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Existence of traveling waves in the fractional bistable equation

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Abstract. We construct traveling waves of the fractional bistable equation by approximating the fractional Laplacian $(D^2)^\alpha$, $\alpha \in (0,1)$, with operators $J * u - (\int_R J)u$, where J is nonsingular. Since the resulting approximating equations are known to have traveling waves, the solutions are obtained by passing to the limit. This provides an answer to the statement (about existence and properties) "This construction will be achieved in a future work" before Assumption 2 in Imbert and Souganidis [6]. With a modification of a part of the argument, we also get the existence of traveling waves for the ignition nonlinearity in the case $\alpha \in (1/2, 1)$.

1. Introduction. We study the equation

$$u_t = (\partial_{xx})^\alpha u - f(u), \tag{1.1}$$

where f is bistable, e.g., $f(u) = (u^2 - 1)(u - a)$, $-1 < a < 1$. The fractional power of the Laplacian in one dimension is defined for $\alpha \in (0, 1]$ as in [2, Appendix A]:

$$(D^2)^\alpha = -\frac{1}{2\cos(\pi\alpha)}({}_{-\infty}D_x^{2\alpha} + {}_xD_\infty^{2\alpha}), \tag{1.2}$$

i.e., a symmetrizing sum of Riemann–Liouville differintegrals. After integration by parts, we get

$$(D^2)^\alpha u(x) = -\frac{1}{2\Gamma(-2\alpha)\cos(\pi\alpha)} \text{ p.v.} \int_R \frac{u(y) - u(x)}{|x - y|^{1+2\alpha}} dy, \tag{1.3}$$

where p.v. denotes the Cauchy principal value. For $\alpha \in (0, 1/2)$ we can usually drop this p.v.. However, this is not optimal since there are functions for which (1.2) and (1.3) are defined but (1.3) without the p.v. is not. (1.3) allows the definition of $(D^2)^{1/2}$ since $\lim_{\alpha\to 1/2}\Gamma(-2\alpha)\cos(\pi\alpha) = -\pi/2$. $(D^2)^\alpha$ is a pseudo-differential operator of symbol $-|\xi|^{2\alpha}$:

$$(D^2)^\alpha u = \mathcal{F}^{-1}(-|\xi|^{2\alpha}(\mathcal{F}u)) \ \forall u \in \mathcal{S},$$

where $\mathcal{S}$ is the Schwartz class.

We seek traveling waves $u(x-ct)$ of (1.1) satisfying $u(-\infty) = -1$ and $u(\infty) = 1$. We assume $f \in C^3$ has only three zeros: $f(\pm 1) = f(a) = 0$, with $-1 < a < 1$ and $f'(\pm 1) > 0$. In particular, f does not have to be balanced, i.e., $\int_{-1}^1 f \neq 0$. In traveling wave coordinates, (1.1) becomes

$$cu' + (D^2)^\alpha u - f(u) = 0. \tag{1.4}$$

To our knowledge the existence of traveling waves has been studied only in the piecewise linear case [9] and in the balanced case $\int_{-1}^1 f = 0$ [3,8]. To give a short solution of the general problem, we use the following limiting argument.

Define $b_\alpha = -\frac{1}{2\Gamma(-2\alpha)\cos(\pi\alpha)}$,

$$J_\varepsilon(x) = \begin{cases} \frac{1}{|x|^{1+2\alpha}}, & |x| \geq \varepsilon, \\ \frac{1}{\varepsilon^{1+2\alpha}}, & |x| < \varepsilon, \end{cases} \tag{1.5}$$

and $j_\varepsilon = \int_R J_\varepsilon = (\frac{1}{\alpha}+2)\frac{1}{\varepsilon^{2\alpha}}$, so that formally $b_\alpha(J_\varepsilon * u - j_\varepsilon u) \to (D^2)^\alpha u$. From [1,4], there exists a unique pair $(u_\varepsilon, c_\varepsilon)$ with $u_\varepsilon' > 0$ solving the approximate equation

$$cu' + b_\alpha(J_\varepsilon * u - j_\varepsilon u) - f(u) = 0. \tag{1.6}$$

To be more precise, in [1] there is an additional assumption that $\int_R |x|J(x)dx < \infty$, however, the arguments requiring it can be adapted to (1.5). If ε is small enough, u_ε cannot be discontinuous with $c_\varepsilon = 0$, since $g_\varepsilon' > 0$, where $g_\varepsilon(u) = b_\alpha j_\varepsilon u + f(u)$, therefore it is smooth with

