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The average size of Ramanujan sums over quadratic number fields

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Abstract. This article is concerned with Ramanujan sums $c_{\mathcal{I}_1}(\mathcal{I})$, where $\mathcal{I}, \mathcal{I}_1$ are integral ideals in an arbitrary quadratic number field $\mathbb{Q}(\sqrt{d})$. In particular, the asymptotic behavior of sums of $c_{\mathcal{I}_1}(\mathcal{I})$, over both $\mathcal{I}$ and $\mathcal{I}_1$, is investigated.

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1. Introduction. Ramanujan sums over the rationals. For positive integers m, n, the classic Ramanujan sum $c_m(n)$ is defined as

$$c_m(n) = \sum_{\substack{1 \le j \le m \\ \gcd(j,m)=1}} e\left(j\,\frac{n}{m}\right) = \sum_{d \,|\, \gcd(m,n)} d\,\mu\left(\frac{m}{d}\right), \tag{1.1}$$

where $e(z) := e^{2\pi i z}$ and μ denotes the Möbius function. See, e.g., the textbook by Krätzel [8, p. 52 and p. 129]. Only quite recently, Chan and Kumchev dealt with the question of the average order of $c_m(n)$, with respect to both variables m, n. They proved in [2, Theorem 1.1] that, for large reals X and $Y \ge X$,

$$\begin{aligned} S_1(X,Y) &:= \sum_{1 \le m \le X} \sum_{1 \le n \le Y} c_m(n) \\ &= Y - \frac{3}{2\pi^2} X^2 + O(XY^{1/3} \log X) + O(X^3 Y^{-1}). \end{aligned} \tag{1.2}$$

Here slight refinements are possible in the first O-term by a more sophisticated application of the method of exponential sums. In fact, (1.2) has a surprising consequence: If $Y \asymp X^\lambda$ with some fixed λ, it readily follows that, as $Y \to \infty$,

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$$S_1(X,Y) \sim \begin{cases} Y & \text{if } \lambda > 2, \\ -\frac{3}{2\pi^2}\, X^2 & \text{if } 1 < \lambda < 2. \end{cases} \tag{1.3}$$

The change in the asymptotic behavior when $Y \asymp X^2$ has been worked out more precisely by Chan and Kumchev [2, Theorem 1.2] by evaluating the second moment

$$S_2(X,Y) := \sum_{1 \le n \le Y} \left(\sum_{1 \le m \le X} c_m(n) \right)^2 . \tag{1.4}$$

This requires a completely different and very deep method using complex integration. The asymptotics obtained is very interesting, but too involved to be cited here in detail.

2. Ramanujan sums in quadratic fields. We consider a quadratic field $\mathbb{F} = \mathbb{Q}(\sqrt{d})$, with $d \notin \{0,1\}$ a square-free integer, and its ring of algebraic integers $\mathcal{O}_{\mathbb{F}}$. For nonzero integral ideals $\mathcal{I}$ in $\mathcal{O}_{\mathbb{F}}$, the Möbius function is classically defined as follows (cf., e.g., Hecke [4, p. 100]): If there exists a prime ideal $\mathcal{P}$ in $\mathcal{O}_{\mathbb{F}}$ such that $\mathcal{P}^2$ divides $\mathcal{I}$, then $\mu(\mathcal{I}) = 0$. If $\mathcal{I}$ is the product of r distinct prime ideals, then $\mu(\mathcal{I}) = (-1)^r$.

Accordingly, for nonzero integral ideals $\mathcal{I}_1, \mathcal{I}$ in $\mathcal{O}_{\mathbb{F}}$, whose norm will be denoted by $N(\cdot)$ throughout, the Ramanujan sum is defined by

$$c_{\mathcal{I}_1}(\mathcal{I}) := \sum_{\mathcal{I}_2:\ \mathcal{I}_2 \,|\, \mathcal{I}_1,\, \mathcal{I}_2 \,|\, \mathcal{I}} N(\mathcal{I}_2)\, \mu\left(\frac{\mathcal{I}_1}{\mathcal{I}_2}\right), \tag{2.1}$$

motivated by the second representation in (1.1). In fact, on the basis of this definition, the concept of Ramanujan sums can be considered in the much more general context of arbitrary arithmetical semigroups: See, e.g., Grytczuk [3] and the monograph by Knopfmacher [7, p. 185]. Returning to $\mathbb{F} = \mathbb{Q}(\sqrt{d})$, the aim of this paper will be to seek for an asymptotics for the sum

$$S_{\mathbb{F}}(X,Y) := \sum_{1 \le N(\mathcal{I}_1) \le X} \ \sum_{1 \le N(\mathcal{I}) \le Y} c_{\mathcal{I}_1}(\mathcal{I}), \tag{2.2}$$

