Arch. Math. 103 (2014), 301–311

0003-889X/14/040301-11
published online October 24, 2014
DOI 10.1007/s00013-014-0696-3

Archiv der Mathematik

Quasi-antichain Chermak–Delgado lattices of finite groups

Ben Brewster, Peter Hauck, and Elizabeth Wilcox

Dedicated to Otto H. Kegel for the occasion of his eightieth birthday.

Abstract. The Chermak–Delgado lattice of a finite group is a dual, modular sublattice of the subgroup lattice of the group. This paper considers groups with a quasi-antichain interval in the Chermak–Delgado lattice, ultimately proving that if there is a quasi-antichain interval between subgroups L and H with $L \leq H$ then there exists a prime p such that H/L is an elementary abelian p-group and the number of atoms in the quasi-antichain is one more than a power of p. In the case where the Chermak–Delgado lattice of the entire group is a quasi-antichain, the relationship between the number of abelian atoms and the prime p is examined; additionally, several examples of groups with a quasi-antichain Chermak–Delgado lattice are constructed.

Mathematics Subject Classification. 20D30.

Keywords. Lattices of subgroups.

This paper pursues the nature of the Chermak–Delgado lattice of a finite group. The Chermak–Delgado lattice was introduced by Chermak and Delgado [4]. Isaacs [6] re-introduced the lattice, sparking further study that resulted in [3] and [2]. In this article, we provide three primary contributions to the study of Chermak–Delgado lattices: a description of the structure of groups with a quasi-antichain (defined below) as an interval in the Chermak–Delgado lattice, results that narrow the possible structure of a quasi-antichain realized as a Chermak–Delgado lattice, and examples to illustrate the breadth of possibilities. Among these contributions is a proof that if a Chermak–Delgado lattice has an interval which is a quasi-antichain, then the width must be a power of a prime plus 1.

The research of the second author was supported by Projecto MTM2010-19938-C03-02, Ministerio de Ciencia e Innovación, Spain. The research of the third author was supported by a Scholarly and Creative Activity Grant from Oswego State University.

Throughout the article, let G be a finite group and p be a prime. The Chermak–Delgado lattice of a finite group G consists of subgroups $H \leq G$ such that $|H||C_G(H)|$ is maximal among all subgroups of G. For any subgroup H of G, the product $|H||C_G(H)|$ is called the *Chermak–Delgado measure of* H (in G) and is denoted by $m_G(H)$; if the group G is clear from the context, then we write simply $m(H)$. To denote the maximum possible Chermak–Delgado measure of a subgroup in G, we write $m^*(G)$ and we refer to the set of all subgroups with measure attaining that maximum as the Chermak–Delgado lattice of G, or $\mathcal{CD}(G)$.

It is known that the Chermak–Delgado lattice is a modular self-dual lattice and if $H, K \in \mathcal{CD}(G)$ then $HK = KH = \langle H, K\rangle$. The duality of the Chermak–Delgado lattice is a result of the fact that if $H \in \mathcal{CD}(G)$ then $C_G(H) \in \mathcal{CD}(G)$ and $H = C_G(C_G(H))$. Moreover, if M is the maximum subgroup in the Chermak–Delgado lattice of a group G, then the Chermak–Delgado lattices of G and M coincide. It is additionally known that the co-atoms in the Chermak–Delgado lattice are normal in M, and consequently the atoms, as centralizers of normal subgroups, are also normal in M. In [2], groups whose Chermak–Delgado lattice is a chain were studied; a *chain of length* n, where n is a positive integer, is a totally ordered lattice with $n + 1$ subgroups. We call a lattice consisting of a maximum, a minimum, and the atoms of the lattice a *quasi-antichain* and the width of the quasi-antichain is the number of atoms. A quasi-antichain of width 1 is also a chain of length 2.

Let $L \leq H \leq G$; we use $[\![L, H]\!]$ to denote the interval from L to H in a sublattice of the lattice of subgroups of G. If $[\![L, H]\!]$ is an interval in $\mathcal{CD}(G)$, then the duality of the Chermak–Delgado lattice tells us that $[\![C_G(H), C_G(L)]\!]$ is an interval in $\mathcal{CD}(G)$. Of course, these intervals may overlap or even coincide exactly. In Sect. 1 we make no assumption about the intersection of $[\![L, H]\!]$ and $[\![C_G(H), C_G(L)]\!]$; in the final two sections, we study the situation where these two intervals not only are equal, but are the entirety of $\mathcal{CD}(G)$.

1. Quasi-antichain intervals in Chermak–Delgado lattices. Let G be a group with $L < H \leq G$ such that $[\![L, H]\!]$ is an interval in $\mathcal{CD}(G)$. The main theorem of this section establishes that if $[\![L, H]\!]$ is a quasi-antichain of width $w \geq 3$, then there exists a prime p and positive integers a, b with $b \leq a$ such that H/L is an elementary abelian p-group with order p^{2a} and $w = p^b + 1$. To make the role of the duality more transparent in the proofs, set $H^* = C_G(L)$ and $L^* = C_G(H)$. Observe that $C_G(H^*) = L$ and $C_G(L^*) = H$.

