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On the setwise convergence of sequences of signed topological measures

Marina Svistula

Abstract. It is proved that the setwise limit of a bounded sequence of signed topological measures is a signed topological measure; here the signed measures and proper signed topological measures which are the components of the decomposition of the members of the sequence setwise converge to the corresponding components of the decomposition of the limit signed topological measure. The results thus obtained give a negative answer to the question concerning the possibility to represent a regular Borel measure (nonzero) as a setwise limit of a sequence of proper topological measures.

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1. Introduction. Throughout the paper, X is a compact Hausdorff space, ζ and τ denote respectively the collection of all closed and open subsets of X, $\alpha = \tau \cup \zeta$ and η is the Borel σ-algebra of X.

Definition 1.1. A set function $\mu : \alpha \to \mathbb{R}$ is called a signed topological measure if the following hold:

1) μ is finitely additive: if $\{E_i\}_{i=1}^n \subset \alpha,\ E_i \cap E_j = \emptyset$ for $i \neq j$, $\cup_{i=1}^n E_i \in \alpha$, then
$$\mu\left(\bigcup_{i=1}^n E_i\right) = \sum_{i=1}^n \mu(E_i);$$
2) μ is bounded: $\sup\{|\mu(E)| : E \in \alpha\} < \infty$;
3) μ is regular: for any $U \in \tau$ and $\varepsilon > 0$ there exists a $C \in \zeta$ such that $C \subset U$, and if $\alpha \ni E \subset U \setminus C$, then $|\mu(E)| < \varepsilon$.

If, in addition, μ is nonnegative, then μ is called a topological measure. It can readily be proved that in this case Definition 1.1 is equivalent to the collection of the following three conditions:

a) if $E, F \in \alpha$ and $E \subset F$, then $\mu(E) \leq \mu(F)$;
b) if $E, F \in \alpha$, $E \cap F = \emptyset$, and $E \cup F \in \alpha$, then $\mu(E \cup F) = \mu(E) + \mu(F)$;
c) if $U \in \tau$, then $\mu(U) = \sup\{\mu(C) : \zeta \ni C \subset U\}$.

Obviously, the signed topological measures form a linear space, and the topological measures form a cone denoted by STM and TM, respectively.

It is known that a topological measure does not necessarily extend to a regular Borel measure. A topological measure which admits such an extension will be called a measure. Denote the cone of measures by M. Similarly, we will define a signed measure. Denote the linear space of signed measures by SM.

The topological measures (formerly called quasi-measures) were first introduced in [1] and the signed topological measures in [4] for representations of certain nonlinear functionals on the space of continuous functions on X. At the same time these set functions are interesting by themselves. So our paper was caused by the Aarnes open question connected with the setwise convergence (see below).

The main results of this paper are contained in Theorems 6.1–6.5, and the material needed for their proof is presented in Sections 2–5.

2. The Aarnes question and proper topological measures. When studying $\mu \in STM$, we use the supremation $\widetilde{\mu}$ given by

$$\widetilde{\mu}(E) = \sup\{|\mu(C)| : \zeta \ni C \subset E\}, \quad E \subset X.$$

Obviously, $\widetilde{\mu}$ is a monotone set function on the collection of all subsets of X, and if $E_1 \cap E_2 = \emptyset$ and both the sets are in τ or ζ, then $\widetilde{\mu}(E_1 \cup E_2) \leq \widetilde{\mu}(E_1) + \widetilde{\mu}(E_2)$. Using the regularity of μ, one can readily show that

$$\widetilde{\mu}(U) = \sup\{|\mu(V)| : \tau \ni V \subset U\} \text{ for } U \in \tau.$$

If $\lambda \in SM$, then λ can be assumed to be extended to η, and one can prove that

$$\widetilde{\lambda}(E) = \sup\{|\lambda(F)| : \eta \ni F \subset E\} \text{ for } E \in \eta,$$

$$\widetilde{\lambda}(E_1 \cup E_2) \leq \widetilde{\lambda}(E_1) + \widetilde{\lambda}(E_2) \text{ for any } E_1, E_2 \in \eta.$$

For $\mu \in TM$, it is clear that $\widetilde{\mu} = \mu$ on α. If $\lambda \in M$, then $\widetilde{\lambda} = \lambda$ on η.

