) \subset S_\alpha$, then $W(A/A_{11}) \subset S_\alpha$.*

Proof. Clearly, if $W(A) \subset S_\alpha$, then $W(A^*) \subset S_\alpha$ and $W(A_{22}) \subset S_\alpha$. Also, for any nonsingular $X \in \mathbb{M}_n$, $W(A) = W(XAX^*)$. Therefore, $W(A^{-1}) = W(AA^{-1}A^*) = W(A^*) \subset S_\alpha$. The desired result follows by observing that $(A/A_{11})^{-1} = (A^{-1})_{22}$. □

In the remaining of this section, we present a few auxiliary results.

Lemma 2.2. *Let $A \in \mathbb{M}_n$ with $W(A) \subset S_\alpha$. Then A can be decomposed as $A = XZX^*$ for some invertible $X \in \mathbb{M}_n$ and $Z = \operatorname{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n})$ with $|\theta_j| \le \alpha$ for all j.*

Remark 2.3. The decomposition appears first in [1, Lemma 1.1]. In [15], it is shown that the diagonal entries of Z are unique up to permutation.

Lemma 2.4. *Let $A \in \mathbb{M}_n$ with $\Re A$ positive definite. Then*

$$(\Re A)^{-1} \ge \Re(A^{-1}).$$

Proof. By [13, Lemma 2.1], $\Re(A^{-1}) = \Big(\Re A + (\Im A)(\Re A)^{-1}(\Im A)\Big)^{-1}$. As $(\Im A)$ $(\Re A)^{-1}(\Im A)$ is positive semidefinite, $\Re(A^{-1}) \le (\Re A)^{-1}$ follows. □

Lemma 2.5. *Let $A \in \mathbb{M}_n$ be partitioned as in (1.1). If $\Re A$ is positive definite, then*

$$\Re(A/A_{11}) \ge (\Re A)/(\Re A_{11}).$$

Proof. The notation $(\Re A)/(\Re A_{11})$ makes sense as $\Re A_{11}$ is the $(1, 1)$ block of $\Re A$. Consider the Cartesian decomposition $A = M + iN$ with $M = \Re A$, $N = \Im A$ being conformally partitioned as A. Then we have the following equality relating the Schur complements [11, Lemma 2.2],

$$A/A_{11} = M/M_{11} + i(N/N_{11}) + Y(M_{11}^{-1} - iN_{11}^{-1})^{-1}Y^*,$$

where $Y = M_{21}M_{11}^{-1} - N_{21}N_{11}^{-1}$.

As $\Re\Big((M_{11}^{-1} - iN_{11}^{-1})^{-1}\Big)$ is positive semidefinite, so is $\Re\Big(Y(M_{11}^{-1} - iN_{11}^{-1})^{-1}Y^*\Big)$. The desired result follows. □

Lemma 2.6. *Let $A \in \mathbb{M}_n$ with $W(A) \subset S_\alpha$. Then*

$$\sec^n(\alpha)\det(\Re A) \ge |\det A|.$$

Proof. Consider the decomposition $A = XZX^*$ as in Lemma 2.2. Then after dividing by $|\det X|^2$, it suffices to show $\sec^n(\alpha)\det(\Re Z) \ge 1$. But each diagonal entry of the diagonal matrix $\sec(\alpha)\Re Z$ is no less than one, implying the result. □

Remark 2.7. The above inequality may be regarded as a complement of the Ostrowski–Taussky inequality (see [9, p. 510]). With some minor modification in the proof of [15, Lemma 3.1], Zhang showed that actually the eigenvalues of $\sec(\alpha)\Re Z$ weakly log majorize the singular values of A.

3. An extension of (1.5). First of all, we remark that a direct extension of (1.5) is not valid. That is, assuming that $A, B \in \mathbb{M}_n$ with $W(A), W(B) \subset S_\alpha$ are comformally partitioned as in (1.1), it does not hold in general that

$$\Re\Big((A + B)/(A_{11} + B_{11})\Big) \ge \Re(A/A_{11}) + \Re(B/B_{11}). \tag{3.1}$$

To see this, take $B = A^*$, then (3.1) contradicts Lemma 2.5.

