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A note on semipositone boundary value problems with nonlocal, nonlinear boundary conditions

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Abstract. We consider the existence of at least one positive solution to a semipositone boundary value problem with nonlocal, nonlinear boundary conditions, which can be quite general since the nonlinearity is realized as a Stieltjes integral. By assuming that the associated Stieltjes measure decomposes in a certain way, the classical Leray-Schauder degree is utilized to derive the existence result.

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1. Introduction. In this note we show that the boundary value problem (BVP)

$$\begin{aligned} -y''(t) &= f(t, y(t)),\ t \in (0,1) \\ y(0) &= H(\varphi(y)) \\ y(1) &= 0 \end{aligned} \tag{1.1}$$

has at least one positive solution when certain growth and structure conditions are imposed on f, H, and φ. In (1.1) the functions $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ and $H : \mathbb{R} \to \mathbb{R}$ will always be assumed to be continuous, with $H([0,+\infty)) \subseteq [0,+\infty)$. Furthermore, the functional $\varphi : \mathcal{C}([0,1]) \to \mathbb{R}$ is linear and, in particular, realizable in the Stieltjes integral representation

$$\varphi(y) = \int_{[0,1]} y(s)\ d\alpha(s), \tag{1.2}$$

with $\alpha : [0,1] \to \mathbb{R}$ of bounded variation on $[0,1]$. We do not assume that α is necessarily monotonically increasing. Consequently, in this note we permit the map $y \mapsto \varphi(y)$ to be possibly negative even if y is nonnegative.

The study of semipositone problems has a long history in the literature, with the work of Anuradha et al. [2] being an early, classic example. More recent papers include those by Goodrich [8], Graef and Kong [11], Sun and Li [21], and Infante and Webb [23]. Furthermore, recently there have been many papers on nonlocal BVPs with either linear or nonlinear boundary conditions. The reader is invited to consult recent papers by Anderson [1], Goodrich [3–7,9,10], Infante et al. [12–17], and Karakostas et al. [18,19]. In particular, the general theory for treating problems with linear, nonlocal boundary conditions realized as Stietjes integrals was developed by Infante and Webb [22] with some earlier work in this direction produced by Yang [25,26]. Finally, an interesting and perhaps not-too-well-known review of the nonlocal BVP theory known in the early 1940s may be found by the interested reader in the paper by Whyburn [24].

In this note we continue these studies by considering problem (1.1). In particular, we are not aware of any contributions concerning the semipositone problem when the boundary conditions are nonlocal, nonlinear and the nonlocality is signed. Moreover, we do not assume that H satisfies any uniform growth behavior but rather that it satisfies merely an asymptotic condition. This strategy works only because we assume that φ decomposes in a particular way. Essentially, we assume that $\varphi(y) = \varphi_1(y) + \varphi_2(y)$, for each y in an appropriate cone, and where φ_1 essentially traps the negativity of φ, whilst φ_2 satisfies an appropriate coercivity condition—see the example in Section 3 as well as [3–7,10] for much more discussion regarding this technique in a variety of contexts for the positone problem.

2. Preliminary lemmata and notation. In order to keep the exposition of this section as brief as possible, we shall recall some necessary facts that are relatively standard and well-known in the existing literature, and we direct the reader to the references for further details. We begin by recalling that Green's function associated to the conjugate problem is the function $G: \ [0,1] \times [0,1] \to \mathbb{R}$ defined by

$$G(t,s) := \begin{cases} t(1-s), & 0 \le t \le s \le 1 \\ s(1-t), & 0 \le s \le t \le 1 \end{cases}. \tag{2.1}$$

Furthermore, if $0 < a < b < 1$ are fixed with $[a,b] \subset [0,1]$, then there exists a constant $\gamma^* = \gamma^*(a,b) := \min_{t\in[a,b]}\{t, 1-t\} \in (0,1)$ such that $\min_{t\in[a,b]} G(t,s) \ge \gamma^* \max_{t\in[0,1]} G(t,s) = \gamma^* G(s,s)$ for each $s \in [0,1]$.

