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On Jordan type bounds for finite groups acting on compact 3-manifolds

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Abstract. By a classical result of Jordan, each finite subgroup of a complex linear group $\mathrm{GL}_n(\mathbb{C})$ has an abelian normal subgroup whose index is bounded by a constant depending only on n. It has been asked whether this remains true for finite subgroups of the diffeomorphism group $\mathrm{Diff}(M)$ of every compact manifold M; in the present paper, using the geometrization of 3-manifolds, we prove it for compact 3-manifolds M.

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1. Introduction. By a classical result of Jordan, each finite subgroup G of a complex linear group $\mathrm{GL}_n(\mathbb{C})$ has an abelian normal subgroup whose index in G is bounded by a constant depending only on n (by [2] the optimal bound is $(n+1)!$, for $n \geq 71$, realized by the symmetric group S_{n+1} occurring as a subgroup of $\mathrm{GL}_n(\mathbb{C})$; this uses the classification of the finite simple groups). Recently there has been much interest in generalizations, replacing $\mathrm{GL}_n(\mathbb{C})$ by more general geometrically interesting groups such as diffeomorphism groups of smooth manifolds and automorphism groups of algebraic varieties ([12–17], [20, Theoreme 3.1], [5], [22, Section 5]).

Following [16,17] we say that a group is a *Jordan group* or has the *Jordan property* if there exists a constant such that every finite subgroup has an abelian normal subgroup whose index is bounded by this constant (note that it is sufficient to find an abelian, not necessarily normal subgroup of bounded index). Denoting by $\mathrm{Diff}(M)$ the diffeomorphism group of a smooth manifold M, the present paper is motivated by the following question:

For which (classes of) smooth manifolds M is $\mathrm{Diff}(M)$ a Jordan group? And for what types of bounds?

Whereas this is in general not the case for noncompact manifolds, it is likely to be true and has been conjectured for compact manifolds (see [12, 14], and [5, 22] for the case of spheres). For example, there is the classical Hurwitz-bound $186(g-1)$ for closed surfaces of genus $g > 1$, the bound $24(g-1)$ for 3-dimensional handlebodies of genus $g > 1$ ([26], [7, Theorem 7.2]), a quadratic polynomial bound for the closed 3-manifolds which are a connected sum of $g > 1$ copies of $S^2 \times S^1$ (which admit S^1-actions; see [23]), and polynomial bounds of higher degrees for higher-dimensional handlebodies ([27]). It has recently been shown by Mundet i Riera ([14]) that $\mathrm{Diff}(S^{n-1})$ and $\mathrm{Diff}(\mathbb{R}^n)$ are Jordan groups; it remains open here whether the bound $(n+1)!$ for the case of finite subgroups of $\mathrm{O}(n) \subset \mathrm{GL}_n(\mathbb{C})$ is still valid.

In the present paper we consider the case of compact 3-manifolds; using the geometrization of compact 3-manifolds after Thurston and Perelman (see [1, 11]), we prove the following.

Theorem 1. *For a compact 3-manifold M,* $\mathrm{Diff}(M)$ *is a Jordan group.*

For compact 2-manifolds this is basically a consequence of the classical formula of Riemann-Hurwitz (see [15]). It is not true for noncompact 2-manifolds, in general: map the fundamental group of some compact surface onto a free group of rank two and then onto a 2-generator group H which contains an isomorphic copy of every finite group; the regular covering corresponding to this projection is a surface (whose fundamental group is not finitely generated) on which H acts as group of covering transformations, and clearly H does not have the Jordan property (an example of such a group H is the group of permutations of the integers $\mathbb{Z}$ generated by the translation $i \to i+1$ and the transposition $(1, 2)$ which contains a copy of every symmetric group S_n and hence of every finite group).

Concerning dimension four, it is shown in [17] that there are noncompact simply-connected smooth 4-manifolds M such that $\mathrm{Diff}(M)$ is not a Jordan group. On the other hand, $\mathrm{Diff}(M)$ is a Jordan group for compact 4-manifolds M with non-zero Euler characteristic ([13]); by [14] the same holds in fact for such manifolds in arbitrary dimensions, using the classification of the finite simple groups.