The main result of this section is a correct version of (3.1).

Theorem 3.1. *Let $A, B \in \mathbb{M}_n$ with $W(A), W(B) \subset S_\alpha$ be comformally partitioned as in (1.1). Then*

$$\sec^2(\alpha)\Re\Big((A+B)/(A_{11}+B_{11})\Big) \geq \Re(A/A_{11}) + \Re(B/B_{11}).$$

Proof. We prove the following claim first, which may be regarded as a reverse complement of Lemma 2.5.

Claim 1. $\sec^2(\alpha)(\Re A)/(\Re A_{11}) \geq \Re(A/A_{11})$.

Proof of Claim 1. We consider the decomposition $A = XZX^*$ as in Lemma 2.2. We further partition X as a 2-by-1 block matrix $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$. Then $A = \begin{bmatrix} X_1ZX_1^* & X_1ZX_2^* \\ X_2ZX_1^* & X_2ZX_2^* \end{bmatrix}$. Let $Y = (X^*)^{-1} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$ be comformally partitioned as X. Then $A^{-1} = \begin{bmatrix} Y_1Z^{-1}Y_1^* & Y_1Z^{-1}Y_2^* \\ Y_2Z^{-1}Y_1^* & Y_2Z^{-1}Y_2^* \end{bmatrix}$. Clearly,

$$\cos^2(\alpha)(\Re Z)^{-1} \leq \Re(Z^{-1}),$$

it follows that

$$\cos^2(\alpha)Y_2(\Re Z)^{-1}Y_2^* \leq \Re(Y_2Z^{-1}Y_2^*),$$

i.e.,

$$\cos^2(\alpha)\Big((\Re A)^{-1}\Big)_{22} \leq \Re(A^{-1})_{22},$$

or

$$\cos^2(\alpha)\Big((\Re A)/(\Re A_{11})\Big)^{-1} \leq \Re\Big((A/A_{11})^{-1}\Big).$$

Taking the inverses of both sides yields

$$\sec^2(\alpha)\Big((\Re A)/(\Re A_{11})\Big) \geq \Big(\Re\Big((A/A_{11})^{-1}\Big)\Big)^{-1} \geq \Re(A/A_{11}),$$

in which the second inequality is by Lemma 2.4. This completes the proof of Claim 1.

To finish the proof of Theorem 3.1, we observe the following chain of inequalities

$$\begin{aligned} \Re\Big((A+B)/(A_{11}+B_{11})\Big) &\geq \Re(A+B)/\Re(A_{11}+B_{11}) \quad \text{by Lemma 2.5} \\ &\geq (\Re A)/(\Re A_{11}) + (\Re B)/(\Re B_{11}) \quad \text{by (1.5)} \\ &\geq \cos^2(\alpha)\Big(\Re(A/A_{11}) + \Re(B/B_{11})\Big) \text{ by Claim 1.} \end{aligned}$$

□

4. An extension of (1.4). As an applicaton of Theorem 3.1, we present the following extension of Haynsworth and Hartfiel's result mentioned in the Introduction.

Theorem 4.1. *Suppose $A, B \in \mathbb{M}_n$ such that $W(A), W(B) \subset S_\alpha$. Let A_k and B_k, $k = 1, \ldots, n-1$, denote the k-th principal submatrices of A and B respectively. Then*

$$\sec^{3n-2}(\alpha)|\det(A+B)| \geq \left(1 + \sum_{k=1}^{n-1}\left|\frac{\det B_k}{\det A_k}\right|\right)|\det A| + \left(1 + \sum_{k=1}^{n-1}\left|\frac{\det A_k}{\det B_k}\right|\right)|\det B| + (2^n - 2n)\sqrt{|\det AB|}.$$

Proof. Clearly, $(A_{k+1} + B_{k+1})/(A_k + B_k) \in \mathbb{C}$, so

$$|(A_{k+1}+B_{k+1})/(A_k + B_k)| \geq \Re\Big((A_{k+1}+B_{k+1})/(A_k + B_k)\Big), \quad k=1, \ldots, n-1.$$