We start with a general statement about subgroups that are in the interval $[\![L, H]\!]$ in $\mathcal{CD}(G)$. Let p be a prime dividing the index of L in H. For this result, we remind the reader that $\Omega_k(M)$, where k is a positive integer and M is any group, denotes the subgroup of M generated by the elements whose order divides p^k.

The hypothesis of Proposition 1 may initially sound restrictive: We require that G be a group with $[\![L, H]\!]$ in $\mathcal{CD}(G)$ such that $[HH^*, HH^*] \leq L \cap L^*$. Note that $L \trianglelefteq H$ and $L^* \trianglelefteq H^*$, so the quotient groups described in Proposition 1 are well-defined. Moreover, notice that the hypotheses of the proposition occur when G is a p-group of nilpotence class 2 and $H = G \in \mathcal{CD}(G)$.

Proposition 1. *Let G be a group with an interval $[\![L, H]\!]$ in $\mathcal{CD}(G)$ such that $[HH^*, HH^*] \leq L \cap L^*$. Suppose that p is a prime dividing $|H/L|$. The subgroups $A_k(H), B_k(H) \leq H$ where $A_k(H)/L = \Omega_k(H/L)$ and $B_k(H) = \langle x^{p^k} \mid x \in H\rangle L$ are members of $\mathcal{CD}(G)$ for all positive values of k, as are the similarly defined subgroups $A_k(H^*), B_k(H^*)$ of H^*.*

Proof. Let k be a positive integer. Without loss of generality, assume that $|A_k(H)/L| \geq |A_k(H^*)/L^*|$. We first show that $C_G(A_k(H)) = B_k(H^*)$ and that $A_k(H) \in \mathcal{CD}(G)$.

Observe that if $x \in H$ and $y \in H^*$ then $[x, y] \in [H, H^*] \leq L \cap L^* = C_G(H^*) \cap C_G(H)$. Therefore $[x^p, y] = [x, y^p]$ whenever $x \in H$ and $y \in H^*$. Moreover, if $x \in A_k(H)$ then $x^{p^k} \in L$, so $[x^{p^k}, y] = 1$ for all $y \in C_G(L) = H^*$. Thus if $x \in A_k(H)$ and y is a generator of $B_k(H^*)$, then $[x, y] = 1$, therefore $B_k(H^*) \leq C_G(A_k(H))$.

Since the quotient H/L is abelian, $|A_k(H)/L| = |H/B_k(H)|$ or equivalently $|A_k(H)/L||B_k(H)| = |H|$. The same is true regarding $|H^*|$ and the subgroups $A_k(H^*)$, $B_k(H^*)$. Thus the measure of $A_k(H)$ in G can be calculated as follows:

$$\begin{aligned} m(A_k(H)) = |A_k(H)||C_G(A_k(H))| &\geq |A_k(H)||B_k(H^*)| \\ &= |A_k(H)/L||L||B_k(H^*)| \\ &\geq |A_k(H^*)/L^*||L||B_k(H^*)| \\ &= |H^*||L| \\ &= |C_G(L)||L| = m^*(G). \end{aligned}$$

Therefore each inequality above is actually an equality, with $C_G(A_k(H)) = B_k(H^*)$ and $|A_k(H)/L| = |A_k(H^*)/L^*|$. Additionally $A_k(H)$, $B_k(H^*)$, $A_k(H^*)$, $B_k(H) \in \mathcal{CD}(G)$. □

For the rest of the paper, we study intervals that are quasi-antichains. Ultimately we will use Proposition 1 to show that $A_1 = H$ and $A_1^* = H^*$ in the case that $[\![L, H]\!]$ is a quasi-antichain of width $w \geq 3$. The next two propositions establish important facts about the atoms of a quasi-antichain interval in $\mathcal{CD}(G)$, as well as show that such an interval satisfies the hypothesis of Proposition 1.

Let $[\![L, H]\!]$ be a quasi-antichain of width w throughout the remainder of the article. Let the w atoms of the quasi-antichain be denoted by $K_1, K_2, \ldots, K_w$. The interval $[\![L^*, H^*]\!]$ is also a quasi-antichain in $\mathcal{CD}(G)$, with atoms $C_G(K_i)$ where $1 \leq i \leq w$. For each i, let $K_i^* = C_G(K_i)$ so that $C_G(K_i^*) = K_i$.

Proposition 2. *If K_1, K_2 are distinct atoms of the quasi-antichain, then $K_i \trianglelefteq H$ for $i = 1, 2$, $L \trianglelefteq H$, and $[K_1, K_2] \leq L$ and analogously for H^*, K_1^*, K_2^*, and L^*. Moreover, $|K_1 : L| = |K_2^* : L^*|$.*

If $w \geq 3$, then $K_i/L \cong K_j/L$ and $K_i^*/L^* \cong K_j^*/L^*$ for all i, j with $1 \leq i, j \leq w$. Furthermore:

$$|H/L| = |H/K_1|^2 = |H^*/K_1^*|^2 = |H^*/L^*|.$$

Proof. Let $K_1, K_2 \in \mathcal{CD}(G)$ with $L < K_i < H$ for $i = 1, 2$. Because the interval $[\![L, H]\!]$ is a quasi-antichain, $H = K_1K_2$ and $K_1 \cap K_2 = L$. From this structure and because H cannot equal $K_1K_1^h$ for $h \in H$, it follows that $K_1 \trianglelefteq H$ (similarly for K_2). Therefore $L \trianglelefteq H$ and $[K_1, K_2] \leq L$. The equality $m^*(G) = m(H) = m(K_2)$ implies

$$\frac{|K_2||K_1|}{|L|}|C_G(H)| = |H||C_G(H)| = |K_2||C_G(K_2)|,$$

and consequently $|K_1 : L| = |C_G(K_2) : C_G(H)| = |K_2^* : L^*|$.