We note that in a preliminary arXiv-version of the present paper ([24]), we proved that $\mathrm{Diff}(\mathbb{R}^n)$ is a Jordan group for $n \leq 6$; shortly after, this was proved in [14] for general n for both S^n and $\mathbb{R}^n$, again on the basis of the classification of the finite simple groups. The proof for the case of S^n in [14] uses also a result in [5] which implies, by the classification, that for each n there are only finitely many finite simple groups which admit an action on a sphere S^n (note that a Jordan group contains only finitely many finite nonabelian simple groups, up to isomorphism). It would be interesting to avoid the classification in these results but it seems likely that it is intrinsically needed.

Stronger results are valid for the sphere S^4 and Euclidean space $\mathbb{R}^4$ which we collect in the following:

Theorem 2. (i) ([8, 9]) *Let G be a finite group of orientation-preserving diffeomorphisms of S^4, or of any homology 4-sphere. Then G, or a subgroup* *of index two if G is solvable, is isomorphic to a subgroup of the orthogonal group* SO(5)*.*

(ii) ([4]) *Every finite group of diffeomorphisms of $\mathbb{R}^4$, or of any acyclic 4-manifold M, is isomorphic to a subgroup of the orthogonal group* O(4)*.*

For the cases of S^3 and $\mathbb{R}^3$ this follows from the geometrization of finite group actions in dimension three (see [3]). For arbitrary dimensions this raises the following:

Problem. Is every finite group of (orientation-preserving) diffeomorphisms of a sphere S^{n-1} or a Euclidean space $\mathbb{R}^n$ isomorphic to a subgroup of the orthogonal group $\mathrm{O}(n)$ $(\mathrm{SO}(n))$?

If this is the case, the classical Jordan bound applies for finite groups of diffeomorphisms of S^{n-1} and $\mathbb{R}^n$, see [2]. We note that this is not true for finite groups of homeomorphisms of S^n and $\mathbb{R}^n$, in general, see [4, Section 7].

Finally, it would be interesting to give a proof of Theorem 1 which avoids the geometrization of closed 3-manifolds (or prove more generally that $\mathrm{Diff}(M)$ is a Jordan group for closed manifolds M with Euler characteristic zero).

2. Proof of Theorem 1. It is easy to see that, if $\tilde{M}$ is a finite covering of M such that $\mathrm{Diff}(\tilde{M})$ is a Jordan group, then also $\mathrm{Diff}(M)$ is a Jordan group. So it is sufficient to consider the case of orientable manifolds, and also of orientation-preserving actions of a finite group G (passing eventually to a subgroup of index two of G). Also, it is sufficient to consider the case of closed manifolds since, for a compact manifold M with non-empty boundary, one can reduce to the closed case by taking the double of M along the boundary and also the double of a finite group action on M.

So let M be a closed orientable 3-manifold and G a finite group of orientation-preserving diffeomorphisms of M. We consider several cases.

2.1. If $\pi_1(M)$ is finite then, by the geometrization of 3-manifolds after Perelman ([1,11]), M is a spherical 3-manifold and finitely covered by S^3; also, any finite group of diffeomorphisms of M is conjugate to a linear (orthogonal) action ([3]). By the classical Jordan bound for linear groups, $\mathrm{Diff}(S^3)$ is a Jordan group, and hence also $\mathrm{Diff}(M)$ is a Jordan group.

2.2. Assume next that M is irreducible and has infinite fundamental group; again by the geometrization of 3-manifolds, we can assume that M is a geometric (admits a decomposition along tori into geometric pieces). Then, if M does not admit a circle action, by [6, Theorem 4.1] there is a bound on the order of finite subgroups of $\mathrm{Diff}(M)$ so we are done (by Mostow's rigidity theorem in the presence of hyperbolic pieces; if all pieces of the torus-decomposition are Seifert fibered, one uses that finite group actions on Seifert fiber spaces are fiber-preserving but that the Seifert fibrations of the pieces of the decomposition don't match up along the boundary tori). For Haken 3-manifolds M this follows also from the stronger results [25, 4.1 and 4.2] stating that, if M is not Seifert fibered, there are only finitely many finite group actions on M up to conjugation.

2.3. Suppose that M is irreducible, has infinite fundamental group and a circle action. Then M is a Seifert fiber space, and by the geometrization of finite group actions on Seifert fiber spaces ([10, Theorems 2.1 and 2.2]), we can assume that the action of the finite group G of diffeomorphisms of M is geometric, in particular fiber-preserving and normalizing the S^1-action of M. Considering a suitable finite covering of M, we can moreover assume that M has no exceptional fibers, and hence that the base space of the Seifert fibration (the quotient of the S^1-action) is a closed orientable surface B without cone points. The finite group G projects to a finite group $\bar{G}$ of diffeomorphisms of the base-surface B, and we can again assume that $\bar{G}$ is orientation-preserving.