Let us next define the function $q: \ [0,1] \to \left[0, \frac{1}{4}\right]$ by $q(t) := t(1-t)$. Observe that q has previously appeared in [21, Lemma 2.4], and it plays a prominent role in the proof of the existence theorem. Equipping $\mathcal{C}([0,1])$ with the max norm, denoted $\|\cdot\|$, we shall work within the cone $\mathcal{K} \subseteq \mathcal{C}([0,1])$ defined by $\mathcal{K} := \{y \in \mathcal{C}([0,1]) \ : \ \varphi_1(y) \ge 0 \text{ and } y(t) \ge q(t)\|y\|, \text{ for } t \in [0,1]\}$.

The general program for analyzing the semipositone problem (with local boundary conditions) may be found essentially in Anuradha et al. [2] and is very well known. In particular, we consider the auxiliary problem

$$-w''(t) = u(t),\ t \in (0,1) \text{ subject to } w(0) = 0 = w(1), \tag{2.2}$$

where $u: [0,1] \to [0,+\infty)$ is a map satisfying $u \in L^1([0,1])$ and is nonzero on a set of positive measure; the purpose of u will be seen in condition (H5) in the sequel. Henceforth, we shall denote by w the unique solution of problem (2.2), and we note that w may be realized in the form

$$w(t) = \int_0^1 G(t,s)u(s)\ ds, \tag{2.3}$$

where G is as in (2.1) above. Additionally, defining the map $y^* : [0,1] \to [0,+\infty)$ by $y^*(t) := \max\{0, y(t) - w(t)\}$ and

$$H^*(t) := H(\max\{0,t\}), \tag{2.4}$$

we also require the modified problem

$$\begin{aligned} &-y'' = f(t, y^*(t)) + u(t),\ t \in (0,1),\\ &y(0) = H^*(\varphi(y-w)), y(1) = 0. \end{aligned} \tag{2.5}$$

To this end, it is standard to show that the completely continuous operator $T : \mathcal{C}([0,1]) \to \mathcal{C}([0,1])$ defined by

$$(Ty)(t) := (1-t)H^*(\varphi(y-w)) + \int_0^1 G(t,s)\left[f(s, y^*(s)) + u(s)\right]\ ds \tag{2.6}$$

has the property that if $Ty = y$, then y is a solution of the modified problem (2.5); it is not difficult to show that $T(\mathcal{K}) \subseteq \mathcal{K}$. The utility of problem (2.5) is due to the following standard lemma, whose proof we omit.

Lemma 2.1. *Suppose both that y is a solution of the modified problem (2.5) and w is the solution of the auxiliary problem (2.2). If it holds both that $\varphi(y-w) \geq 0$ and that $(y-w)(t) \geq 0$ for each $t \in [0,1]$, then $\Upsilon(t) := (y-w)(t)$ is a positive solution of the original problem (1.1).*

Proof. Omitted. □

The following simple lemma will be required in Section 3.

Lemma 2.2. *For each $(t,s) \in [0,1] \times [0,1]$ it holds that $G(t,s) \leq q(t)$.*

Proof. We simply observe that

$$G(t,s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1\\ s(1-t), & 0 \leq s \leq t \leq 1 \end{cases} \leq \begin{cases} t(1-t), & 0 \leq t \leq s \leq 1\\ t(1-t), & 0 \leq s \leq t \leq 1 \end{cases} = q(t), \tag{2.7}$$

which completes the proof. □

We next list the hypotheses imposed on problem (1.1) and then provide some notation to be utilized in Section 3. Some discussion of these hypotheses will be provided at the conclusion of Section 3; in particular, we provide an example to illustrate their applicability and use. The decomposition hypotheses (H0)–(H1) originated in [5].

H0: There are two linear functionals $\varphi_1, \varphi_2 : \mathcal{C}([0,1]) \to \mathbb{R}$ such that $\varphi(y) = \varphi_1(y) + \varphi_2(y)$. Moreover, assume that there exists a constant $C_0 > 0$ such that $\varphi_2(y) \geq C_0\|y\|$ holds for each $y \in \mathcal{K}$. (Note that since φ is continuous, there is another constant, say $C_1 \geq 0$, such that $|\varphi(y)| \leq C_1\|y\|$, for each $y \in \mathcal{C}([0,1])$. Henceforth, C_1 shall denote this constant.)

H1: The functionals $\varphi(y)$, $\varphi_1(y)$, and $\varphi_2(y)$ are linear and are realized as

$$\varphi(y) := \int_{[0,1]} y(t)\, d\alpha(t),\ \varphi_i(y) := \int_{[0,1]} y(t)\, d\alpha_i(t),\ i = 1, 2$$

where $\alpha, \alpha_1, \alpha_2 : [0,1] \to \mathbb{R}$ satisfy $\alpha, \alpha_1, \alpha_2 \in BV([0,1])$.