Definition 2.1. A signed topological measure μ is said to be proper if for any $\varepsilon > 0$ there exists a finite family $\{U_i\}_{i=1}^n \subset \tau$ such that

$$X = \bigcup_{i=1}^{n} U_i \quad \text{and} \quad \sum_{i=1}^{n} \widetilde{\mu}(U_i) < \varepsilon.$$

In the case of a topological measure, we replace $\widetilde{\mu}$ by μ.

Denote by $PSTM$ (respectively PTM) the class of all proper signed topological measures (respectively proper topological measures).

Remark 2.2. In [5] a topological measure μ is called proper if whenever λ is a measure and $0 \leq \lambda \leq \mu$, then $\lambda = 0$. The equivalence of the two definitions was proved in [6]. The concept of a proper signed topological measure was considered in [7]. Our approach is more convenient for the results here. It was also used to prove the nontrivial fact that if $\mu_1, \mu_2 \in PSTM$, then $\mu_1 + \mu_2 \in PSTM$ (see [6,7]). Using this, everyone can show that $PSTM$ is a linear space and PTM is a cone.

Below we also need the following result [7].

Theorem 2.3. *For any $\mu \in STM$ there is a unique decomposition of the form $\mu = \lambda + \nu$, where $\lambda \in SM$ and $\nu \in PSTM$. If $\mu \in TM$, then $\lambda \in M$ and $\nu \in PTM$.*[1]

We say that a sequence $\{\mu_n\} \subset STM$ converges setwise if for any set $E \in \alpha$ there is a finite $\lim_n \mu_n(E) = \mu(E)$. It is true that $\mu \in PSTM$ if $\mu_n \in PSTM$ for any n? This problem remained open even for topological measures. Let us consider the example given by Aarnes [2, Sec. 6].

Example. Let $X = [0;1] \times [0;1]$, and let λ be the Lebesgue measure on X.[2] Let $n \in \mathbb{N}$ and $q = 2^n + 1$. Define

$$I_k = [k/q;\ (k+1)/q) \ \text{ for } \ k = 0, 1, \ldots, 2^n - 1, \quad I_{2^n} = [2^n/q; 1].$$

Let C be a closed solid in X, i.e., a closed set such that both the set and its complement are connected. Put

$$\mu_n(C) = k/2^n \quad \text{if} \quad \lambda(C) \in I_k.$$

Obviously, $\mu_n(C) \in I_k$ for $\lambda(C) \in I_k$, $\mu_n(C) = 0$ for $\lambda(C) < 1/q$, and $\mu_n(C) = 1$ for $\lambda(C) \geq 2^n/q$. It was proved in [2] that every μ_n is a solid set-function and hence it can uniquely be extended to a topological measure on α, and for any closed set C with a finite number of connected components

$$\lim_n \mu_n(C) = \lambda(C). \tag{2.1}$$

Aarnes asks whether or not this equality holds for any closed set C. Our results lead to a negative answer. Obviously, there is a family of closed solids $\{C_i\}_{i=1}^m$ such that

$$X = \bigcup_{i=1}^m C_i \quad \text{and} \quad \lambda(C_i) < 1/q.$$

In this case $\mu_n(C_i) = 0$ for any $i = 1, \ldots, m$. This implies that $\mu_n \in PTM$. Assuming an affirmative answer, we can readily deduce the equality (2.1) for all sets from α and obtain by Theorem 6.4 that $\lambda \in PTM$, whereas λ is not proper.

[1] The result for $\mu \in TM$ was proved in [5].
[2] In [2] the conditions on X and λ are more general.

3. Singularity. We noticed the following property of a proper signed topological measure.

Proposition 3.1. *Let $\nu \in PSTM$ and $\lambda \in SM$. Then ν is singular to λ in the sense that for any set $\Theta \in \tau$ and for any $\varepsilon > 0$, there exists a set $C \in \zeta$ such that $C \subset \Theta$, $\widetilde{\lambda}(\Theta \setminus C) < \varepsilon$, and $\widetilde{\nu}(C) < \varepsilon$.*

Proof. Since $\nu \in PSTM$, there is a finite family $\{U_i\}_{i=1}^n \subset \tau$ such that

$$\Theta = \bigcup_{i=1}^{n} U_i \quad \text{and} \quad \sum_{i=1}^{n} \widetilde{\nu}(U_i) < \varepsilon.$$

Put

$$E_1 = U_1,\ E_2 = U_2 \setminus U_1,\ \ldots,\ E_n = U_n \setminus \bigcup_{i=1}^{n-1} U_i.$$

