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Hölder estimates for the noncommutative Mazur maps

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Abstract. For any von Neumann algebra $\mathcal{M}$, the noncommutative Mazur map $M_{p,q}$ from $L_p(\mathcal{M})$ to $L_q(\mathcal{M})$ with $1 \leqslant p, q < \infty$ is defined by $f \mapsto f|f|^{\frac{p-q}{q}}$. In analogy with the commutative case, we gather estimates showing that $M_{p,q}$ is $\min\{\frac{p}{q}, 1\}$-Hölder on balls.

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1. Introduction. In the integration theory, the Mazur map $M_{p,q}$ from $L_p(\Omega)$ to $L_q(\Omega)$ is defined by $f \mapsto f|f|^{\frac{p-q}{q}}$. It is an easy exercise to check that it is $\min\{\frac{p}{q}, 1\}$-Hölder. These maps also make sense in the noncommutative L_p-setting, for which one should expect a similar behavior. We refer to [8] for the definitions of L_p-spaces for semifinite von Neumann algebras or more general ones. Having a quantitative result on Mazur maps may be useful when dealing with the structure of noncommutative L_p-spaces (see also [10]). By the way, these maps are used implicitly in the definition of L_p. It is known that $M_{p,q}$ is locally uniformly continuous in full generality [10, Lemma 3.2]. The lack of references for quantitative estimates motivates this note. When dealing with the Schatten classes (when $\mathcal{M} = B(\ell_2)$), some can be found in [1], more precisely $M_{p,q}$ is $\frac{p}{q}$-Hölder when $1 < p < q$. The techniques developed there can be adapted to semifinite von Neumann algebras but can't reach the case $p = 1$. An estimate when $q = p'$ and $1 < p < \infty$ can also be found in [5]. Here we aim to give the best possible estimates especially for $p = 1$.

Theorem *Let $\mathcal{M}$ be a von Neumann algebra, for $1 \leqslant p, q < \infty$, $M_{p,q}$ is $\min\{\frac{p}{q}, 1\}$-Hölder on the unit ball of $L_p(\mathcal{M})$.*

When $p > q$, the Lipschitz constant is of order $\frac{p}{q}$ as in the commutative case. But when $p < q$, the proofs provide a strange behavior of the Hölder constants $c_{p,q}$ as $c_{p,q} \to \infty$ if $q \to 1$ or $p \to \infty$. This reflects the fact that the absolute value is not Lipschitz on L_1 or L_∞, or that the main triangular projection is not bounded on S_1 or $\mathbb{B}(\ell_2)$. We don't know if the result holds with an absolute constant for $p < q$ as in the commutative case.

We follow a basic approach, showing first the results for semifinite von Neumann algebras in section 2. We start by looking at positive elements and then use some commutator or anticommutator estimates. The ideas here are inspired by [2,6]. In section 3, we explain briefly how the Haagerup reduction technique from [7] can be used to get the theorem in full generality.

2. Semifinite case. In this section $\mathcal{M}$ is assumed to be semifinite with a nsf trace τ. We refer to [8] for definitions. We denote by $L_0(\mathcal{M},\tau)$ the set of τ-measurable operators, and

$$L_p(\mathcal{M},\tau) = \Big\{ f \in L_0(\mathcal{M},\tau) \mid \|f\|_p^p = \tau\big(|f|^p\big) < \infty \Big\}.$$

We drop the reference to τ in this section.

First we focus on the Mazur maps for positive elements using some basic inequalities. The first one can be found in [4, Lemma 1.2]. An alternative proof can be obtained by adapting the arguments of [2, Theorem X.1.1] to semifinite von Neumann algebras.