H2: There exist $0 \leq \xi_1 \leq \xi_2 \leq 1$ such that $\lim_{y\to+\infty} \frac{f(t,y)}{y} = +\infty$ uniformly for $t \in [\xi_1, \xi_2]$.

H3: There is a number $C_2 \geq 0$ such that $\lim_{z\to+\infty} \frac{|H(z)-C_2z|}{z} = 0$.

H4: It holds that

$$\int_{[0,1]} (1-t)\, d\alpha_1(t),\ \int_{[0,1]} G(t,s)\, d\alpha_1(t) \geq 0,$$

where the latter inequality holds for each $s \in [0,1]$.

H5: There exists a function $u : [0,1] \to [0,+\infty)$ which is not identically zero on any subinterval of $[0,1]$ and satisfies $u \in L^1([0,1])$, such that $f(t,y) \geq -u(t)$ for each $(t,y) \in [0,1] \times \mathbb{R}$.

H6: Assume both that $\left[\int_0^1 1-t\, d\alpha(t)\right]^{-1} - C_2 > 0$ and that $\int_0^1 1-t\, d\alpha(t) \neq 0$. Then there exists a number $\varepsilon > 0$ satisfying $0 < \varepsilon < \left[\int_0^1 1-t\, d\alpha(t)\right]^{-1} - C_2$ and a number $r_2 \neq 0$ satisfying

$$r_2 > \inf\left\{z \in (0,+\infty)\ :\ \frac{|H(x)-C_2x|}{x} \leq \varepsilon,\ \forall x \in [z-\varphi(w), +\infty) \cap (0,+\infty)\right\}$$

such that for each $(t,y) \in [0,1] \times \left[0, \frac{r_2}{C_0}\right]$ it holds that $f(t,y) \leq r_2\vartheta(t)$, where $\vartheta : [0,1] \to [0,+\infty)$ is continuous.

Remark 2.3. Henceforth, we shall utilize the following notation.

- For each $r > 0$, $\Omega_r := \{y \in \mathcal{C}([0,1])\ :\ \|y\| < r\}$.
- For each $r > 0$, $V_r := \left\{y \in \mathcal{K}\ :\ \min_{t\in[\xi_1,\xi_2]} y(t) < r\right\}$.

Note that the idea of using sets of the form V_r rather than merely Ω_r appears to have been introduced first by Lan [20].

Remark 2.4. We remark that due to condition (H0), the requirement that $\varphi_2(y) \geq 0$ need not be incorporated into the definition of $\mathcal{K}$ as it is superfluous. Consequently, we did not include this condition when we earlier defined the cone $\mathcal{K}$. We also note that condition (H3) permits H to be linear; thus, our result includes the possibility that $y(0) = M\varphi(y)$ for some $M > 0$.

Finally, we state the fixed point result that we utilize in the proof of Theorem 3.1—see Infante et al. [16] or, for more general results on the Leray-Schauder index, Zeidler [27].

Lemma 2.5. *Let D be a bounded open set and, with $\mathcal{K}$ a cone in a Banach space $\mathcal{X}$, suppose both that $D\cap\mathcal{K}\neq\emptyset$ and that $\overline{D}\cap\mathcal{K}\neq\mathcal{K}$. Let D_1 be open in $\mathcal{X}$ with $\overline{D}_1\subseteq D\cap\mathcal{K}$. Assume that $T:\overline{D}\cap\mathcal{K}\to\mathcal{K}$ is a compact map such that $Tx\neq x$ for $x\in\mathcal{K}\cap\partial D$. If $i_{\mathcal{K}}(T,D\cap\mathcal{K})=1$ and $i_{\mathcal{K}}(T,D_1\cap\mathcal{K})=0$, then T has a fixed point in $(D\cap\mathcal{K})\setminus\left(\overline{D_1\cap\mathcal{K}}\right)$. Moreover, the same result holds if $i_{\mathcal{K}}(T,D\cap\mathcal{K})=0$ and $i_{\mathcal{K}}(T,D_1\cap\mathcal{K})=1$.*

3. Proof of the existence theorem and discussion. We now state and prove the main result of this paper. We then conclude with some brief comments on both its applicability and use.