Obviously, $E_i \in \eta,\ E_i \cap E_j = \emptyset$ for $i \neq j,\ E_i \subset U_i$, and $\Theta = \cup_{i=1}^n E_i$. Assume that λ is extended to a signed regular Borel measure on η. By the regularity, there are sets $C_i \in \zeta$ such that $C_i \subset E_i$ and $\widetilde{\lambda}(E_i \setminus C_i) < \varepsilon/n,\ i = 1, \ldots, n$. Put

$$C = \bigcup_{i=1}^{n} C_i.$$

Obviously, $C \in \zeta,\ C \subset \Theta$, and $\Theta \setminus C = \cup_{i=1}^n (E_i \setminus C_i)$. We see that

$$\widetilde{\lambda}(\Theta \setminus C) \leq \sum_{i=1}^{n} \widetilde{\lambda}(E_i \setminus C_i) < \varepsilon \quad \text{and} \quad \widetilde{\nu}(C) \leq \sum_{i=1}^{n} \widetilde{\nu}(C_i) \leq \sum_{i=1}^{n} \widetilde{\nu}(U_i) < \varepsilon.$$

This proves the proposition. □

Now we can obtain two inequalities that are useful below.

Proposition 3.2. *Let $\mu = \lambda + \nu$, where $\lambda \in SM$ and $\nu \in PSTM$ (and hence $\mu \in STM$). Then $\widetilde{\lambda}(U) \leq \widetilde{\mu}(U)$ and $\widetilde{\nu}(U) \leq \widetilde{\mu}(U)$ for any $U \in \tau$.*

Proof. Let $\Theta \in \tau$ and $\varepsilon > 0$. Find $C \in \zeta$ as in Proposition 3.1. Then

$$\begin{aligned} |\lambda(\Theta)| &\leq |\lambda(\Theta \setminus C)| + |\lambda(C)| \leq \varepsilon + |\lambda(C)| \\ &\leq \varepsilon + |\lambda(C) + \nu(C)| + |\nu(C)| \leq 2\varepsilon + |\mu(C)| \leq 2\varepsilon + \widetilde{\mu}(\Theta). \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we see that $|\lambda(\Theta)| \leq \widetilde{\mu}(\Theta)$ for any $\Theta \in \tau$. If $U, \Theta \in \tau$, and $\Theta \subset U$, then $|\lambda(\Theta)| \leq \widetilde{\mu}(\Theta)| \leq \widetilde{\mu}(U)$. Passing to the supremum over all Θ of the above kind in this inequality, we obtain $\widetilde{\lambda}(U) \leq \widetilde{\mu}(U)$. We can carry out similar bounds for the function ν. □

4. Uniform s-boundedness. The concept of uniform s-boundedness, which is widely used in measure theory, turns out to be fruitful when studying signed topological measures as well.

Definition 4.1. Let ξ be a collection of subsets of a set T. A family of set functions $\{\mu_\gamma\}_{\gamma\in\Gamma}$, where $\mu_\gamma : \xi \to \mathbb{R}$, is said to be uniformly s-bounded on ξ if for any sequence of pairwise disjoint members $\{E_n\}$ in ξ we have

$$\lim_n \sup_{\gamma\in\Gamma} |\mu_\gamma(E_n)| = 0.$$

In the case of a single function, we say that it is s-bounded.

Proposition 4.2. *If $\mu \in STM$, then it is s-bounded on α.*

Proof. Assume the contrary. Then there is an $\varepsilon > 0$ and a sequence $\{E_n\}$ of pairwise disjoint members in τ or in ζ for which $|\mu(E_n)| > \varepsilon$ holds for all n and μ has the same signs on all E_n. It is readily seen that this contradicts the boundedness of μ. □

Proposition 4.3. *Let $\{\mu_\gamma\}_{\gamma\in\Gamma} \subset STM$. The family $\{\mu_\gamma\}$ is uniformly s-bounded on τ if and only if the family $\{\widetilde{\mu}_\gamma\}$ has this property.*

Proof. The sufficiency is obvious. Let us prove the necessity. Suppose the contrary. Then there are an $\varepsilon > 0$, a sequence $\{U_n\}$ of pairwise disjoint members in τ, and a sequence $\{\gamma_n\} \subset \Gamma$ such that $\widetilde{\mu}_{\gamma_n}(U_n) > \varepsilon$ for any $n \in \mathbb{N}$. By the properties of supremation, for any $n \in \mathbb{N}$ there is a $V_n \in \tau$ such that $V_n \subset U_n$ and $|\mu_{\gamma_n}(V_n)| > \varepsilon$. We obtain a sequence $\{V_n\}$ of pairwise disjoint members in τ for which $\lim_n \mu_{\gamma_n}(V_n) \neq 0$. This contradicts the uniform s-boundedness property of the family $\{\mu_\gamma\}$ on τ. □

Propositions 3.2 and 4.3 immediately imply the following proposition.