Lemma 2.1. *If* $p \geqslant 1$, $0 < \theta \leqslant 1$, *for any* $x, y \in L_{\theta p}^+(\mathcal{M})$, *we have*

$$\big\|x^\theta - y^\theta\big\|_p \leqslant \big\|x - y\big\|_{\theta p}^\theta.$$

Its proof relies on the fact that $s \mapsto s^\theta$ is operator monotone and has an integral representation

$$s^\theta = c_\theta \int_{\mathbb{R}_+} \frac{t^\theta s}{s+t}\,\frac{dt}{t} \qquad \text{with} \qquad c_\theta = \left(\int_{\mathbb{R}_+} \frac{u^\theta}{u(1+u)}\,du \right)^{-1}.$$

Lemma 2.2. *If* $p \geqslant 1$, $0 < \theta \leqslant 1$, *for any* $x, y \in L_{(1+\theta)p}^+(\mathcal{M})$, *we have:*

$$\big\|x^{1+\theta} - y^{1+\theta}\big\|_p \leqslant 3\big\|x - y\big\|_{(1+\theta)p} \max\Big\{ \big\|x\big\|_{(1+\theta)p}, \big\|y\big\|_{(1+\theta)p} \Big\}^\theta.$$

Proof. By standard arguments, cutting x and y by some of their spectral projections, we may assume that τ is finite x and y are bounded and invertible to avoid differentiability issues. We use

$$s^{1+\theta} = c_\theta \int_{\mathbb{R}_+} \frac{t^\theta s^2}{s+t}\,\frac{dt}{t}.$$

On bounded and invertible elements, the maps $f_t : s \mapsto \frac{s^2}{s+t} = s(s+t)^{-1}s$ are differentiable and

$$D_s f_t(\delta) = \delta(s+t)^{-1}s + s(s+t)^{-1}\delta - s(s+t)^{-1}\delta(s+t)^{-1}s.$$

Hence putting $\delta = x - y$, we get the integral representation

$$x^{1+\theta} - y^{1+\theta} = c_\theta \int_0^1 \int_{\mathbb{R}_+} t^\theta D_{y+u\delta} f_t(\delta) \, \frac{dt}{t} du.$$

We get, letting $g_t(s) = s(s+t)^{-1}$,

$$x^{1+\theta} - y^{1+\theta} = \int_0^1 \Big((y+u\delta)^\theta \delta + \delta (y+u\delta)^\theta \Big) \, du$$

$$-c_\theta \int_0^1 \int_{\mathbb{R}_+} t^\theta g_t(y+u\delta)\delta g_t(y+u\delta) \, \frac{dt}{t} du.$$

The first term is easily handled by the Hölder inequality. When u is fixed, note that $g_t(y+u\delta)$ is an invertible positive contraction. Put

$$\gamma^2 = c_\theta \int_{\mathbb{R}_+} t^\theta g_t(y+u\delta)^2 \frac{dt}{t} \leqslant (y+u\delta)^\theta,$$

and write $g_t(y+u\delta) = v_t \gamma$ so that v_t and $y+u\delta$ commute and

$$c_\theta \int_{\mathbb{R}_+} t^\theta v_t^2 \frac{dt}{t} = 1.$$

Therefore the map defined on $\mathcal{M}$, $x \mapsto c_\theta \int_{\mathbb{R}_+} t^\theta v_t x v_t \frac{dt}{t} = 1$ is unital completely positive and trace preserving. Hence it is a contraction on both L_1 and L_∞, thus it extends to a contraction on L_q when $1 \leqslant q \leqslant \infty$ by interpolation (see [7, section 5] for a general version of this fact). Applying it to $x = \gamma\delta\gamma$, we deduce

$$\left\| c_\theta \int_{\mathbb{R}_+} t^\theta g_t(y+u\delta)\delta g_t(y+u\delta) \, \frac{dt}{t} \right\|_p \leqslant \big\|\gamma\delta\gamma\big\|_p \leqslant \big\|\delta\big\|_{(1+\theta)p} . \big\|\gamma\big\|^2_{\frac{2(1+\theta)p}{\theta}}$$

$$\leqslant \big\|\delta\big\|_{(1+\theta)p} . \big\|y+u\delta\big\|^\theta_{(1+\theta)p}$$

thanks to the Hölder inequality again, this is enough to get the conclusion. □

Corollary 2.3. *Let $\alpha > 1$, $p \geqslant 1$, for any x, $y \in L^+_{\alpha p}(\mathcal{M})$:*

$$\big\|x^\alpha - y^\alpha\big\|_p \leqslant 3\alpha \big\|x-y\big\|_{\alpha p} \max\Big\{\big\|x\big\|_{\alpha p}, \big\|y\big\|_{\alpha p}\Big\}^{\alpha - 1}.$$