Theorem 3.1. *Suppose that conditions (H0)–(H6) hold. In addition, assume that each of the inequalities*

$$(C_2+\varepsilon)\int_0^1 1-t\ d\alpha(t)+\int_0^1\int_0^1 G(t,s)\left[\frac{1}{r_2}u(s)+\vartheta(s)\right]\ d\alpha(t)\ ds<1,$$

$$C_1\int_0^1 u(s)\ ds<r_2 \tag{3.1}$$

is satisfied. Then problem (1.1) has at least one positive solution.

Proof. Begin by putting $q_0:=\min_{t\in[\xi_1,\xi_2]}q(t)$. Then $q_0\in(0,1)$. Let $\eta>0$ be a number selected sufficiently large such that

$$\eta q_0\int_{\xi_1}^{\xi_2} G(s,s)\ ds>\frac{1}{q_0}+\int_0^1 u(s)\ ds. \tag{3.2}$$

Condition (H2) implies the existence of a number $r_1>0$ such that $f(t,y)>\eta y$, for each $t\in[\xi_1,\xi_2]$, whenever $y\in[r_1,+\infty)$. Since we may assume without loss that $r_1\geq 1$, this together with the choice of η in (3.2) implies that

$$\eta q_0\int_{\xi_1}^{\xi_2} G(s,s)\ ds>\frac{1}{q_0}+\int_0^1 u(s)\ ds\geq\frac{1}{q_0}+\frac{1}{r_1}\int_0^1 u(s)\ ds. \tag{3.3}$$

Now, for notional convenience in the sequel, put

$$\widehat{r}_1:=\frac{r_1}{q_0}+\int_0^1 u(s)\ ds,$$

and let $y \in \partial V_{\widehat{r}_1}$ be fixed but arbitrary. Observe that by Lemma 2.2, for each $t \in [\xi_1, \xi_2]$ we estimate

$$\begin{aligned}(y-w)(t) &\geq q(t)\|y\| - \int_0^1 G(t,s)u(s)\ ds\\ &\geq q(t)\left[\|y\| - \int_0^1 u(s)\ ds\right]\\ &\geq q(t)\left[\frac{r_1}{q_0} + \int_0^1 u(s)\ ds - \int_0^1 u(s)\ ds\right] \geq r_1.\end{aligned} \tag{3.4}$$

Inequality (3.4) implies that for any $y \in \partial V_{\widehat{r}_1}$ we have

$$f\left(t, y^*(t)\right) = f(t,(y-w)(t)) \geq \eta(y-w)(t) \tag{3.5}$$

for $t \in [\xi_1, \xi_2]$. Moreover, since without loss we may assume that

$$\frac{r_1}{q_0} + \int_0^1 u(s)\ ds > \frac{r_2}{C_0}$$

by simply selecting r_1 sufficiently large, which will not affect the previous estimates, we henceforth assume that this is so. This inequality will be used at the end of this proof.

We now conclude the first part of the proof by demonstrating that

$$i_{\mathcal{K}}\left(T, V_{\widehat{r}_1}\right) = 0. \tag{3.6}$$

To this end, assume for contradiction the existence of $y \in \partial V_{\widehat{r}_1}$ such that $(Ty)(t) \leq y(t)$ for each $t \in [0,1]$; we shall show that this leads to an absurdity when $t \in [\xi_1, \xi_2]$. Indeed, we estimate, for each $t \in [\xi_1, \xi_2]$,

$$\begin{aligned}(Ty)(t) &\geq \int_{\xi_1}^{\xi_2} G(t,s)\left[f\left(s, y^*(s)\right) + u(s)\right]\ ds\\ &\geq \eta q_0 \int_{\xi_1}^{\xi_2} G(s,s)(y-w)(s)\ ds\\ &\geq r_1\eta q_0 \int_{\xi_1}^{\xi_2} G(s,s)\ ds > \frac{r_1}{q_0} + \int_0^1 u(s)\ ds,\end{aligned} \tag{3.7}$$

where we have utilized estimates (3.3)–(3.5); note that in (3.7) we have used the fact that $G(t,s) \geq \gamma^*\left(\xi_1, \xi_2\right) G(s,s) \geq q_0 G(s,s)$ for each $(t,s) \in [\xi_1, \xi_2] \times [\xi_1, \xi_2]$. But then since $y \geq Ty$, by the contradiction hypothesis, we obtain from (3.7) that

$$y(t) \geq (Ty)(t) > \frac{r_1}{q_0} + \int_0^1 u(s)\, ds = \widehat{r}_1, \tag{3.8}$$

which contradicts the assumption that $y \in \partial V_{\widehat{r}_1}$. Consequently, it follows that $Ty > y$, and so, (3.6) holds, as claimed—see the proof of [27, Theorem 13.D] for a precise justification of this index calculation.