Here we set $A_n = A, B_n = B$. By Proposition 2.1, $W(A_{k+1}/A_k), W(B_{k+1}/B_k) \subset S_\alpha$; then by Theorem 3.1 and Lemma 2.6,

$$\begin{aligned}\sec^2(\alpha)\Re\Big((A_{k+1} + B_{k+1})/(A_k + B_k)\Big) &\geq \Re(A_{k+1}/A_k) + \Re(B_{k+1}/B_k)\\ &\geq \cos(\alpha)\Big(|A_{k+1}/A_k| + |B_{k+1}/B_k|\Big).\end{aligned}$$

Hence,

$$\sec^3(\alpha)|(A_{k+1} + B_{k+1})/(A_k + B_k)| \geq |A_{k+1}/A_k| + |B_{k+1}/B_k|,$$

that is,

$$\sec^3(\alpha)\left|\frac{\det(A_{k+1} + B_{k+1})}{\det(A_k + B_k)}\right| \geq \left|\frac{\det A_{k+1}}{\det A_k}\right| + \left|\frac{\det B_{k+1}}{\det B_k}\right| \tag{4.1}$$

for $k = 1, \ldots, n-1$.

Taking the product for k from 1 to $n-1$ in (4.1) yields

$$\sec^{3(n-1)}(\alpha)|\det(A+B)| \geq |A_1 + B_1| \prod_{k=1}^{n-1}\left(\left|\frac{\det A_{k+1}}{\det A_k}\right| + \left|\frac{\det B_{k+1}}{\det B_k}\right|\right).$$

As $|A_1 + B_1| \geq \cos(\alpha)(|A_1| + |B_1|)$, we therefore arrive at

$$\begin{aligned}\sec^{3n-2}(\alpha)|\det(A+B)| &\geq (|A_1| + |B_1|)\prod_{k=1}^{n-1}\left(\left|\frac{\det A_{k+1}}{\det A_k}\right| + \left|\frac{\det B_{k+1}}{\det B_k}\right|\right)\\ &= \prod_{k=1}^{n}\left(\left|\frac{\det A_k}{\det A_{k-1}}\right| + \left|\frac{\det B_k}{\det B_{k-1}}\right|\right),\end{aligned}$$

where, by convention, $\det A_0 = \det B_0 = 1$.

The conclusion follows by taking $a_k = |\det A_k|$, $b_k = |\det B_k|$, $k = 0, 1, \ldots, n$, in Claim 2.

Claim 2. Let $a_k, b_k > 0$, $k = 1, \ldots, n$, also let $a_0 = b_0 = 1$. Then

$$\prod_{k=1}^{n}\left(\frac{a_k}{a_{k-1}} + \frac{b_k}{b_{k-1}}\right) \geq a_n\left(1 + \sum_{s+1}^{n-1}\frac{b_s}{a_s}\right) + b_n\left(1 + \sum_{s+1}^{n-1}\frac{a_s}{b_s}\right) + (2^n - 2n)\sqrt{a_n b_n}.$$

Proof of Claim 2. Let $\mathbb{N}_n = \{1, 2, \ldots, n\}$, and let $\mathcal{P}(\mathbb{N}_n)$ be the set of subsets of $\mathbb{N}_n$. We consider special subsets $(\mathcal{B}_s)_{1 \le s \le n}$ and $(\mathcal{B}'_s)_{2 \le s \le n}$ defined by

$$\mathcal{B}_s = \{1, 2, \ldots, s\}, \quad \mathcal{B}'_s = \{s, s+1, \ldots, n\}.$$

Finally we define $\Omega = \{\emptyset\} \cup \{\mathcal{B}_s : 1 \le s \le n\} \cup \{\mathcal{B}'_s : 2 \le s \le n\}$ and $\Omega' = \mathcal{P}(\mathbb{N}_n) \setminus \Omega$. Note that $|\Omega'| = 2^n - 2n$, and that each $k \in \mathbb{N}_n$ belongs to exactly n of the subsets of Ω.