Proof. When $\alpha = n \in \mathbb{N}$, the result is obvious with constant n. For the general case, put $n = [\alpha]$, so that $\alpha = n(1+\delta)$ with $0 \leqslant \delta < 1$, then use the result for n and then Lemma 2.2. □

Coming back to the Mazur map $M_{p,q}$, Corollary 2.3 says that $M_{p,q}$ is Lipschitz on the positive unit ball of $L_p(\mathcal{M})$ if $q < p$. On the other hand, Lemma 2.1 says that it is $\frac{p}{q}$-Hölder if $q > p$. To release the positivity assumption, we will need a couple of Lemmas, but we start by reducing the problem to selfadjoint elements by a well known 2×2-trick.

If x, $y \in L_p(\mathcal{M})$ are in the unit ball with polar decompositions $x = u|x|$ and $y = v|y|$, we want to prove that with $\theta = \min\{\frac{p}{q}, 1\}$

$$\Big\| u|x|^{\frac{p}{q}} - v|y|^{\frac{p}{q}}\Big\|_q \leqslant c_{p,q}\Big\|x-y\Big\|_p^{\theta}. \tag{1}$$

In $\mathbb{M}_2(\mathcal{M})$ equipped with the tensor trace, let

$$\tilde{x} = \begin{pmatrix} 0 & x \\ x^* & 0\end{pmatrix} \quad\text{and}\quad \tilde{y} = \begin{pmatrix} 0 & y \\ y^* & 0\end{pmatrix}.$$

They are selfadjoint with polar decompositions

$$\tilde{x} = \tilde{u}|\tilde{x}| = \begin{pmatrix} 0 & u \\ u^* & 0\end{pmatrix}\cdot\begin{pmatrix} u|x|u^* & 0 \\ 0 & |x|\end{pmatrix} \quad\text{and}\quad \tilde{y} = \tilde{v}|\tilde{y}| = \begin{pmatrix} 0 & v \\ v^* & 0\end{pmatrix}\cdot\begin{pmatrix} v|y|v^* & 0 \\ 0 & |y|\end{pmatrix}.$$

The estimates for $\tilde{x}$ and $\tilde{y}$ imply that for x and y as

$$\tilde{u}|\tilde{x}|^{\frac{p}{q}} = \begin{pmatrix} 0 & u|x|^{\frac{p}{q}} \\ |x|^{\frac{p}{q}}u^* & 0\end{pmatrix} \quad\text{and}\quad \tilde{v}|\tilde{y}|^{\frac{p}{q}} = \begin{pmatrix} 0 & v|y|^{\frac{p}{q}} \\ |y|^{\frac{p}{q}}v^* & 0\end{pmatrix},$$

we have

$$\Big\|\tilde{x}-\tilde{y}\Big\|_p = 2^{\frac{1}{p}}\Big\|x-y\Big\|_p \qquad \Big\|\tilde{u}|\tilde{x}|^{\frac{p}{q}} - \tilde{v}|\tilde{y}|^{\frac{p}{q}}\Big\|_q = 2^{\frac{1}{q}}\Big\|u|x|^{\frac{p}{q}} - v|y|^{\frac{p}{q}}\Big\|_q.$$

Next, we reduce the theorem to a commutator estimate by using the 2×2-trick again. We use the commutator notation $[x, b] = xb - bx$. Put

$$\tilde{x} = \begin{pmatrix} x & 0 \\ 0 & y\end{pmatrix} \quad\text{and}\quad \tilde{b} = \begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}.$$