On the other hand, we next show that

$$i_{\mathcal{K}}\left(T, \Omega_{\frac{r_2}{C_0}}\right) = 1. \tag{3.9}$$

So, suppose for contradiction that $\mu y = Ty$ for some $y \in \mathcal{K} \cap \partial\Omega_{\frac{r_2}{C_0}}$ with $\mu \geq 1$. Now, it can be shown that the second inequality in (3.1) together with the fact that $w \in \mathcal{K}$ jointly imply that $r_2 - \varphi(w) \geq 0$. This combined with the observation that $\|y\| = \frac{r_2}{C_0}$ implies the estimate

$$\varphi(y - w) \geq C_0\|y\| - \varphi(w) = r_2 - \varphi(w) \geq 0,$$

which by condition (H6) implies that $H^*(\varphi(y-w)) = H(\varphi(y-w)) \leq (\varepsilon + C_2)$ $\varphi(y - w)$. In addition, since $y(s) \in \left[0, \frac{r_2}{C_0}\right]$ for each $s \in [0,1]$, condition (H6) implies that $f(s, y^*(s)) \leq r_2\vartheta(s)$ for each $s \in [0,1]$. Putting this together and using the fact that $0 \leq \varphi(y - w) \leq \varphi(y)$, we compute

$$\begin{aligned}
\mu\varphi(y) &= H^*(\varphi(y-w)) \int_0^1 1 - t\, d\alpha(t) \\
&\quad + \int_0^1 \int_0^1 G(t,s)\left[f(s, y^*(s)) + u(s)\right]\, d\alpha(t)\, ds \\
&\leq \left[C_2\varphi(y) + \varepsilon\varphi(y)\right] \int_0^1 1 - t\, d\alpha(t) \\
&\quad + \int_0^1 \int_0^1 G(t,s)\left[r_2\vartheta(s) + u(s)\right]\, d\alpha(t)\, ds.
\end{aligned} \tag{3.10}$$

Since $\varphi(y) \geq C_0\|y\| > 0$, we may divide both sides of (3.10) by $\varphi(y)$ and thus obtain

$$\begin{aligned}
\mu &\leq (C_2 + \varepsilon) \int_0^1 1 - t\, d\alpha(t) + \frac{1}{\varphi(y)} \int_0^1 \int_0^1 G(t,s)\left[r_2\vartheta(s) + u(s)\right]\, d\alpha(t)\, ds \\
&\leq (C_2 + \varepsilon) \int_0^1 1 - t\, d\alpha(t) + \frac{1}{C_0\|y\|} \int_0^1 \int_0^1 G(t,s)\left[r_2\vartheta(s) + u(s)\right]\, d\alpha(t)\, ds \\
&= (C_2 + \varepsilon) \int_0^1 1 - t\, d\alpha(t) + \frac{1}{r_2} \int_0^1 \int_0^1 G(t,s)\left[r_2\vartheta(s) + u(s)\right]\, d\alpha(t)\, ds,
\end{aligned} \tag{3.11}$$

using the fact that $\|y\| = \frac{r_2}{C_0}$. But then combining (3.11) with assumption (3.1) implies that

$$\mu \le (C_2+\varepsilon)\int_0^1 1-t\ d\alpha(t)+\frac{1}{r_2}\int_0^1\int_0^1 G(t,s)\left[r_2\vartheta(s)+u(s)\right]\ d\alpha(t)\ ds<1, \tag{3.12}$$

which contradicts the fact that $\mu \ge 1$. Thus, (3.9) holds.

Now, recall that

$$\widehat{r}_1=\frac{r_1}{q_0}+\int_0^1 u(s)\ ds>\frac{r_2}{C_0}$$

so that

$$\mathcal{K}\cap\left(V_{\widehat{r}_1}\backslash\overline{\Omega}_{\frac{r_2}{C_0}}\right)\ne\emptyset, \tag{3.13}$$

where we have used the simple fact that $\Omega_\rho \subseteq V_\rho$ for each $\rho > 0$. Consequently, combining (3.6) and (3.9), we obtain from an application of Lemma 2.5 that there exists $y_0 \in V_{\widehat{r}_1}\backslash\overline{\Omega}_{\frac{r_2}{C_0}}$ such that $Ty_0 = y_0$.