In the situation that $w \geq 3$, then $H = K_1K_2 = K_3K_2$ where $K_1 \cap K_2 = K_2 \cap K_3 = L$ and thus $|K_i| = |K_j|$ for all i, j with $1 \leq i, j \leq w$. This additionally yields $K_i/L \cong K_j/L$ for all i, j. From the Isomorphism Theorems, it follows that $|H/K_1| = |K_1/L|^2$.

The same arguments applied to the quasi-antichain $[\![L^*, H^*]\!]$ yield the remaining assertions. □

Proposition 3. *If $w \geq 3$, then $[H, H^*] \leq L \cap L^* = C_G(HH^*)$. Additionally, H/L and H^*/L^* are isomorphic elementary abelian p-groups. In particular, if $G = H$ then $G/Z(G)$ and $[G, G]$ are elementary abelian p-groups.*

Proof. Since $w \geq 3$, there exist at least three distinct atoms K_1, K_2, and K_3 in the interval $[\![L, H]\!]$ in $\mathcal{CD}(G)$. By Proposition 2,

$$[K_1, K_2K_3] \leq \langle [K_1, K_3][K_1, K_2] \rangle \leq L.$$

Therefore K_1/L centralizes $K_2K_3/L = H/L$. By symmetry, the same holds for K_2/L; consequently $H = K_1K_2$ centralizes H/L and H/L is abelian. Similarly H^* centralizes H^*/L^*, and the latter is abelian.

Since K_i normalizes every K_j, it also normalizes every K_j^*. Therefore $[K_i, H^*] = [K_i, K_1^*K_2^*] \leq K_2^*$ and $[K_i, H^*] = [K_i, K_1^*K_3^*] \leq K_3^*$, so that $[K_i, H^*] \leq L^*$. Similarly $[K_2, H^*] \leq L^*$ and thus $[H, H^*] \leq L^*$. In the same way, $[H, H^*] \leq L$. By the duality of the Chermak-Delgado lattice, $L \cap L^* = C_G(HH^*)$ so $[H, H^*] \leq L \cap L^* = C_G(HH^*)$, as claimed.

Applying Proposition 1, the subgroup A_1, where $A_1/L = \Omega_1(H/L)$, is a member of $\mathcal{CD}(G)$. Since $\mathcal{CD}(G)$ is a quasi-antichain, either $A_1 = H$ or there exists i such that $1 \leq i \leq w$ where $K_i = A_1$. At minimum, K_i/L is an elementary abelian p-group but, as $K_i/L \cong K_j/L$ for all i, j, we have H/L is an elementary abelian p-group. With similar reasoning, H^*/L^* is an elementary abelian p-group and, since $|H/L| = |H^*/L^*|$, these quotients are isomorphic elementary abelian p-groups.

If $H = G$ and $\mathrm{Z}(G) = L$, then $G/\mathrm{Z}(G)$ is an elementary abelian p-group. Thus, for $x, y \in G$, we have $[x, y]^p = [x^p, y] = 1$; therefore $[G, G]$ is elementary abelian. □

Theorem 4. *Let G be a group such that $L, H \in \mathcal{CD}(G)$ with the interval $[\![L, H]\!]$ in $\mathcal{CD}(G)$ a quasi-antichain of width $w \geq 3$. There exists a prime p and positive integers a, b with $b \leq a$ such that H/L and $C_G(L)/C_G(H)$ are elementary abelian p-groups of order p^{2a} and $w = p^b + 1$.*

Proof. The existence of the prime p and the fact that H/L and H^*/L^* are elementary abelian p-groups were established in Proposition 3. From Proposition 2, we know $H/L = K_1/L \times K_2/L$. Let $i \geq 3$; the subgroup K_i/L projects onto each coordinate under the natural projection maps and intersects each of K_1/L and K_2/L trivially. Thus K_i/L is a subdirect product and there exists an isomorphism $\overline{\beta_i} : K_1/L \to K_2/L$ such that $K_i/L = \{(kL)\overline{\beta_i}(kL) \mid k \in K_1\}$. Choose $\beta_i(k) \in K_2$ with $\overline{\beta_i}(kL) = \beta_i(k)L$; then $K_i/L = \{k\beta_i(k)L \mid k \in K_1\}$. Similarly, there exists an isomorphism $\overline{\alpha_i} : K_1^*/L^* \to K_2^*/L^*$ where $\overline{\alpha_i}(mL^*) = \alpha_i(m)L^*$ for each $m \in K_1^*$ and $K_i^*/L^* = \{m\alpha_i(m)L^* \mid m \in K_1^*\}$.