Proposition 4.4. *Let $\{\mu_\gamma\}_{\gamma\in\Gamma} \subset STM$. Let $\mu_\gamma = \lambda_\gamma + \nu_\gamma$, where $\lambda_\gamma \in SM$ and $\nu_\gamma \in PSTM$. The family $\{\mu_\gamma\}$ is uniformly s-bounded on τ if and only if both the families $\{\lambda_\gamma\}$ and $\{\nu_\gamma\}$ have this property.*

We need the following well-known variant of the Vitali–Hahn–Saks–Nikodým theorem. For a proof we refer to [3, I.4.8].[3]

Theorem 4.5. *A sequence of finitely additive, bounded, real valued set functions on a σ-algebra which converges setwise is uniformly s-bounded.*

Now we can readily obtain the main result of this section.

Proposition 4.6. *Let $\lambda_n \in SM$, $\nu_n \in PSTM$, and $\mu_n = \lambda_n + \nu_n$, $n \in \mathbb{N}$. Let a finite $\lim_n \mu_n(E)$ exists for any $E \in \alpha$. Then the sequences $\{\mu_n\}$, $\{\lambda_n\}$, and $\{\nu_n\}$ are uniformly s-bounded on τ.*

Proof. Let $\{U_k\}$ be an arbitrary sequence of pairwise disjoint members in τ. Obviously, the collection of sets $\Sigma = \{\cup_{k\in I} U_k : I \subset \mathbb{N}\}$ is a σ-algebra of the space $\cup_{k=1}^{\infty} U_k$, and the restriction of an arbitrary signed topological measure to Σ is a finitely additive, bounded, real valued set function. Thus, $\{\mu_n|_\Sigma\}$ is a setwise convergent sequence of set functions with the above properties. By Theorem 4.5, it is uniformly s-bounded on Σ.

[3] Note that in [3, I.4.8] uniform strong additivity can be replaced by uniform s-boundedness in view of their equivalence (see [3, I.1.17 and I.1.18]).

We have that $\lim_k \sup_{n\in\mathbb{N}} |\mu_n(U_k)| = 0$, and hence the sequence $\{\mu_n\}$, together with the sequences $\{\lambda_n\}$ and $\{\nu_n\}$ (by Proposition 4.4), is uniformly s-bounded on τ. □

5. Uniform regularity. Now we establish a relationship between the uniform s-boundedness and uniform regularity for a family of signed topological measures and also for a family of signed regular Borel measures. Although the results related to the last are known [3, VI.2.13], we construct proofs in a somewhat different way, with regard to results concerning signed topological measures.

For a family $\{\mu_\gamma\}_{\gamma\in\Gamma} \subset STM$, we interrelate the following statements:

(a) $\{\mu_\gamma\}$ is uniformly s-bounded on ζ;
(b) $\{\mu_\gamma\}$ is uniformly s-bounded on τ;
(c) for any $U \in \tau$ and $\varepsilon > 0$, there is a $C \in \zeta$ such that $C \subset U$ and $\widetilde{\mu}_\gamma(U \setminus C) \le \varepsilon$ for all $\gamma \in \Gamma$ simultaneously (obviously, this is equivalent to the following: for any $C \in \zeta$ and $\varepsilon > 0$, there is a $U \in \tau$ such that $C \subset U$ and $\widetilde{\mu}_\gamma(U \setminus C) \le \varepsilon$ for all $\gamma \in \Gamma$).

If, moreover, $\{\mu_\gamma\}_{\gamma\in\Gamma} \subset SM$, then we also consider:

(d) $\{\mu_\gamma\}$ is uniformly s-bounded on η;
(e) for any $E \in \eta$ and $\varepsilon > 0$, there is a $C \in \zeta$ such that $C \subset E$ and $\widetilde{\mu}_\gamma(E \setminus C) \le \varepsilon$ for all $\gamma \in \Gamma$ (obviously, this is equivalent to what follows: for any $E \in \eta$ and $\varepsilon > 0$, there is a $U \in \tau$ such that $E \subset U$ and $\widetilde{\mu}_\gamma(U \setminus E) \le \varepsilon$ for all $\gamma \in \Gamma$).