It remains only to show that the function $\Upsilon : [0,1] \to \mathbb{R}$ defined by $\Upsilon(t) := (y_0 - w)(t)$ is nonnegative for each $t \in [0,1]$. Since $y_0 \in \mathcal{K}\backslash\Omega_{\frac{r_2}{C_0}}$, using the fact that $C_1 > C_0$, we may estimate for each $t \in [0,1]$

$$\begin{aligned}(y_0-w)(t)&\ge q(t)\left[\|y_0\|-\int_0^1 u(s)\ ds\right]\\&\ge q(t)\left[\frac{r_2}{C_0}-\int_0^1 u(s)\ ds\right]\\&\ge q(t)\underbrace{\left(\frac{C_1}{C_0}-1\right)}_{\ge 0}\int_0^1 u(s)\ ds\ge 0,\end{aligned} \tag{3.14}$$

where the second-to-last inequality is a consequence of assumption (3.1). Consequently, due to (3.14) and recalling that $\varphi(y-w)\ge 0$ for $y \in \mathcal{K}\backslash\overline{\Omega}_{\frac{r_2}{C_0}}$, an application of Lemma 2.1 implies that Υ is a positive solution of (1.1). And this completes the proof. □

Example. Suppose that $\varphi(y) := \frac{1}{2}y\left(\frac{1}{3}\right)-\frac{1}{5}y\left(\frac{1}{2}\right)$. Then we may use the decomposition

$$\varphi(y)=\underbrace{\frac{1}{3}y\left(\frac{1}{3}\right)-\frac{1}{5}y\left(\frac{1}{2}\right)}_{:=\varphi_1(y)}\underbrace{+\frac{1}{6}y\left(\frac{1}{3}\right)}_{:=\varphi_2(y)}. \tag{3.15}$$

It is easy to compute that (3.15) implies that $C_0 := \frac{1}{27}$ and $C_1 := \frac{7}{10}$. If, for definiteness, we put $H(z) := z^{\frac{1}{3}}$, then $C_2 = 0$ and a simple calculation shows that r_2 may be taken to be $\left(\frac{7}{29}\right)^{\frac{3}{2}}$. It may be checked, furthermore, that conditions (H1) and (H4) hold.

Now, assume for the sake of simplicity that $\vartheta(t) \equiv \vartheta_0$ and $u(t) \equiv u_0$. Then provided condition (3.1) holds, which in this case is

$$\frac{29}{30} + \frac{119}{3240}\left[\left(\frac{7}{29}\right)^{-\frac{3}{2}} u_0 + \vartheta_0\right] < 1$$

$$u_0 < \frac{10}{7}\left(\frac{7}{29}\right)^{\frac{3}{2}}, \tag{3.16}$$

problem (1.1) would have at least one positive solution provided that $(t, y) \mapsto f(t, y)$ satisfies both the superlinear growth condition (H2) as well as $f(t, y) < \left(\frac{7}{29}\right)^{\frac{3}{2}} \vartheta_0$, for each $(t, y) \in [0, 1] \times \left[0, 27\left(\frac{7}{29}\right)^{\frac{3}{2}}\right)$. Finally, it is worth noting that the inequalities presented in (3.16) are easily attained. Indeed, this system of inequalities can be satisfied for infinitely many choices of ϑ_0, $u_0 > 0$.

Remark 3.2. It is also worth observing that, regarding the first of the two inequalities presented in (3.1), it holds that

$$(C_2 + \varepsilon)\int_0^1 1 - t \, d\alpha(t) < C_2 \int_0^1 1 - t \, d\alpha(t) + \underbrace{1 - C_2 \int_0^1 1 - t \, d\alpha(t)}_{> \varepsilon \int_0^1 1 - t \, d\alpha(t)} = 1, \tag{3.17}$$

by means of condition (H6). We conclude from (3.17) that the first addend in the first inequality presented in (3.1) cannot alone violate the inequality. Consequently, in some loose sense, as long as the maps $s \mapsto \frac{1}{r_2}u(s)$ and $s \mapsto \vartheta(s)$ can be kept sufficiently small, then the first inequality of (3.1) can always be satisfied.

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