For i, j such that $3 \leq i, j \leq w$, let $\Delta_{i,j} = \{k\beta_i(k)\beta_j(k) \mid k \in K_1\}$ and $\Delta^*_{i,j} = \{m\alpha_i(m)\alpha_j(m) \mid m \in K_1^*\}$; additionally define $K_{i,j} = \Delta_{i,j}L$ and $K^*_{i,j} = \Delta^*_{i,j}L^*$. Since $[K_1, K_2] \leq L$ and the functions $\overline{\beta_i}$, $\overline{\beta_j}$, are homomorphisms, it follows that $K_{i,j} \leq H$. Also observe that if $k\beta_i(k)\beta_j(k)L = k'\beta_i(k')\beta_j(k')L$, then $kL = k'L$ because $K_1 \cap K_2 = L$. Therefore $|K_{i,j}/L| = |K_1/L|$. Corresponding facts are true regarding $K^*_{i,j}$.

Our goal is to show that $K_{i,j}$ is one of the atoms in $[\![L, H]\!]$, so we calculate $m(K_{i,j})$. From the definitions, clearly $[k_1, m_1] = [k_2, m_2] = 1$ when $k_i \in K_i$ and $m_i \in K_i^*$ for $i = 1, 2$. By this information and the fact that $[H, H^*]$ is centralized by H and H^*, if $k \in K_1$ and $m \in K_1^*$ then we obtain

$$1 = [k\beta_i(k), m\alpha_i(m)] = [k, \alpha_i(m)][\beta_i(k), m]$$

for all i such that $3 \leq i \leq w$. Given $k \in K_1$ and $m \in K_1^*$, if $3 \leq i, j \leq w$ then

$$\begin{aligned}
&[k\beta_i(k)\beta_j(k), m\alpha_i(m)\alpha_j(m)] \\
&\quad = [k, \alpha_j(m)][\beta_i(k), \alpha_j(m)][\beta_j(k), \alpha_i(m)][\beta_j(k), m] \\
&\quad = [k, \alpha_j(m)][\beta_j(k), m] \\
&\quad = 1.
\end{aligned}$$

By $[H, L^*] = [H^*, L] = 1$ and the above calculation, $K^*_{i,j} \leq C_G(K_{i,j})$. Because $|K_{i,j}| = |K_1|$ and $|K^*_{i,j}| = |K_1^*|$, therefore $m(K_{i,j}) = m(K_1)$ and $K_{i,j} \in \mathcal{CD}(G)$. Thus for each i, j with $3 \leq i, j \leq w$, either $K_{i,j} = K_h$ for some h such that $3 \leq h \leq w$ or $K_{i,j} = K_1$. Setting $\beta_2(k) = 1$ for all $k \in K_1$, it follows that $\{k\beta_i(k)\beta_j(k)L \mid k \in K_1\} = \{k\beta_h(k)L \mid k \in K_1\}$ for some h with $2 \leq h \leq w$. Notice that if $k\beta_i(k)\beta_j(k)L = k'\beta_h(k')L$, then $kL = k'L$ and $\beta_i(k)\beta_j(k)L = \beta_h(k)L$ because $K_1 \cap K_2 = L$.

Fix a $k_1 \in K_1 \backslash L$. Let $\Lambda = \{\beta_2(k_1), \beta_3(k_1), \ldots \beta_w(k_1)\}$ and $R = \Lambda \cdot L$. By what we have shown in the preceding paragraphs, $R \leq H$ and, as $K_i \cap K_j = L$ for $i \neq j$, the set Λ is a transversal for L in R. Hence $|R/L| = |\Lambda| = w - 1$. Since $R \leq K_2$, it follows that $w - 1$ divides p^a. □

2. Quasi-antichain Chermak–Delgado lattices. We study here the groups G such that $\mathcal{CD}(G)$ is a quasi-antichain and $G \in \mathcal{CD}(G)$, meaning that $G = H = H^*$ and $\mathrm{Z}(G) = L = L^*$ in the notation of the first section. Additionally, the subgroups K_i^* are now atoms of $[\![L, H]\!]$; notice $K_i^* = K_i$ if and only if K_i is abelian.

When studying groups of this type, the added condition that $[G, G]$ be cyclic imposes very strong restrictions on the structure of the group, as seen in the next proposition.

Proposition 5. *Let $G \in \mathcal{CD}(G)$ and $[G, G]$ be cyclic. Then $\mathcal{CD}(G)$ is a quasi-antichain of width $w \geq 3$ with $G \in \mathcal{CD}(G)$ if and only if there exists a prime p such that $|[G, G]| = p$ and $G/Z(G) \cong C_p \times C_p$. In this case $w = p + 1$.*

Proof. Let $[G, G]$ be cyclic and $G \in \mathcal{CD}(G)$. Suppose first that $\mathcal{CD}(G)$ is a quasi-antichain of width $w \geq 3$. We know that there exists a prime p such that $G/\mathrm{Z}(G)$ and $[G, G]$ are elementary abelian p-groups by Proposition 3. Therefore $[G, G]$ has order p but, more importantly, all $U \leq G$ such that $\mathrm{Z}(G) \leq U$ are centralizers by [7, Satz]. In particular, a maximal subgroup M with $\mathrm{Z}(G) \leq M$ is a centralizer, so there exists $U > \mathrm{Z}(P)$ with $M = C_G(U)$ and $m(M) = \frac{|G|}{p}|C_G(X)| \geq |G||\mathrm{Z}(G)|$. Yet $G \in \mathcal{CD}(G)$, so $M, U \in \mathcal{CD}(G)$. The Chermak–Delgado lattice of G is a quasi-antichain of width at least 3, so by Proposition 2, $|M| = |U|$ and thus $|G/\mathrm{Z}(P)| = p^2$.