With this notation, for every $x_1, x_2, \ldots, x_n > 0$, we infer that $\prod_{\mathcal{B} \in \Omega} \prod_{k \in \mathcal{B}} x_k = \prod_{k=1}^{n} x_k^n$ and so $\prod_{\mathcal{B} \in \Omega'} \prod_{k \in \mathcal{B}} x_k = \prod_{k=1}^{n} x_k^{2^{n-1}-n}$, moreover,

$$\begin{aligned}\prod_{k=1}^{n}(1 + x_k) &= \sum_{\mathcal{B} \in \mathcal{P}(\mathbb{N}_n)} \prod_{k \in \mathcal{B}} x_k \\ &= \sum_{\mathcal{B} \in \Omega} \prod_{k \in \mathcal{B}} x_k + \sum_{\mathcal{B} \in \Omega'} \prod_{k \in \mathcal{B}} x_k.\end{aligned}$$

But

$$\sum_{\mathcal{B} \in \Omega} \prod_{k \in \mathcal{B}} x_k = 1 + \sum_{s=1}^{n} x_1 x_2 \cdots x_s + \sum_{s=2}^{n} x_s x_{s+1} \cdots x_n$$

and using the arithemtic mean-geometric mean inequality

$$\begin{aligned}\sum_{\mathcal{B} \in \Omega'} \prod_{k \in \mathcal{B}} x_k &\ge |\Omega'| \left(\prod_{\mathcal{B} \in \Omega'} \prod_{k \in \mathcal{B}} x_k \right)^{1/|\Omega'|} \\ &= (2^n - 2n) \left(\prod_{k=1}^{n} x_k^{2^{n-1}-n} \right)^{1/(2^n - 2n)} \\ &= (2^n - 2n)\sqrt{x_1 x_2 \cdots x_n}.\end{aligned}$$

So we have

$$\prod_{k=1}^{n}(1 + x_k) \ge 1 + \sum_{s=1}^{n} x_1 x_2 \cdots x_s + \sum_{s=2}^{n} x_s x_{s+1} \cdots x_n + (2^n - 2n)\sqrt{x_1 x_2 \cdots x_n}.$$

Taking $x_k = \frac{a_{k-1} b_k}{b_{k-1} a_k}$, for $k = 1, \ldots, n$, gives

$$\begin{aligned}\prod_{k=1}^{n}\left(1 + \frac{a_{k-1} b_k}{b_{k-1} a_k}\right) &\ge 1 + \sum_{s=1}^{n} \frac{b_s}{a_s} + \frac{b_n}{a_n} \sum_{s=2}^{n} \frac{a_{s-1}}{b_{s-1}} + (2^n - 2n)\sqrt{b_n / a_n} \\ &= 1 + \sum_{s=1}^{n-1} \frac{b_s}{a_s} + \frac{b_n}{a_n}\left(1 + \sum_{s=1}^{n-1} \frac{a_s}{b_s}\right) + (2^n - 2n)\sqrt{b_n / a_n}.\end{aligned}$$

Multiplying both sides of the inequality by $\prod_{k=1}^{n} \frac{a_k}{a_{k-1}} = a_n$ yields the desired inequality. This completes the proof of Claim 2. □

Apparently, Theorem 4.1 reduces to (1.4) when $\alpha = 0$. A matrix $A \in \mathbb{M}_n$ is accretive-dissipative if both $\Re A$, $\Im A$ are positive definite (see [3]). Note that if A is accretive-dissipative, then $W(e^{-i\pi/4}A) \subset S_{\pi/4}$. Thus, we have the following corollary.

Corollary 4.2. *Suppose $A, B \in \mathbb{M}_n$ are accretive-dissipative. Let A_k and B_k, $k = 1, \ldots, n-1$, denote the k-th principal submatrices of A and B respectively. Then*

$$2^{\frac{3}{2}n-1}|\det(A+B)| \geq \left(1 + \sum_{k=1}^{n-1}\left|\frac{\det B_k}{\det A_k}\right|\right)|\det A| + \left(1 + \sum_{k=1}^{n-1}\left|\frac{\det A_k}{\det B_k}\right|\right)|\det B| + (2^n - 2n)\sqrt{|\det AB|}.$$

Note added in proof. After the acceptance of the paper, the author is aware of that Lemma 2.5 has also appeared in Theorem 7 of (J. Liu, J. Wang, Linear Algebra Appl 293:233–241, 1999).

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