Proposition 5.1. 1) *Let $\{\mu_\gamma\}_{\gamma\in\Gamma} \subset STM$. Then (a) $\Rightarrow$ (b) $\Leftrightarrow$ (c).*

2) *If $\{\mu_\gamma\}_{\gamma\in\Gamma} \subset SM$, then each of the statements (a) – (e) above implies all the others.*

Proof. 1) Let us prove that (a) $\Rightarrow$ (b). Suppose the contrary. Then there are an $\varepsilon > 0$, a sequence $\{U_n\}$ of pairwise disjoint members in τ, and a sequence $\{\gamma_n\} \subset \Gamma$ such that $|\mu_{\gamma_n}(U_n)| > \varepsilon$ for any $n \in \mathbb{N}$. By the regularity of each μ_{γ_n}, there is a $C_n \in \zeta$ such that $C_n \subset U_n$ and $|\mu_{\gamma_n}(C_n)| > \varepsilon$. This contradicts (a).

The proof that (b) $\Rightarrow$ (c) is analogous to [3, VI.2.13]. Suppose that there are $U \in \tau$ and $\varepsilon > 0$ for which there exists no desired set C. Put $\Theta_1 = U$. By assumption, the empty set is not a desired set. Therefore, there is a $\gamma_1 \in \Gamma$ such that $\widetilde{\mu}_{\gamma_1}(\Theta_1) > \varepsilon$. By the definition of the supremation, there is a $C_1 \in \zeta$ such that $C_1 \subset \Theta_1$ and $|\mu_{\gamma_1}(C_1)| > \varepsilon$. Since the compact Hausdorff space X is normal, there are $U_1 \in \tau$ and $K_1 \in \zeta$ with $C_1 \subset U_1 \subset K_1 \subset \Theta_1$. Using Definition 1.1, one can readily prove the existence of a set $V_1 \in \tau$ such that

$$C_1 \subset V_1 \subset U_1 \quad \text{and} \quad |\mu_{\gamma_1}(V_1)| > \varepsilon.$$

Put $\Theta_2 = U \setminus K_1$. By assumption, the set K_1 is not a desired one. Therefore, there are $\gamma_2 \in \Gamma$ and $C_2 \in \zeta$ such that $C_2 \subset \Theta_2$ and $|\mu_{\gamma_2}(C_2)| > \varepsilon$. We further find $V_2 \in \tau$ and $K_2 \in \zeta$ for which

$$C_2 \subset V_2 \subset K_2 \subset \Theta_2 \quad \text{and} \quad |\mu_{\gamma_2}(V_2)| > \varepsilon.$$

Put $\Theta_3 = U \setminus (K_1 \cup K_2)$, and so on. Continuing the process, we obtain a sequence $\{V_n\}$ of pairwise disjoint members in τ and a sequence $\{\mu_{\gamma_n}\}$ such that $|\mu_{\gamma_n}(V_n)| > \varepsilon$ for any $n \in \mathbb{N}$. This contradicts the (b).

To prove that (c) $\Rightarrow$ (b), let $\{U_n\}$ be a sequence of pairwise disjoint members in τ and $\varepsilon > 0$. Put $U = \cup_{n=1}^{\infty} U_n$, and find a set C as in (c). Since C is compact, there is n_0 such that $C \subset \cup_{n=1}^{n_0} U_n$. Thus, for $m > n_0$ we have $U_m \subset U \setminus C$ and, consequently, $|\mu_\gamma(U_m)| < \varepsilon$ for all γ.

2) To complete the proof, it is sufficient to show that (c) $\Rightarrow$ (a), (a) $\Rightarrow$ (e), and (a) $\Rightarrow$ (d).