If B is a hyperbolic surface (that is, of genus $g \geq 2$) then, by the formula of Riemann-Hurwitz, the order of the finite group $\bar{G}$ of diffeomorphisms of B is bounded, and hence G has a finite cyclic subgroup of bounded index (the intersection of G with the S^1-action).

If B is a torus T^2, then there are two cases. First, M may be a 3-dimensional torus T^3; this acts by rotations on itself. Since the action of G is geometric, the subgroup G_0 of G acting trivially on the fundamental group is a subgroup of the T^3-action and hence abelian of rank at most three (see [19] for the geometries of 3-manifolds and their isometry groups). The factor group G/G_0 acts faithfully on the fundamental group $\mathbb{Z}^3$ of the 3-torus and is isomorphic to a subgroup of $\mathrm{GL}_3(\mathbb{Z})$. Since, by a well-known result of Minkowski, there is a bound on the finite subgroups of $\mathrm{GL}_n(\mathbb{Z})$ for each n, the group G has an abelian subgroup G_0 of bounded index.

If M fibers over T^2 but is not a 3-torus, then it belongs to the nilpotent geometry Nil given by the Heisenberg group (see again [19]). Now the subgroup G_0 of G acting trivially on the fundamental group (up to inner automorphisms) is a cyclic subgroup of the S^1 action on M, and G/G_0 injects into the outer automorphism group $\mathrm{Out}(\pi_1 M)$ of the fundamental group. The fundamental group of M has a presentation

$$\pi_1 M = \langle a, b, t \mid [a, b] = t^k, \;\; [a, t] = [b, t] = 1\rangle,$$

with $k \neq 0$. Now an easy calculation shows that the subgroup of the outer automorphism group of $\pi_1 M$ inducing the identity of the factor group $\pi_1 M / < t > \cong \mathbb{Z}^2$ is finite. Since the orders of finite subgroups of $\mathrm{GL}_2(\mathbb{Z})$ are also bounded, G has a finite cyclic subgroup G_0 of bounded index.

Finally, if the base-surface is the 2-sphere then either M has finite fundamental group and is a spherical manifold, or homeomorphic to $S^2 \times S^1$ (and hence non-irreducible). We note that $S^2 \times S^1$ belongs to the $(S^2 \times \mathbb{R})$-geometry, one of Thurston's eight 3-dimensional geometries; this is the easiest of the eight geometries and can be easily handled directly, see [19] for the isometry group of this geometry.

Summarizing, we have shown for any closed irreducible 3-manifold M (and also for $S^2 \times S^1$) that $\mathrm{Diff}(M)$ is a Jordan group.

2.4. Suppose that M is non-irreducible but not $S^2 \times S^1$. If M has a summand other than lens spaces and $S^2 \times S^1$ then, by [6, Theorem 4.2], the orders of finite diffeomorphism groups of M are bounded and we are done.

Suppose next that M is a connected sum $\sharp_g(S^2 \times S^1)$ of g copies of $S^2 \times S^1$, with $g > 1$. Such manifolds M admit circle actions with global fixed points (cf. [18]); by [23], G has a finite cyclic normal subgroup C (the subgroup acting trivially on the fundamental group) such that the order of the factor group G/C is bounded by a quadratic polynomial in g, so we are done also in this case. Alternatively, the factor group G/C is isomorphic to a subgroup of the outer automorphism group $\mathrm{Out}(F_g)$ of a free group F_g of rank g, and by [21] the maximal order of a finite subgroup of $\mathrm{Out}(F_g)$ is (exponential) $2^g g!$, for $g > 2$ (and 12 for g=2).

Finally, if M is a connected sum of lens spaces including $S^2 \times S^1$, then M admits again circle actions with global fixed points and has a finite covering by a 3-manifold of type $\tilde{M} = \sharp_g(S^2 \times S^1)$ as considered before. Now $\mathrm{Diff}(\tilde{M})$ is a Jordan group and hence also $\mathrm{Diff}(M)$.

We have considered all possibilities for M. This completes the proof of Theorem 1.

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