$$c_\varepsilon = \frac{\int_{-1}^1 f}{||u_\varepsilon'||_2^2}. \tag{1.7}$$

We expect $u_0 = \lim_{\varepsilon\to 0} u_\varepsilon$ to be the solution of (1.4). This limit of a subsequence exists by the Helly Theorem, suggesting passing to the limit in distributions. The problem with this approach is that bootstrapping a weak solution requires some effort since we know only that u_0 is monotonic, so if ϕ is a test function, then

$$\int_R u_\varepsilon b_\alpha(J_\varepsilon * \phi - j_\varepsilon\phi) \to \int_R u_0(D^2)^\alpha\phi,$$

and it is not clear if $\int_R u_0(D^2)^\alpha\phi = \int_R[(D^2)^\alpha u_0]\phi$. In Section 2 we show

Theorem 1.1. *Let $c \neq 0$ and $\alpha \in (0, 1/2)$. A monotonic weak solution of (1.4) with $u(-\infty) = -1$ and $u(\infty) = 1$ is also a strong solution.*

The proof requires that $(D^2)^\alpha$ be decomposed into a sum of two operators. This idea led us to estimates enabling using a strong formulation to get a solution. Namely, let $J_\varepsilon = S_\varepsilon + K$, with $S_\varepsilon > 0$, and let $K \in W^{1,1}(R)$ be chosen so that $g_k' > 0$, where $g_k(u) = b_\alpha ku + f(u)$, $k = \int_R K$. Let $s_\varepsilon = \int_R S_\varepsilon$. (1.6) differentiated can be written as

$$cu'' + b_\alpha(S_\varepsilon * u' - s_\varepsilon u') + b_\alpha K' * u = g_k'(u)u'. \tag{1.8}$$

Let $u_\varepsilon'(m_\varepsilon) = \max_{x\in R} u_\varepsilon'(x)$. Then since on the left hand side of (1.8) at m_ε the first term is 0, the second < 0, and the third uniformly bounded, u_ε' is uniformly bounded. Differentiating (1.8) once and twice more and using similar arguments shows that $|u_\varepsilon''|$ and $|u_\varepsilon'''|$ are also uniformly bounded, which implies after some work that

$$b_\alpha(J_\varepsilon * u_\varepsilon - j_\varepsilon u_\varepsilon) \to (D^2)^\alpha u_0 \tag{1.9}$$

pointwise, getting

Theorem 1.2. *Let f be of bistable type and $\alpha \in (0, 1)$. There exists a solution of (1.4) with sgn c =sgn $\int f$, $u(-\infty) = -1$, $u(\infty) = 1$, and $u' > 0$.*

The proofs of Theorems 1.1 and 1.2 are provided in Section 2. In Section 3 we adapt the existence argument to (1.1) with the ignition nonlinearity, i.e., $f|_{(-1,\rho)} \equiv 0$, $f|_{(\rho,1)} < 0$ and $f'(1) > 0$, getting

Theorem 1.3. *Let f be of ignition type and $\alpha \in (1/2, 1)$. There exists a solution of (1.4) with $c < 0$, $u(-\infty) = -1$, $u(\infty) = 1$, and $u' > 0$.*

This result was also announced in [7].

2. Bistable. We neglect subsequences in the notation.
Proof of Theorem 1.1. u_0 satisfies

$$-c_0 \int_R u_0\phi' + \int_R u_0(D^2)^\alpha\phi - \int_R f(u_0)\phi = 0.$$

Let $J_2 \in L^1(R)$ be such that

$$J_1(x) \begin{cases} = \frac{1}{|x|^{1+2\alpha}} - J_2(x), & x \in R, \\ = \frac{1}{|x|^{1+2\alpha}}, & |x| \le 1, \\ \le \frac{1}{|x|^{1+2\alpha}}, & |x| > 1, \end{cases}$$

is compactly supported. Then dropping p.v.

$$(D^2)^\alpha\phi = J_1 \circledast \phi + J_2 * \phi - j_2\phi,$$

where

$$J_1 \circledast \phi(x) = \int_R J_1(x-y)(\phi(y) - \phi(x))dx$$

and $j_2 = \int_R J_2$. Let $u_1(x) = \int_0^x u_0$. u_1 is Lipschitz continuous with $u_1' = u_0$ a.e., so we can use

$$u_1(x+y) - u_1(x) = \int_0^1 yu_0(x+ty)dt,$$

not

$$u_0(x+y)-u_0(x)=\int_0^1 yu_0'(x+ty)dt, \tag{2.1}$$

since u_0 is not absolutely continuous, only monotonic, to get a.e.

$$c_0u_0'+(J_1\circledast u_1)'+J_2*u_0-j_2u_0-f(u_0)=0, \tag{2.2}$$

where we used $u_1(x)J_1\circledast\phi(x)=O(x^{-2\alpha})\to 0$ as $x\to\pm\infty$.