So that

$$\big\|[M_{p,q}(\tilde{x}), \tilde{b}]\big\|_q = \big\|M_{p,q}(x) - M_{p,q}(y)\big\|_q \qquad\text{and}\qquad \big\|[\tilde{x}, \tilde{b}]\big\|_p = \big\|x-y\big\|_p.$$

Lemma 2.4. *If* $p \geqslant 1$, $0 < \theta \leqslant 1$ *and* $x \in L_p^+(\mathcal{M})$ *and* $b \in \mathcal{M}$, *then*

$$\Big\|\big[x^\theta, b\big]\Big\|_{\frac{p}{\theta}} \leqslant 2^\theta \|b\|_\infty^{1-\theta}\big\|[x,b]\big\|_p^\theta,$$

$$\big\|[x,b]\big\|_p \leqslant \frac{12}{\theta}\,\|x\|_p^{1-\theta}\Big\|\big[x^\theta, b\big]\Big\|_{\frac{p}{\theta}}.$$

Proof. We start with the first inequality. We may assume $\|b\|_\infty = 1$ by homogeneity. Using the 2×2-trick with

$$\tilde{x} = \begin{pmatrix} x & 0 \\ 0 & x\end{pmatrix} \quad\text{and}\quad \tilde{b} = \begin{pmatrix} 0 & b \\ b^* & 0\end{pmatrix},$$

we may assume $b = b^*$ (without loosing on the constant).

Next, as $b = b^*$, we may use the Cayley transform defined by

$$u = (b-i)(b+i)^{-1}, \qquad b = 2i(1-u)^{-1} - i.$$

Clearly u is unitary, and the functional calculus gives that $\|(1-u)^{-1}\|_\infty \leqslant \frac{1}{\sqrt{2}}$. We have, using Lemma 2.1,

$$\begin{aligned}\big\|[x^\theta, b]\big\|_{\frac{p}{\theta}} &\leqslant 2\big\|x^\theta(1-u)^{-1} - (1-u)^{-1}x^\theta\big\|_{\frac{p}{\theta}}\\ &\leqslant 2\big\|(1-u)^{-1}\big\|_\infty^2\big\|x^\theta(1-u) - (1-u)x^\theta\big\|_{\frac{p}{\theta}}\\ &\leqslant \big\|u^*x^\theta u - x^\theta\big\|_{\frac{p}{\theta}}\\ &\leqslant \big\|xu - ux\big\|_p^\theta\\ &\leqslant \big\|(b+i)^{-1}\big\|_\infty^{2\theta}\big\|(b+i)x(b-i) - (b-i)x(b+i)\big\|_p^\theta\\ &\leqslant 2^\theta\,\big\|xb - bx\big\|_p^\theta.\end{aligned}$$

For the second one, we proceed similarly using Corollary 2.3. □

Lemma 2.5. *If $p \geqslant 1$, $0 < \theta \leqslant 1$, there are constants C and C_t $(t \geqslant 1)$ so that for any x, $y \in L_p^+(\mathcal{M})$ and $b \in \mathcal{M}$ then*

$$\Big\|x^\theta b + by^\theta\Big\|_{\frac{p}{\theta}} \leqslant C_{\frac{p}{\theta}}\big\|b\big\|_\infty^{1-\theta}\big\|xb + by\big\|_p^\theta,$$

$$\big\|xb + by\big\|_p \leqslant C\big\|x\big\|_p^{1-\theta}\Big\|x^\theta b + by^\theta\Big\|_{\frac{p}{\theta}}.$$

Proof. Using the 2×2-trick, we may assume $x = y$. Moreover, we may assume that x has full support in $\mathcal{M}$. Indeed let $e = 1_{(0,\infty)}(x)$ and $e^\perp = 1 - e$:

$$\big\|xb + bx\big\|_p \sim \big\|xebe + ebex\big\|_p + \big\|exbe^\perp\big\|_p + \big\|e^\perp bxe\big\|_p$$
$$\big\|x^\theta b + bx^\theta\big\|_{\frac{p}{\theta}} \sim \big\|x^\theta ebe + ebex^\theta\big\|_{\frac{p}{\theta}} + \big\|ex^\theta be^\perp\big\|_{\frac{p}{\theta}} + \big\|e^\perp bx^\theta e\big\|_{\frac{p}{\theta}}$$