Now suppose there exists a prime p such that $|[G, G]| = p$ and $G/\mathrm{Z}(G) \cong C_p \times C_p$. In this case, all $p+1$ subgroups U such that $\mathrm{Z}(G) < U < G$ are abelian, have order $p|\mathrm{Z}(G)|$, and have measure $p^2|\mathrm{Z}(G)|^2$, which also equals the measure of G. Therefore $\mathcal{CD}(G) = \{U \leq G \mid \mathrm{Z}(G) \leq U\}$ is a quasi-antichain of width $p + 1$ with $G \in \mathcal{CD}(G)$. □

The next theorem justifies our attention on p-groups while studying groups with a quasi-antichain Chermak–Delgado lattice. The proof of Theorem 6 requires the following observation: Let M and N be any pair of finite groups. The modularity of the Chermak–Delgado lattice implies that every maximal chain in the lattice has the same length. For example, all maximal chains in $\mathcal{CD}(G)$ have length 2 because $\mathcal{CD}(G)$ is a quasi-antichain. That $\mathcal{CD}(M \times N) \cong \mathcal{CD}(M) \times \mathcal{CD}(N)$ as lattices [3] gives that the length of a maximal chain in $M \times N$ is the sum of the lengths of maximal chains in M and N.

Theorem 6. *If G is a group with $\mathcal{CD}(G)$ a quasi-antichain of width $w \geq 3$ and $G \in \mathcal{CD}(G)$, then G is nilpotent of class 2; in fact, there exists a prime p, a nonabelian Sylow p-subgroup P with nilpotence class 2, and an abelian Hall p'-subgroup Q such that $G = P \times Q$, $P \in \mathcal{CD}(P)$, and $\mathcal{CD}(G) \cong \mathcal{CD}(P)$ as lattices. Moreover, there exist positive integers a, b with $b \leq a$ such that $|G/Z(G)| = |P/Z(P)| = p^{2a}$ and $w = p^b + 1$.*

Proof. Note that G is nilpotent, by Proposition 3, but nonabelian and the length of a maximal chain in $\mathcal{CD}(G)$ is 2. If $G = Q_1 \times Q_2$ where Q_1 and Q_2 are Hall π-, π'-subgroups of G, respectively, then $\mathcal{CD}(G) \cong \mathcal{CD}(Q_1 \times Q_2)$; as a consequence of the additivity of chain length over a direct product, if both Q_1 and Q_2 are nonabelian, then $\mathcal{CD}(Q_i) = \{Q_i, \mathrm{Z}(Q_i)\}$ for $i = 1, 2$. However, this implies that $\mathcal{CD}(Q_1 \times Q_2)$ is a quasi-antichain of width 2. Consequently, exactly one of Q_1 or Q_2 is abelian and the Chermak–Delgado lattice of the nonabelian factor is isomorphic (as lattices) to $\mathcal{CD}(G)$. Therefore there exists a unique prime p such that $G = P \times Q$, where P is a nonabelian Sylow p-subgroup of G and Q is an abelian Hall p'-subgroup of G, with $\mathcal{CD}(G) \cong \mathcal{CD}(P)$ as lattices. The rest follows from Proposition 2 and Theorem 4. □

We investigated the number of abelian atoms that is permitted in a quasi-antichain Chermak–Delgado lattice. The final theorem of this section records our contributions in this direction. Recall that the nonabelian atoms must come in pairs, K and $C_G(K) = K^*$; our theorem therefore examines the number of abelian atoms and the number of pairs of nonabelian atoms, where each pair consists of a subgroup and its centralizer.

Theorem 7. *Let G be a p-group with $\mathcal{CD}(G)$ a quasi-antichain of width $w \geq 3$ and suppose $|G/Z(G)| = p^{2a}$ for a positive integer a. Let t be the number of abelian atoms in $\mathcal{CD}(G)$ and u be the number of pairs of the form K, $C_G(K)$ where K is a nonabelian atom.*

1. *If $t = 0$ then p is odd.*
2. *If $t = 1$ then $p = 2$.*
3. *If $t \geq 2$, then there exists a positive integer $c \leq a$ such that $t = p^c + 1$. In particular, $p-1$ divides $t-2$; if p is odd then p^c divides u and if $p = 2$ then $t \geq 3$ and 2^{c-1} divides u.*
4. *If $t \geq 2$ and $u \geq 1$, then $3 \leq t \leq 2u+1$ when $p = 2$ and $2 \leq t \leq u+1$ when p is odd.*
5. *If $t \geq 3$ then $t \geq p + 1$.*

Proof. Theorem 4 tells us that $w = p^b + 1$ for some positive integer $b \leq a$, but also $w = t + 2u$ as set up by the notation. If $t = 0$ then $w = 2u = p^b + 1$, necessarily forcing p to be odd. If $t = 1$ then $2u = p^b$; clearly p must equal 2 in this case.