Since $c_0u_0+J_1\circledast u_1\in W^{1,\infty}(R)$, it is Lipschitz continuous, so that

$$-c_0u_0(x)=-c_0u_0(0)+J_1\circledast u_1(x)-J_1\circledast u_1(0)+\int_0^x[J_2*u_0-j_2u_0-f(u_0)].$$

$J_1\circledast u_1$ is absolutely continuous from

$$\begin{aligned} J_1\circledast u_1(x) &= \int_{R\backslash(-1,1)} J_1(y)[u_1(x+y)-u_1(x)]dy \\ &\quad +\int_{-1}^1 \frac{y^\beta(\int_0^1 u_0(x+ty)dt)^\beta(u_1(x+y)-u_1(x))^{1-\beta}}{|y|^{1+2\alpha}}dy, \end{aligned}$$

with $\beta>2\alpha$, so u_0 is also absolutely continuous and (2.1) used in the Leibniz criterion yields $(J_1\circledast u_1)'=J_1\circledast u_0$. From the Fundamental Theorem of Calculus, (2.2) holds everywhere. □

Proof of Theorem 1.2. From a diagonal argument and the Arzelà-Ascoli theorem (AAT), $u_\varepsilon\to u_0$ and $u_\varepsilon'\to u_0'$ pointwise on R. Using (1.9) and passing to the limit in (1.6), we get

$$c_0u_0'+(D^2)^\alpha u_0-f(u_0)=0. \tag{2.3}$$

Let $S(\varepsilon)=[-1,1]\backslash(-\varepsilon,\varepsilon)$. To show (1.9) we have

$$\begin{aligned} J_\varepsilon*u_\varepsilon-j_\varepsilon u_\varepsilon &= \int_{R\backslash(-1,1)}\frac{u_\varepsilon(x+y)-u_\varepsilon(x)}{|y|^{1+2\alpha}}dy+\int_{S(\varepsilon)}\frac{u_0(x+y)-u_0(x)}{|y|^{1+2\alpha}}dy \\ &\quad +\int_{S(\varepsilon)}\frac{u_\varepsilon(x+y)-u_\varepsilon(x)-[u_0(x+y)-u_0(x)]}{|y|^{1+2\alpha}}dy \\ &\quad +\int_{-\varepsilon}^{\varepsilon}\frac{u_\varepsilon(x+y)-u_\varepsilon(x)}{\varepsilon^{1+2\alpha}}dy=I_1(\varepsilon)+I_2(\varepsilon)+I_3(\varepsilon)+I_4(\varepsilon). \end{aligned} \tag{2.4}$$

In $I_1(\varepsilon)$ $\left|\frac{u_\varepsilon(x+y)-u_\varepsilon(x)}{|y|^{1+2\alpha}}\right|\le\frac{2}{|y|^{1+2\alpha}}$, so we pass to the limit using the Lebesgue dominated convergence theorem (LDCT). Then

$$\lim_{\varepsilon\to 0}I_2(\varepsilon)=\text{p.v.}\int_{[-1,1]}\frac{u_0(x+y)-u_0(x)}{|y|^{1+2\alpha}}dy.$$

In $I_3(\varepsilon)$ for $\alpha \in (0, 1/2)$ we use (2.1), and for $\alpha \in [1/2, 1)$ we use the Taylor formula

$$u(x+y) - u(x) = u'(x)y + \int_0^1 y^2(1-t)u''(x+ty)dt,$$

getting for $\alpha \in (0, 1/2)$

$$|I_2(\varepsilon)| \le \int_{S(\varepsilon)} \frac{\int_0^1 |u'_\varepsilon(x+ty) - u'_0(x+ty)|dt}{|y|^{2\alpha}}$$
$$\le \sup_{s \in S_x} |u'_\varepsilon(s) - u'_0(s)| \int_{S(\varepsilon)} \frac{dy}{|y|^{2\alpha}} \to 0$$