To prove that (c) $\Rightarrow$ (a), let $\{C_n\}$ be a sequence of pairwise disjoint members in ζ and $\varepsilon > 0$. According to (c), for any $n \in \mathbb{N}$ there is $U_n \in \tau$ such that $C_n \subset U_n$ and $\widetilde{\mu}_\gamma(U_n \setminus C_n) < \varepsilon/2^n$ for all γ. Put $U = \cup_{n=1}^{\infty} U_n$, and find a set $C \subset U$ such that $\widetilde{\mu}_\gamma(U \setminus C) < \varepsilon$ for all γ. Since C is compact, there is n_0 such that $C \subset \cup_{n=1}^{n_0} U_n$. For $m > n_0$ we have

$$C_m \subset (U \backslash C) \cup \left(\bigcup_{n=1}^{n_0} (U_n \setminus C_n) \right).$$

For any $\gamma \in \Gamma$ we obtain $\widetilde{\mu}_\gamma(C_m) \le \varepsilon + \Sigma_{n=1}^{n_0} \varepsilon/2^n \le 2\varepsilon$ as supremation of a signed measure is subadditive.

Let us prove that (a) $\Rightarrow$ (e). Suppose the contrary. Then there are $E \in \eta$ and $\varepsilon > 0$ for which there exists no desired set C. Find $\gamma_1 \in \Gamma$ and $C_1 \in \zeta$ such that $C_1 \subset E$ and $|\mu_{\gamma_1}(C_1)| > \varepsilon$. By assumption, the set C_1 is not a desired one. Therefore, there are $\gamma_2 \in \Gamma$ and $C_2 \in \zeta$ such that $C_2 \subset E \setminus C_1$ and $|\mu_{\gamma_2}(C_2)| > \varepsilon$. Find $\gamma_3 \in \Gamma$ and $C_3 \in \zeta$ such that $C_3 \subset E \setminus (C_1 \cup C_2)$ and $|\mu_{\gamma_3}(C_3)| > \varepsilon$. Continuing the process, we obtain a sequence $\{C_n\}$ of pairwise disjoint members in ζ and a sequence $\{\mu_{\gamma_n}\}$ such that $|\mu_{\gamma_n}(C_n)| > \varepsilon$ for any $n \in \mathbb{N}$. This contradicts (a).

We obtain the proof that (a) $\Rightarrow$ (d) if in the proof that (a) $\Rightarrow$ (b) we replace $U_n \in \tau$ by $E_n \in \eta$. ☐

Remark 5.2. We do not know whether the implication (b) $\Rightarrow$ (a) is true for a family $\{\mu_\gamma\}_{\gamma \in \Gamma} \subset STM$.

Now we can generalize Proposition 3.1 as follows.

Proposition 5.3. *Let the family* $\{\lambda_\gamma\}_{\gamma \in \Gamma} \subset SM$ *be uniformly s-bounded on* τ. *Let* $\nu \in PSTM$. *Let* $\Theta \in \tau$ *and* $\varepsilon > 0$. *Then there is a* $C \in \zeta$ *such that* $C \subset \Theta$, $\widetilde{\nu}(C) < \varepsilon$, *and* $\widetilde{\lambda}_\gamma(\Theta \setminus C) < \varepsilon$ *for all* $\gamma \in \Gamma$ *simultaneously.*

Proof. Let us repeat the proof of Proposition 3.1 by taking $C_i \in \zeta$ in such a way that $C_i \subset E_i$ and $\widetilde{\lambda}_\gamma(E_i \setminus C_i) < \varepsilon/n$ for all $\gamma \in \Gamma$, which is possible by Proposition 5.1. ☐

6. Main results.

Theorem 6.1. *Let* $\lambda_n \in SM$, $\nu_n \in PSTM$, $\mu_n = \lambda_n + \nu_n$, $n \in \mathbb{N}$. *Further, let* $\lim_n \mu_n(E) = 0$ *for any* $E \in \alpha$. *Then* $\lim_n \lambda_n(E) = 0$, *and consequently* $\lim_n \nu_n(E) = 0$ *for any* $E \in \alpha$.

Proof. Note that, by Proposition 4.6, the sequences $\{\lambda_n\}$ and $\{\nu_n\}$ are uniformly s-bounded on τ.

Take $\Theta \in \tau$. Suppose that the condition $\lim_n \lambda_n(\Theta) = 0$ fails to hold.

Put $\Theta_1 = \Theta$. Then one can find an $\varepsilon > 0$ and a $\delta_1 > 0$ such that for any n there is an $m > n$ for which $|\lambda_m(\Theta_1)| > \varepsilon + 3\delta_1$. Since $\lim_n \mu_n(\Theta_1) = 0$, we can find an n_1 such that $|\mu_{n_1}(\Theta_1)| < \delta_1$ and $|\lambda_{n_1}(\Theta_1)| > \varepsilon + 3\delta_1$. Therefore, $|\nu_{n_1}(\Theta_1)| > \varepsilon + 2\delta_1$.