If we apply the result in $e\mathcal{M}e$ where $xe \in e\mathcal{M}e$ has full support, we get control for the first terms. For the two middle terms, this is clear by interpolation as $\big\|ex^\theta be^\perp\big\|_{\frac{p}{\theta}} \leqslant \big\|exbe^\perp\big\|_p^\theta\|b\|_\infty^{1-\theta}$ and $\big\|exbe^\perp\big\|_p \leqslant \big\|ex^\theta be^\perp\big\|_{\frac{p}{\theta}}$ $\|ex\|_p^{1-\theta}$.

We will use techniques from [11] based on Schur multiplier estimates and interpolation. We use M_{cb} for the completely bounded norm of a Schur multiplier on $\mathbb{B}(\ell_2)$. By an obvious approximation, we may also assume that x has a finite spectrum. Let $(\lambda_i)_{i=1\ldots n}$ be the spectrum of x with associated projections $(p_i)_{i=1\ldots n}$. We start by the second inequality. For any $\alpha \in [0,1]$, the matrix $\left(\frac{\lambda_i^\alpha\lambda_j^{1-\alpha}+\lambda_i^{1-\alpha}\lambda_j^\alpha}{\lambda_i+\lambda_j}\right)_{i,j}$ defines a unital completely positive Schur multiplier on $\mathbb{B}(\ell_2^n)$, see the computation in [11, Corollary 2.5]; this implies that

$$\Big\|x^{1-\alpha}bx^\alpha + x^\alpha bx^{1-\alpha}\Big\|_p \leqslant \Big\|xb + bx\Big\|_p.$$

We use

$$xb + bx = x^{1-\theta}(x^\theta b + bx^\theta) + (x^\theta b + bx^\theta)x^{1-\theta} - (x^{1-\theta}bx^\theta + x^\theta bx^{1-\theta}).$$

Assume $\theta \geqslant \frac{1}{3}$, by the Hölder inequality

$$\Big\|xb+bx\Big\|_p \leqslant \|x\|_p^{1-\theta}\left(2\Big\|x^\theta b+bx^\theta\Big\|_{\frac{p}{\theta}}+\Big\|x^{\frac{1-\theta}{2}}bx^{\frac{3\theta-1}{2}}+x^{\frac{3\theta-1}{2}}bx^{\frac{1-\theta}{2}}\Big\|_{\frac{p}{\theta}}\right).$$

Using the above argument with $\alpha=\frac{1-\theta}{2}$:

$$\Big\|xb+bx\Big\|_p \leqslant 3\big\|x\big\|_p^{1-\theta}\Big\|x^\theta b+bx^\theta\Big\|_{\frac{p}{\theta}}.$$

When $\theta < \frac{1}{3}$, we use

$$\Big\|x^{1-\theta}bx^\theta+x^\theta bx^{1-\theta}\Big\|_p \leqslant 2\big\|x\big\|_p^{1-\theta}\Big\|x^{\frac{\theta}{2}}bx^{\frac{\theta}{2}}\Big\|_{\frac{p}{\theta}}.$$

And one corrects again by the above argument with $\alpha=\frac{1}{2}$ to get

$$\Big\|x^{1-\theta}bx^\theta+x^\theta bx^{1-\theta}\Big\|_p \leqslant \big\|x\big\|_p^{1-\theta}\Big\|x^\theta b+bx^\theta\Big\|_{\frac{p}{\theta}}.$$

For the first inequality, the result is a particular case of the main theorem of [11] assuming x has full support. The latter says the Banach spaces defined by norms $\|b\|_{L_q(x^\alpha)}=\|x^\alpha b+bx^\alpha\|_q$ interpolate, so that $L_{\frac{p}{\theta}}(x^\theta)=(L_\infty(x^0),L_p(x))_\theta$. As a corollary,

$$\Big\|x^\theta b+bx^\theta\Big\|_{\frac{p}{\theta}} \leqslant C_{\frac{p}{\theta}}\big\|b\big\|_\infty^{1-\theta}\big\|xb+bx\big\|_p^\theta.$$