Suppose that $t \geq 2$; we continue here with the same notation and set up as in the proof of Theorem 4 except that we add the condition that K_1 and K_2 are abelian atoms. Recall the fixed $k_1 \in K_1 \backslash L$ and that $\beta_2(k) = 1$ for all $k \in K_1$. Set $\Gamma = \{\beta_i(k_1) \mid 2 \leq i \leq w \text{ and } K_i = K_i^*\}$, a subset of Λ.

Let $i, j \geq 3$ and assume that K_i and K_j are abelian atoms; we show that $K_{i,j}$ is also abelian. We use the functions α_i defined in the proof of Theorem 4. Observe that $\alpha_i(k)$ now differs from $\beta_i(k)$ only by an element in $\mathrm{Z}(G)$, for all $k \in K_1$. The calculation below follows:

$$[k\beta_i(k)\beta_j(k), k'\beta_i(k')\beta_j(k')] = [k\beta_i(k)\beta_j(k), k'\alpha_i(k')\alpha_j(k')] = 1$$

for all $k, k' \in K_1$. Therefore $K_{i,j}$ is also an abelian atom in $\mathcal{CD}(G)$. Since $K_{i,j} \neq K_2$, it follows that $\beta_i(k_1)\beta_j(k_1)\mathrm{Z}(G) = \beta_h(k_1)\mathrm{Z}(G)$ for some $\beta_h(k_1) \in \Gamma$. Consequently $\Gamma\mathrm{Z}(G) \leq K_2$ and since $\Gamma \subseteq \Lambda$, the elements of Γ are distinct. Therefore $|\Gamma\mathrm{Z}(G)/\mathrm{Z}(G)| = |\Gamma| = t - 1$ divides p^a and there exists a positive integer c such that $t - 1 = p^c$.

Since $t = p^c + 1$, clearly if $p = 2$ and $t \geq 2$, then $t = 2^c + 1 \geq 3$ but, for all primes p, it is true that $p - 1$ divides $p^c - 1 = t - 2$. Observe, for part (5), that if $t \geq 3$ then $t \geq p + 1$ is necessary for $t - 2$ to be a multiple of $p - 1$. To complete the proof of part (3), notice that $2u = w - t = p^c(p^{b-c} - 1)$, so that $u = \frac{1}{2}p^c(p^{b-c} - 1)$. If $p = 2$ then 2^{c-1} divides u, and if p is odd, then u must be divisible by p^c. This completes the assertions in part (3).

Continuing with part (4), suppose that $t \geq 2$ and $u \geq 1$. Then $b > c$, so $p^{b-c} - 1 > p - 1$. If p is odd, this implies $\frac{1}{2}(p^{b-c} - 1) \geq 1$ and consequently

$$t = p^c + 1 \leq p^c \left(\frac{p^{b-c} - 1}{2}\right) + 1 = u + 1.$$

If $p = 2$ then $2^{b-c} - 1 \geq 1$, so that

$$t = 2^c + 1 \leq 2^c(2^{b-c} = 1) + 1 = 2u + 1.$$

Thus when $t \geq 2$ and $u \geq 1$, the inequalities asserted in part (4) of the theorem are true. □

Corollary 8. *While there exist finite groups with Chermak–Delgado lattice a quasi-antichain of width 6, there does not exist such a group with exactly 4 abelian atoms in its Chermak–Delgado lattice.*

Proof. An extraspecial group of order 5^3 has a Chermak–Delgado lattice that is a quasi-antichain of width 6 by Proposition 5. If we assume that G is a finite group with $\mathcal{CD}(G)$ a quasi-antichain of width 6 having exactly 4 abelian atoms, then we know that G has a Sylow 5-subgroup P with $\mathcal{CD}(G) \cong \mathcal{CD}(P)$ as lattices by Proposition 6 and Theorem 4. Theorem 7 forces $4 = t \leq u+1 = 2$; thus G cannot exist. □

3. Examples. In this section we construct several examples of p-groups having a quasi-antichain for their Chermak–Delgado lattice. The first two examples show that every possible quasi-antichain of width 2 can be realized as the Chermak–Delgado lattice of a p-group.

1. **A group G where $\mathcal{CD}(G)$ is a quasi-antichain of width 2 with no abelian atoms**: Let H be any group with $\mathcal{CD}(H) = \{H, \mathrm{Z}(H)\}$. A family of p-groups, each member of which having such a Chermak–Delgado lattice, was described in [2].

   Define $G = H \times H$. In [3] it was established that $\mathcal{CD}(G)$ is a quasi-antichain of width 2 with atoms $\mathrm{Z}(H) \times H$ and $H \times \mathrm{Z}(H)$. Clearly $H \times \mathrm{Z}(H) = C_G(\mathrm{Z}(H) \times H)$.
2. **A group P such that $\mathcal{CD}(P)$ is a quasi-antichain of width 2 with both atoms abelian**: Let $P = \langle m_1, m_2, n_1, n_2\rangle$ where each element has order p and

   $$[m_1, m_2] = [n_1, n_2] = 1, \quad [m_i, n_j] = z_{ij} \in \mathrm{Z}(P) \text{ for } i, j \in \{1, 2\},$$