and for $\alpha \in [1/2, 1)$

$$I_2(\varepsilon) = \int_{S(\varepsilon)} \frac{\int_0^1 (1-t)[u''_\varepsilon(x+ty) - u''_0(x+ty)]dt}{|y|^{2\alpha-1}},$$

so

$$|I_2(\varepsilon)| \le \frac{1}{2} \sup_{s \in S_x} |u''_\varepsilon(s) - u''_0(s)| \int_{S(\varepsilon)} \frac{dy}{|y|^{2\alpha-1}} \to 0,$$

both from AAT, since $S_x = x + [-1, 1]$. In $I_4(\varepsilon)$ the same Taylor formulas yield: for $\alpha \in (0, 1/2)$ $|I_4(\varepsilon)| \le C_1(\varepsilon)\varepsilon^{1-2\alpha} \to 0$ and for $\alpha \in [1/2, 1)$ $|I_4(\varepsilon)| \le \frac{2}{3}C_2(\varepsilon)\varepsilon^{2-2\alpha} \to 0$, where $C_1(\varepsilon) = \sup|u'_\varepsilon|$ and $C_2(\varepsilon) = \sup|u''_\varepsilon|$.

This proves the Theorem after we rule out degenerate cases and show that $u'_0 > 0$.

Let $u_\varepsilon(0) = r$ where $-1 < r < a$ if $\int_{-1}^1 f \le 0$ and $a < r < 1$ otherwise. We can have the following: (a) $u_0 \equiv r$, (b) $u_0(-\infty) = -1$, $u_0(\infty) = r$ (or $u_0(-\infty) = r$, $u_0(\infty) = 1$), and (c) $c_\varepsilon \to \pm\infty$.

(a) This is not possible since $f(r) \ne 0$.

(b) Let $-1 < r < a$. As in (1.9) we get $b_\alpha(J_\varepsilon * u_0 - j_\varepsilon u_0) \to (D^2)^\alpha u_0$. It is a standard calculation that $\int_R b_\alpha(J_\varepsilon * u_0 - j_\varepsilon u_0)u'_0 = 0$, namely

$$\int_R (J_\varepsilon * u_0)u'_0 = {u_0}^2(\infty) - {u_0}^2(-\infty) - \int_R (J_\varepsilon * u'_0)u_0,$$

and since $\int_R (J_\varepsilon * u'_0)u_0 = \int_R (J_\varepsilon * u_0)u'_0$, we get

$$\int_R (J_\varepsilon * u_0)u'_0 = \frac{1}{2}[u_0^2(\infty) - u_0^2(-\infty)].$$

Since $b_\alpha(J_\varepsilon * u_0 - j_\varepsilon u_0)$ is uniformly bounded by a constant, using LDCT we get $\int_R [(D^2)^\alpha u_0]u'_0 = 0$. Multiplying (1.4) by u'_0 and integrating over R yields $c_0 > 0$, a contradiction with $c_\varepsilon \le 0$. We also get

$$c_0 = \frac{\int_{-1}^{1} f}{||u_0'||_2^2}.$$

(c) Here the proof is very similar to [1, Lemma 2.6]. Assuming that $c_\varepsilon \to \infty$ let $\eta > 0$ and $f(u_\varepsilon(\xi_\varepsilon)) < -\eta$. Then at ξ_ε

$$\eta < c_\varepsilon u_\varepsilon' - f(u_\varepsilon) = -b_\alpha(J_\varepsilon * u_\varepsilon - j_\varepsilon u_\varepsilon)$$

with a contradiction after showing that $|b_\alpha(J_\varepsilon * u_\varepsilon - j_\varepsilon u_\varepsilon)| \to 0$ uniformly. Namely, similarly to (2.4) let

$$J_\varepsilon * u_\varepsilon - j_\varepsilon u_\varepsilon = I_1(\varepsilon) + I_2(\varepsilon) + I_3(\varepsilon), \tag{2.5}$$

where $I_1(\varepsilon)$ is as in (2.4),

$$I_2(\varepsilon) = \int\limits_{S(\varepsilon)} \frac{u_\varepsilon(x+y) - u_\varepsilon(y)}{|y|^{1+2\alpha}} dy, \;\; I_3(\varepsilon) = \int\limits_{-\varepsilon}^{\varepsilon} \frac{u_\varepsilon(x+y) - u_\varepsilon(y)}{\varepsilon^{1+2\alpha}} dy.$$