To avoid the use of [11], we provide an alternate proof of the latter inequality with a better constant only when $p=1$ and $\theta\leqslant\frac{1}{2}$. Assuming $\|b\|_\infty\leqslant 1$, we use the Jensen inequality from [3] for the convex function $x\mapsto x^{\frac{1}{2\theta}}$ (for us it follows easily from the operator convexity of x^α for $\alpha\in[1,2]$ and an iteration argument):

$$\begin{aligned}\Big\|x^\theta b+bx^\theta\Big\|_{\frac{1}{\theta}}^{\frac{1}{\theta}} &\leqslant 2^{\frac{1}{\theta}}\left(\big\|x^\theta b\big\|_{\frac{1}{\theta}}^{\frac{1}{\theta}}+\big\|bx^\theta\big\|_{\frac{1}{\theta}}^{\frac{1}{\theta}}\right)\\ &\leqslant 2^{\frac{1}{\theta}}\tau\Big(\big(b^*x^{2\theta}b\big)^{\frac{1}{2\theta}}+\big(bx^{2\theta}b^*\big)^{\frac{1}{2\theta}}\Big)\\ &\leqslant 2^{\frac{1}{\theta}}\tau\Big(b^*xb+bxb^*\Big)\\ &\leqslant 2^{\frac{1}{\theta}}\big\|xb+bx\big\|_1.\end{aligned}$$

□

Lemma 2.6. *There is an absolute constant $C>0$ and constants C_t $(t\geqslant 1)$ so that:*

- *If $q>p\geqslant 1$, and $x\in L_p(\mathcal{M})$, $x=x^*$ and $b\in\mathcal{M}$, then*

$$\Big\|\big[M_{p,q}(x),b\big]\Big\|_q \leqslant C_q\big\|b\big\|_\infty^{1-\frac{p}{q}}\big\|[x,b]\big\|_p^{\frac{p}{q}}. \tag{2}$$

- *If $p>q\geqslant 1$, and $x\in L_p(\mathcal{M})$, $x=x^*$ and $b\in\mathcal{M}$, then*

$$\Big\|\big[M_{p,q}(x),b\big]\Big\|_q \leqslant C\frac{p}{q}\big\|x\big\|_p^{\frac{p}{q}-1}\big\|[x,b]\big\|_p. \tag{3}$$

Proof. For (2), write $e_+ = 1_{[0,\infty)}(x)$ and $e_- = 1_{(-\infty,0)}(x)$ and put $b_{\pm,\pm} = e_\pm b e_\pm$. So that

$$\big[M_{p,q}(x), b\big] = \big[x_+^{\frac{p}{q}}, b_{+,+}\big] - \big[x_-^{\frac{p}{q}}, b_{-,-}\big] + \big(x_+^{\frac{p}{q}} b_{+,-} + b_{+,-} x_-^{\frac{p}{q}}\big) \\ - \big(x_-^{\frac{p}{q}} b_{-,+} + b_{-,+} x_+^{\frac{p}{q}}\big).$$

We can apply either Lemmas 2.4 or 2.5 to each term. In any case, the upper bound we get is smaller than the right side of (2).

A similar argument works for (3). □

Remark 2.7. The techniques developed here work if one replaces $M_{p,q}$ by any function $f : \mathbb{R} \to \mathbb{R}$. With such a general function f, 2.6 boils down to the boundedness of some Schur multipliers on $S_p[L_p(\mathcal{M})]$ (by the discretization from [11]), this is the argument of [6]. This also explains why the results of [1,6,9] remain true for semifinite von Neumann algebras.