   and $\mathrm{Z}(P) = \langle z_{i,j} \mid i, j \in \{1, 2\}\rangle$ is elementary abelian of order p^4. Clearly P is nilpotent of class 2 with order p^8 and Chermak–Delgado measure p^{12}. Let $M = \langle m_1, m_2\rangle \mathrm{Z}(P)$ and $N = \langle n_1, n_2\rangle \mathrm{Z}(P)$. It's a straightforward calculation to show that $C_P(m) = M$ whenever $m \in M\backslash \mathrm{Z}(P)$ and $C_P(n) = N$ whenever $n \in N\backslash \mathrm{Z}(P)$, whereas $C_P(x) = \langle x\rangle \mathrm{Z}(P)$ for all $x \in P\backslash(M \cup N)$. Thus of all subgroups containing $\mathrm{Z}(P)$, only M and N have the largest measure, which is p^{12}. Since $m_P(M) = m_P(N) = m_P(\mathrm{Z}(P)) = p^{12}$ and no other subgroups have this measure, $\mathcal{CD}(P)$ is a quasi-antichain of width 2 (containing P) such that both atoms are abelian.

We now show that for every prime p and every positive integer n, there exists a p-group whose Chermak–Delgado lattice is a quasi-antichain of width $p^n + 1$ with all atoms abelian.

Proposition 9. *Let p be a prime and n a positive integer. Let P be the group of all 3×3 lower triangular matrices over $\mathbf{GF}(p^n)$ with 1s along the diagonal.*

The Chermak–Delgado lattice of P is a quasi-antichain of width $p^n + 1$, and all subgroups in the middle antichain are abelian.

Proof. By Exercise 39 in [5, III.16], P has exactly $p^n + 1$ abelian subgroups of maximal order equal to p^{2n}; these subgroups have measure $p^{4n} = m(P)$. If $x \in P\backslash \mathrm{Z}(P)$, it is easy to check that $|C_P(x)| = p^{2n}$. Therefore if $U \in \mathcal{CD}(P)$ with $\mathrm{Z}(P) < U < P$, then $|C_P(U)| \leq p^{2n}$ and $|U| = |C_P(C_P(U))| \leq p^{2n}$. It follows that $m^*(P) = p^{4n}$ and $|U| = |C_P(U)| = p^{2n}$. If $U \neq C_P(U)$ then $U \cap C_P(U) = \mathrm{Z}(P)$ since $U \cap C_P(U) \in \mathcal{CD}(P)$. But then for $x \in U\backslash \mathrm{Z}(P)$ we have $C_P(x) = C_P(U)$ by order considerations and therefore $x \in U \cap C_P(U) = \mathrm{Z}(P)$, a contradiction. Thus $U = C_P(U)$ is one of the abelian subgroups of maximal order and the assertion follows. □

Extraspecial groups of order p^3 are examples where each of the $p+1$ atoms in the quasi-antichain is abelian (Proposition 5); the next two propositions construct p-groups where the Chermak–Delgado lattice is a quasi-antichain of width $p+1$ and, depending on the value of p modulo 4, the number of abelian atoms is either 0, 1, or 2.

Proposition 10. *Given any prime p, there exists a group P of order p^9 such that $\mathcal{CD}(P)$ is a quasi-antichain of width $p+1$. In this example: if $p = 2$, then exactly one of the three atoms of $\mathcal{CD}(P)$ is abelian, when $p \equiv 1$ modulo 4 then exactly two of the $p+1$ atoms of $\mathcal{CD}(P)$ are abelian, and if $p \equiv 3$ modulo 4 then none of the atoms in $\mathcal{CD}(P)$ are abelian.*

Proof. Let P be generated by $\{x_1, x_2, x_3, y_1, y_2, y_3\}$ with defining relationships $x_i^p = y_i^p = 1$ and $[x_i, y_j] = 1$ for all i, j such that $1 \leq i, j \leq 3$, $\mathrm{Z}(P) = \langle z_{1,2}, z_{1,3}, z_{2,3}\rangle$ is elementary abelian with order p^3, and $[x_i, x_j] = [y_i, y_j] = z_{ij}$ for every i, j with $1 \leq i < j \leq 3$. Let $M_0 = \langle x_1, x_2, x_3\rangle \mathrm{Z}(P)$ and $M_p = \langle y_1, y_2, y_3\rangle \mathrm{Z}(P)$. For $1 \leq i \leq p-1$, let $M_i = \langle x_1 y_1^i, x_2 y_2^i, x_3 y_3^i\rangle \mathrm{Z}(P)$. We show that $\mathcal{CD}(P) = \{P, \mathrm{Z}(P), M_i \mid 0 \leq i \leq p\}$.

Observe that P is the central product of M_0 with M_p and $\mathrm{Z}(P) = M_0 \cap M_p$. Additionally $C_P(M_0) = M_p$ and vice versa, yielding $m_P(P) = m_P(M_0) = m_P(M_p) = p^{12}$. It is easy to show that if $x \in M_0\backslash \mathrm{Z}(P)$, then $C_{M_0}(x) = \langle x\rangle \mathrm{Z}(P)$ and $C_P(x) = \langle x\rangle M_p$. It follows that $C_P(\langle x, y\rangle) = \langle x, y\rangle \mathrm{Z}(P)$ whenever $x \in M_0\backslash \mathrm{Z}(P)$ and $y \in M_p\backslash \mathrm{Z}(P)$.