Both $||u_\varepsilon'||_\infty \to 0$ and $||u_\varepsilon''||_\infty \to 0$ since in (1.4) all terms other than $c_\varepsilon u_\varepsilon'$ are uniformly bounded, and in differentiated (1.4) all terms other than $c_\varepsilon u_\varepsilon''$ are uniformly bounded. From this all $I_i(\varepsilon) \to 0$ uniformly, e.g., in the case $\alpha \in (0, 1/2]$:

$$I_1(\varepsilon) = \int\limits_{R\setminus(-1,1)} \frac{y^\beta (\int_0^1 u_\varepsilon'(x+ty)dt)^\beta (u_\varepsilon(x+y) - u_\varepsilon(x))^{1-\beta}}{|y|^{1+2\alpha}} dy,$$

where $\beta < 2\alpha$, giving

$$|I_1(\varepsilon)| \le 2^{1-\beta} ||u_\varepsilon'||_\infty^\beta \int\limits_{R\setminus(-1,1)} \frac{dy}{|y|^{1+2\alpha-\beta}} \to 0$$

uniformly.

The argument that $u_0' > 0$ is standard. We have $u_0' \ge 0$. If $u_0'(x_0) = 0$, then after we differentiate (1.4), at x_0 the middle term is > 0, the other two are 0. $\square$

3. Ignition.

Proof of Theorem 1.3. Here we use a couple of calculations from [5], where the author constructed traveling waves for (1.6) with the ignition nonlinearity.

In the approximating equations, we let bistable $f_\varepsilon \to f$ with f'_ε uniformly bounded and normalize $u_\varepsilon(0) = \rho$.

$c_\varepsilon \not\to 0$ since after pairing (1.6) with u'_ε, $\int_R u'^2_\varepsilon$ is uniformly bounded and $\int_{-1}^1 f_\varepsilon \to \int_{-1}^1 f < 0$.

$c_\varepsilon \not\to -\infty$ since after integrating (1.6) over $(-\infty, 0)$, we get

$$-c_\varepsilon(\rho+1) < -c_\varepsilon(\rho+1) + \int_{-\infty}^{0} f(u_\varepsilon) = \int_{-\infty}^{0} b_\alpha(J_\varepsilon * u_\varepsilon - j_\varepsilon u_\varepsilon), \tag{3.1}$$

so that with (2.5) again, since $\alpha \in (1/2, 1)$, all $\int_{-\infty}^0 I_i(\varepsilon)$ are uniformly bounded, e.g.,

$$\left|\int_{-\infty}^{0} I_1(\varepsilon)\right| = \left|\int_{-\infty}^{0}\int_{R\backslash(-1,1)} \frac{y\int_0^1 u'_\varepsilon(x+ty)dt}{|y|^{1+2\alpha}}dydx\right| \le 2\int_{R\backslash(-1,1)} \frac{dy}{|y|^{2\alpha}}.$$

Passing to the limit in (1.6), we get (2.3) with $c_0 < 0$. To show that $u_0(-\infty) = -1$ and $u_0(\infty) = 1$, we first pass to the limit in (3.1), getting

$$-c_0(\rho+1) \le \int_{-\infty}^{0} (D^2)^\alpha u_0, \tag{3.2}$$

then integrate (2.3) over $(-\infty, 0)$, getting

$$-c_0(\rho - u_0(-\infty)) = \int_{-\infty}^{0} (D^2)^\alpha u_0. \tag{3.3}$$

(3.2) and (3.3) give $u_0(-\infty) \le -1$. We pair (2.3) with u'_0 to get $\int_{-1}^{u(\infty)} f < 0$, thus $u(\infty) > \rho$. Passing to the limit $x \to \infty$ in (2.3), we get $f(u(\infty)) = 0$, thus $u(\infty) = 1$.

Getting (3.2) from (3.1) is done with a bit of care, e.g.,

$$\int_{-\infty}^{0} I_1(\varepsilon) = \int_{R\backslash(-1,1)}\int_0^1 \frac{y(u_\varepsilon(ty)+1)}{|y|^{1+2\alpha}}dtdy \to \int_{R\backslash(-1,1)}\int_0^1 \frac{y(u_0(ty)+1)}{|y|^{1+2\alpha}}dtdy$$

$$= \int_{-\infty}^{0}\int_{R\backslash(-1,1)} \frac{u_0(x+y)-u_0(y)}{|y|^{1+2\alpha}}dydx.$$

□

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