By Proposition 5.3, there is a $C_1 \in \zeta$ with $C_1 \subset \Theta_1$, $\widetilde{\lambda}_n(\Theta_1 \setminus C_1) < \delta_1$ for all $n \in \mathbb{N}$ simultaneously and $\widetilde{\nu}_{n_1}(C_1) < \delta_1$.

Let us find $K_1 \in \zeta$ and $\Theta_2 \in \tau$ for which $C_1 \subset \Theta_2 \subset K_1 \subset \Theta_1$ and $|\nu_{n_1}(K_1)| < \delta_1$.

Put $U_1 = \Theta_1 \setminus K_1$. Obviously, $U_1 \in \tau$, $U_1 \cap \Theta_2 = \emptyset$, and

$$|\nu_{n_1}(U_1)| \geq |\nu_{n_1}(\Theta_1)| - |\nu_{n_1}(K_1)| > \varepsilon + \delta_1.$$

Note that $\Theta_1 = \Theta_2 \cup (\Theta_1 \setminus \Theta_2)$, where $\widetilde{\lambda}_n(\Theta_1 \setminus \Theta_2) < \delta_1$ for all $n \in \mathbb{N}$ simultaneously. Therefore, if $|\lambda_m(\Theta_1)| > \varepsilon + 3\delta_1$ for some m, then we necessarily have $|\lambda_m(\Theta_2)| > \varepsilon + 2\delta_1$.

Let $3\delta_2 = 2\delta_1$. We have $\Theta_2 \in \tau$, and for any n there is an $m > n$ for which $|\lambda_m(\Theta_2)| > \varepsilon + 3\delta_2$. Consider Θ_2 instead of Θ_1, and so on. Continuing the process, we obtain a sequence $\{U_k\}$ of pairwise disjoint members in τ and a sequence $\{n_k\}$ for which $|\nu_{n_k}(U_k)| > \varepsilon$, $k \in \mathbb{N}$. This contradicts the uniform s-boundedness of the sequence $\{\nu_n\}$ on τ. □

Theorem 6.2. *Let $\{\mu_n\} \subset STM$, and assume that for any $E \in \alpha$ there exists a finite $\lim_n \mu_n(E) = \mu(E)$. Let the sequence $\{\mu_n\}$ be uniformly bounded on α, i.e.,*

$$\sup\{|\mu_n(E)| : E \in \alpha,\ n \in \mathbb{N}\} < \infty. \tag{6.1}$$

Then $\mu \in STM$.

If we have here $\mu = \lambda + \nu$ and $\mu_n = \lambda_n + \nu_n$, where $\lambda, \lambda_n \in SM$ and $\nu, \nu_n \in PSTM$, then

$$\lim_n \lambda_n(E) = \lambda(E) \quad \text{and} \quad \lim_n \nu_n(E) = \nu(E).$$

Proof. It is clear that μ is finitely additive on α. By Proposition 4.6, the sequence $\{\mu_n\}$ is uniformly s-bounded on τ and hence, by Proposition 5.1, the statement (c) is true. We immediately obtain that μ is regular. The uniform boundedness of $\{\mu_n\}$ implies the boundedness of μ. Thus, $\mu \in STM$ by definition.

We have

$$\mu_n - \mu = (\lambda_n - \lambda) + (\nu_n - \nu),$$

where $\lambda_n - \lambda \in SM$ and $\nu_n - \nu \in PSTM$. Further, $\lim_n(\mu_n - \mu)(E) = 0$ for any $E \in \alpha$. By Theorem 6.1, $\lim_n(\lambda_n - \lambda)(E) = 0$ and $\lim_n(\nu_n - \nu)(E) = 0$ for any $E \in \alpha$. □

Theorem 6.3. *Let $\{\mu_n\} \subset PSTM$, and assume that for any $E \in \alpha$ there exists a finite $\lim_n \mu_n(E) = \mu(E)$. If the sequence $\{\mu_n\}$ is uniformly bounded on α, then $\mu \in PSTM$.*

Proof. By Theorem 6.2, $\mu \in STM$. Then there are $\lambda \in SM$ and $\nu \in PSTM$ such that $\mu = \lambda + \nu$. One can consider the decomposition $\mu_n = \lambda_n + \nu_n$, where $\lambda_n = 0 \in SM$ and $\nu_n = \mu_n \in PSTM$. By Theorem 6.2, $\lim_n \lambda_n(E) = \lambda(E)$ for any $E \in \alpha$. Thus, $\lambda = 0$ and $\mu = \nu$. □

Let us now consider a sequence $\{\mu_n\} \subset TM$. In this case, the setwise convergence obviously implies the uniform boundedness, and condition (6.1) should be omitted in the statements of Theorems 6.2 and 6.3. For example, the following assertion holds.