Let $a_i, b_i \in \mathbb{Z}/p\mathbb{Z}$ with $(a_1, a_2, a_3) \neq (0, 0, 0) \neq (b_1, b_2, b_3)$. Set $x = x_1^{a_1} x_2^{a_2} x_3^{a_3} \in M_0\backslash \mathrm{Z}(P)$ and $y = y_1^{b_1} y_2^{b_2} y_3^{b_3} \in M_p\backslash \mathrm{Z}(P)$. For x', y' with similar structure, $x'y' \in C_P(xy)$ if and only if $[x, x'] = [y', y]$. Further decomposing the commutators reveals

$$[x, x'] = \prod_{1\leq i<j\leq 3} z_{ij}^{a_i a_j' - a_j a_i'} \quad \text{and} \quad [y', y] = \prod_{1\leq i<j\leq 3} z_{ij}^{b_i' b_j - b_j' b_i}.$$

Thus $[xy, x'y'] = 1$ if and only if each of the three equations $a_i a_j' - a_j a_i' - b_i' b_j + b_j' b_i = 0$ hold where $1 \leq i < j \leq 3$. If (a_1, a_2, a_3) and (b_1, b_2, b_3) are not scalar multiples, then $\langle x, y, x_1^{b_1} x_2^{b_2} x_3^{b_2} y_1^{a_1} y_2^{a_2} y_3^{a_3}\rangle \mathrm{Z}(P) = C_P(xy)$. On the other hand, if there exists k such that $(b_1, b_2, b_3) = k(a_1, a_2, a_3)$, then $C_P(xy) =$

$\langle x, x_i y_i^{-k^{-1}} \mid 1 \le i \le 3 \rangle \mathrm{Z}(P) = \langle x \rangle M_{-k^{-1}}$. Therefore $C_P(M_k) = M_{-k^{-1}}$ for $1 \le k \le p-1$ and $m_P(M_k) = p^{12}$.

It follows then that $m(U) < m(M_k)$ whenever $\mathrm{Z}(P) < U < M_k$. Additionally, if $U \le P$ and there exist $u_1, u_2 \in U$ where $u_1 \in M_k$ and $u_2 \in M_{k'}$ with $k \ne k'$, then $C_P(U) \le \mathrm{Z}(P)$. Hence $m^*(P) = p^{12}$ and $\mathcal{CD}(P) = \{P, \mathrm{Z}(P), M_k \mid 0 \le k \le p\}$.

Since $C_P(M_k) = M_{-k^{-1}}$ for $1 \le k \le p-1$, there exists an abelian atom of $\mathcal{CD}(P)$ if and only if $p = 2$ or $p \equiv 1$ modulo 4. When $p = 2$ then M_1 is the unique abelian atom, and if $p \equiv 1$ modulo 4 then M_1 and M_{p-1} are both abelian, but no other atom in $\mathcal{CD}(P)$ is abelian. When $p \equiv 3$ modulo 4, then there do not exist any abelian atoms in $\mathcal{CD}(P)$. □

Proposition 11. *Let p be a prime. There exists a group Y of order p^9 such that $\mathcal{CD}(Y)$ is a quasi-antichain of width $p+1$. In this example, if $p = 2$ then exactly one of the three atoms of $\mathcal{CD}(Y)$ is abelian, and if p is odd, then exactly two of the $p+1$ atoms are abelian.*

Proof. Define P as in Proposition 10 except designate that $[n_i, n_j] = z_{ij}^{-1}$. The same arguments made earlier will now show that $C_P(M_i) = M_{i^{-1}}$ for $i = 1, 2, \ldots, p-1$. This forces M_1 and M_{p-1} to be abelian, yet the remaining facts still stand. □

In lattice theory [1, Chapter 1, Section 2], a lattice $\mathcal{L}$ has a duality $\theta : \mathcal{L} \to \mathcal{L}$ if θ is a bijection and if $A, B \in \mathcal{L}$ with $A \le B$ implies $\theta(B) \le \theta(A)$. Such a duality need not have order 2 as a function; in the case of the Chermak–Delgado lattice, the duality *does* have order 2. In particular, the examples in this section show that all possible types of quasi-antichains of width 4 with duality of order 2 occur as Chermak–Delgado lattices of 3-groups and those of width 3 with duality of order 2 occur as Chermak–Delgado lattices of 2-groups.

This leaves several questions open for investigation, including: Which values of t (in the notation of Theorem 7) are possible in quasi-antichain Chermak–Delgado lattices of width $w = p^n + 1$ when $n > 1$? That is, which dualities can be realized by the centralizer map? The first open case is when $w = 5$ and $t = 3$. And, are there examples of groups G with $G \in \mathcal{CD}(G)$ and $\mathcal{CD}(G)$ a quasiantichain where $t = 0$ and $p \equiv 1$ modulo 4?

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Ben Brewster
Department of Mathematical Sciences,
Binghamton University,
Binghamton, NY,
USA
e-mail: ben@math.binghamton.edu

Peter Hauck
Fachbereich Informatik,
Eberhard-Karls-Universität Tübingen,
Tübingen,
Germany
e-mail: hauck@informatik.uni-tuebingen.de

Elizabeth Wilcox
Mathematics Department,
Oswego State University,
Oswego, NY,
USA
e-mail: elizabeth.wilcox@oswego.edu

Received: 24 July 2014