3. General case. In the general case, we use the Haagerup definition of L_p-spaces [12] and the Haagerup reduction technique from [7] (see [4] for an extension from states to weights). As the construction is very technical, we only give a sketch to keep the paper short. Let $\mathcal{M}$ be a general von Neumann algebra with a fixed faithful normal semifinite weight φ. As usual σ^φ denotes the automorphism group of φ and we use the classical notation $\mathfrak{n}_\varphi$, $\mathfrak{m}_\varphi$,... for other constructions associated to φ. We let $\hat{\mathcal{M}} = \mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R}$ be the core of $\mathcal{M}$. It is a semifinite von Neumann algebra with a distinguished trace τ such that $\tau \circ \hat{\sigma}_s = e^{-s}\tau$, where $\hat{\sigma}$ is the dual action of $\mathbb{R}$ on $\hat{\mathcal{M}}$. The definition is then

$$L_p^\varphi(\mathcal{M}) = \Big\{ f \in L_0(\hat{\mathcal{M}}, \tau) \mid \hat{\sigma}_s(x) = e^{-\frac{s}{p}} x \Big\}.$$

Then $L_1^\varphi(\mathcal{M})$ is order isometric to $\mathcal{M}_*$, and the evaluation at 1 is denoted by tr. The L_p^φ norm is given by $\|x\|_p^p = \mathrm{tr}|x|^p$. We also denote by D_φ the Radon–Nykodym derivative of the dual weight $\hat{\varphi}$ with respect to τ.

These L_p^φ spaces are disjoint, and the norm topology coincides with the measure topology of $L_0(\hat{\mathcal{M}}, \tau)$ (Proposition 26 in [12]). The construction does not depend on the choice of φ up to $*$-topological isomorphisms (see below) so that we may drop the superscript φ when no confusion can arise.

The Haagerup reduction theorem is (see [7, Theorem 2.1] or [4, Theorem 7.1]):

Theorem 3.1. *For any $(\mathcal{M}, \varphi)$ there is a bigger von Neumann algebra $(\mathcal{R}, \tilde{\varphi})$, where $\tilde{\varphi}$ is a nfs weight extending φ, a family a_n in the center of the centralizer of $\tilde{\varphi}$ so that*

i) *there is a conditional expectation $\mathcal{E} : \mathcal{R} \to \mathcal{M}$ such that*

$$\varphi \circ \mathcal{E} = \tilde{\varphi} \quad \text{and} \quad \mathcal{E} \circ \sigma_s^{\tilde{\varphi}} = \sigma_s^{\varphi} \circ \mathcal{E} \quad \text{for all} \quad s \in \mathbb{R},$$

ii) *the centralizer $\mathcal{R}_n$ of $\varphi_n(.) = \tilde{\varphi}(e^{-a_n}.)$ is semifinite for all $n \geqslant 1$ (with trace φ_n),*

iii) *there exist conditional expectations $\mathcal{E}_n : \mathcal{R} \to \mathcal{R}_n$ such that*

$$\tilde{\varphi} \circ \mathcal{E}_n = \tilde{\varphi} \quad \text{and} \quad \mathcal{E}_n \circ \sigma_s^{\tilde{\varphi}} = \sigma_s^{\tilde{\varphi}} \circ \mathcal{E}_n \quad \text{for all} \quad s \in \mathbb{R},$$

iv) $\mathcal{E}_n(x) \to x$ *σ-strongly for* $x \in \mathfrak{n}_{\tilde{\varphi}}$ *and* $\bigcup_{n\geqslant 1} \mathcal{R}_n$ *is σ-strongly dense in* $\mathcal{R}$.