Theorem 6.4. *If $\{\mu_n\} \subset PTM$ and if for any $E \in \alpha$ there exists a finite $\lim_n \mu_n(E) = \mu(E)$, then $\mu \in PTM$.*

Obviously, Theorems 6.2 and 6.3 remain valid if condition (6.1) is replaced by the boundedness of μ. It is not known to the author whether or not μ is bounded in the general case if condition (6.1) is omitted. However, the following assertion holds.

Theorem 6.5. *Let $\{\mu_n\} \subset STM$, and assume that for any $E \in \alpha$ there exists a finite $\lim_n \mu_n(E) = \mu(E)$. Further, let $\mu_n = \lambda_n + \nu_n$, where $\lambda_n \in SM$ and $\nu_n \in PSTM$. Then for any $E \in \alpha$ there exists a finite $\lim_n \lambda_n(E) = \lambda(E)$ and hence the finite $\lim_n \nu_n(E) = \nu(E)$ exists as well. Here $\lambda \in SM$, and the functions μ and ν are finitely additive and regular on α.*

Proof. Let $A \in \alpha$. We claim that $\{\lambda_n(A)\}$ is a Cauchy sequence. Suppose the contrary. Then there are an $\varepsilon > 0$ and an increasing sequence $\{n_m\}$ with

$$|\lambda_{n_{2k-1}}(A) - \lambda_{n_{2k}}(A)| > \varepsilon, \quad k \in \mathbb{N}.$$

Put

$$\lambda'_k = \lambda_{n_{2k-1}} - \lambda_{n_{2k}}, \quad \nu'_k = \nu_{n_{2k-1}} - \nu_{n_{2k}}, \quad \mu'_k = \mu_{n_{2k-1}} - \mu_{n_{2k}}.$$

Obviously, $\lambda'_k \in SM$, $\nu'_k \in PSTM$, $\mu'_k = \lambda'_k + \nu'_k$, and $\lim_k \mu'_k(E) = 0$ for any $E \in \alpha$. In this case, $\lim_k \lambda'_k(A) = 0$ by Theorem 6.1. This contradicts the condition that $|\lambda'_k(A)| > \varepsilon$ for any $k \in \mathbb{N}$.

Extending all functions λ_n to η, we obtain a sequence of signed regular Borel measures with setwise convergence on τ. By the well-known Dieudonné theorem, for any $E \in \eta$ there exists a finite $\lim_n \lambda_n(E) = \lambda(E)$, and λ is a signed regular Borel measure (this also can be obtained by using Propositions 4.6 and 5.1).

It is clear that μ and ν are finitely additive, and their regularity can be proved in just the same way as the regularity of μ in Theorem 6.2. □

References

[1] J. F. Aarnes, Quasi-states and quasi-measures, Adv. Math. **86** (1991), 41–67.

[2] J. F. Aarnes, Construction of non-subadditive measures and discretization of Borel measures, Fund. Math. **147** (1995), 213–237.

[3] J. Diestel and J. J. Uhl, Jr., Vector measures, Amer. Math. Soc., Providence, RI, 1977.

[4] D. J. Grubb, Signed quasi-measures, Trans. Amer. Math. Soc. **349** (1997), 1081–1089.

[5] D. J. Grubb and T. LaBerge, Additivity of quasi-measures, Proc. Amer. Math. Soc. **126** (1998), 3007–3012.

[6] M. G. Svistula, Proper quasi-measure criterion, Math. Notes **81** (2007), 671–680.

[7] M. G. Svistula, A signed quasi-measure decomposition, Vestn. Samar. Gos. Univ. Estestvennonauchn. Ser. **62** (2008), 192–207.

Marina Svistula
Department of Functional Analysis and Theory of Functions,
Samara State University,
Ak. Pavlova st. 1,
443011 Samara,
Russia
e-mail: marinasvistula@mail.ru

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