The modular conditions for the conditional expectations imply that we can view $L_p(\mathcal{M})$ and $L_p(\mathcal{R}_n)$ as subspaces of $L_p(\mathcal{R})$ and there are extensions

$$\mathcal{E}^p : L_p(\mathcal{R}) \to L_p(\mathcal{M}) \quad \text{and} \quad \mathcal{E}_n^p : L_p(\mathcal{R}) \to L_p(\mathcal{R}_n).$$

Moreover from iv), for any $x \in L_p(\mathcal{R})$ $(1 \leqslant p < \infty)$ we have (see [4, Lemma 7.3] for instance):

$$\lim_{n\to\infty} \big\|\mathcal{E}_n^p(x) - x\big\|_p = 0.$$

Now we make explicit the independence of $L_p(\mathcal{R}_n)$ relative the choice of the weight. Considering $\mathcal{R}_n$ with φ_n or $\tilde{\varphi}_n$ gives two constructions, the corresponding spaces of measurable operators $N_{\varphi_n} = L_0(\mathcal{R}_n \rtimes_{\sigma^{\varphi_n}} \mathbb{R}, \hat{\varphi}_n)$ and $N_{\tilde{\varphi}} = L_0(\mathcal{R}_n \rtimes_{\sigma^{\tilde{\varphi}}} \mathbb{R}, \tau)$ in which the L_p-spaces live. By [12, Corollary 38], there is a topological $*$-homomorphism $\kappa : N_{\tilde{\varphi}} \to N_{\varphi_n}$ so that $\kappa(L_p^{\tilde{\varphi}}(\mathcal{R}_n)) = L_p^{\varphi_n}(\mathcal{R}_n)$ and that is isometric on L_p.

As φ_n is a trace, we know that $\mathcal{R}_n \rtimes_{\sigma^{\varphi_n}} \mathbb{R} \simeq \mathcal{R}_n \otimes L_\infty(\mathbb{R})$ and the identification $\iota_p : L_p(\mathcal{R}_n, \varphi_n) \to L_p^{\varphi_n}(\mathcal{R}_n)$ is $\iota_p(x) = x \otimes e^{\frac{\cdot}{p}}$. Hence we get isometric isomorphisms $\kappa_p = \iota_p^{-1} \circ \kappa : L_p(\mathcal{R}_n) \to L_p(\mathcal{R}_n, \varphi_n)$ that are compatible with left and right multiplications by elements of $\mathcal{R}_n$ and powers in the sense that for $1 \leqslant q, p < \infty$ and $x \in L_p^+(\mathcal{R}_n)$

$$\kappa_p(x)^{\frac{p}{q}} = \kappa_q\big(x^{\frac{p}{q}}\big). \tag{4}$$

One can check that κ_p is formally given by $\kappa_p(D_{\tilde{\varphi}}^{\frac{1}{2p}} x D_{\tilde{\varphi}}^{\frac{1}{2p}}) = e^{-\frac{a_n}{2p}} x e^{-\frac{a_n}{2p}}$ for $x \in \mathfrak{m}_{\varphi_n}$.

Now we can conclude the proof of the theorem in the general case. Take x and y in $L_p(\mathcal{M})$, then

$$\big\|x-y\big\|_p = \lim_{n\to\infty} \big\|\mathcal{E}_n(x) - \mathcal{E}_n(y)\big\|_{L_p(\mathcal{R}_n)} = \lim_{n\to\infty} \big\|\kappa_p(\mathcal{E}_n(x)) - \kappa_p(\mathcal{E}_n(y))\big\|_{L_p(\mathcal{R}_n,\varphi_n)}.$$

By Lemma 3.2 in [10], the map $M_{p,q}$ is continuous on $N_{\tilde{\varphi}}$, thus also $L_p \to L_q$, hence

$$\big\|M_{p,q}(x) - M_{p,q}(y)\big\|_q = \lim_{n\to\infty} \big\|\kappa_q(M_{p,q}(\mathcal{E}_n(x))) - \kappa_q(M_{p,q}(\mathcal{E}_n(y)))\big\|_{L_q(\mathcal{R}_n,\varphi_n)}.$$

But thanks to (4), $\kappa_q(M_{p,q}(\mathcal{E}_n(x))) = M_{p,q}(\kappa_p(\mathcal{E}_n(x))$, so that we can use the estimate for semifinite von Neumann algebras to conclude.

In the same way, all inequalities from section 2 can be extended to arbitrary von Neumann algebras (except Remark 2.7 as one can not make sense of $f(x) \in L_q$ when $x \in L_p^{sa}$ for general functions other than powers).

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