in particular, to determine a sufficient condition on the relative size of X and Y (as less stringent as possible) which ensures that the exact asymptotic behavior of $S_{\mathbb{F}}(X,Y)$ can be determined. By an obvious device,

$$\begin{aligned} S_{\mathbb{F}}(X,Y) &= \sum_{1 \le N(\mathcal{I}) \le Y} \left(\sum_{\mathcal{I}_1, \mathcal{I}_2:\ 1 \le N(\mathcal{I}_1 \mathcal{I}_2) \le X,\ \mathcal{I}_1 \,|\, \mathcal{I}} N(\mathcal{I}_1)\mu(\mathcal{I}_2) \right) \\ &= \sum_{\mathcal{I}_1, \mathcal{I}_2:\ 1 \le N(\mathcal{I}_1 \mathcal{I}_2) \le X} N(\mathcal{I}_1)\mu(\mathcal{I}_2) \left(\rho\, \frac{Y}{N(\mathcal{I}_1)} + P_{\mathbb{F}}\left(\frac{Y}{N(\mathcal{I}_1)} \right) \right) \\ &= \rho Y + \sum_{\mathcal{I}_1, \mathcal{I}_2:\ 1 \le N(\mathcal{I}_1 \mathcal{I}_2) \le X} N(\mathcal{I}_1)\mu(\mathcal{I}_2)\, P_{\mathbb{F}}\left(\frac{Y}{N(\mathcal{I}_1)} \right). \end{aligned} \tag{2.3}$$

Here the well known fact[1] has been used that the number of nonzero integral ideals in $\mathcal{O}_{\mathbb{F}}$ of norm $\leq$ a large real parameter t, is approximated in first order by $\rho_d\, t$, with a remainder term $P_{\mathbb{F}}(t)$, where ρ_d is the residue of the Dedekind zeta-function $\zeta_{\mathbb{F}}(s)$ at $s = 1$. Explicitly,

$$\rho_d = \begin{cases} \frac{h(D)}{w_D}\frac{2\pi}{\sqrt{|D|}} & \text{if } d < 0, \\ \frac{h(D)}{\sqrt{D}}\log(\epsilon_0) & \text{if } d > 0. \end{cases}$$

$h(D)$ denotes the class number (in the narrow sense), the discriminant D equals d if $d \equiv 1 \mod 4$, and $D = 4d$ for $d \not\equiv 1 \mod 4, \epsilon_0$ is a fundamental unit, and w_D is the number of roots of unity in $\mathcal{O}_{\mathbb{F}}$, i.e.,

$$w_D = \begin{cases} 6 & \text{for } D = -3, \\ 4 & \text{for } D = -4, \\ 2 & \text{for } D < -4. \end{cases}$$

Let $\mathcal{R} = \mathcal{R}_{\mathbb{F}}(X, Y)$ denote the last sum in (2.3), then readily

$$\mathcal{R} \ll \sum_{1 \leq N(\mathcal{I}_2) \leq X} \left| \sum_{1 \leq N(\mathcal{I}_1) \leq X/N(\mathcal{I}_2)} N(\mathcal{I}_1)\, P_{\mathbb{F}}\left(\frac{Y}{N(\mathcal{I}_1)}\right) \right|. \tag{2.4}$$

In fact, $P_{\mathbb{F}}(t)$ can be estimated according to the classic examples of the circle and divisor problems: See Huxley [5] whose nowadays sharpest bound readily gives $P_{\mathbb{F}}(t) \ll t^{131/416+\epsilon}$. Using this in (2.4), we immediately infer that

$$\mathcal{R} \ll X^{701/416}\, Y^{131/416+\epsilon}. \tag{2.5}$$

This implies that, as $Y \to \infty$,

$$S_{\mathbb{F}}(X, Y) \sim \rho Y, \tag{2.6}$$

provided that $Y \gg X^{\lambda}$ for some

$$\lambda > \frac{701}{285} = 2.4596\ldots.$$

In a previous paper [13], the author has established the asymptotics (2.6) in the slightly wider range

$$\lambda > \frac{29}{12} = 2.41\dot{6},$$

for the special case of the Gaussian field $\mathbb{F} = \mathbb{Q}(i)$. The objective of the present paper is to prove a stronger result, for any quadratic field $\mathbb{F}$.

Theorem 1. *For each fixed quadratic field $\mathbb{F} = \mathbb{Q}(\sqrt{d})$, the asymptotic formula (2.6) is true provided that $Y \gg X^{\lambda}$ for some*

$$\lambda > \frac{1973}{820} = 2.40609\ldots.$$

[1] For this and other basics in the theory of quadratic fields, the reader is referred to the monograph by Zagier [15].

More precisely, for $Y > X$ and arbitrary fixed $\epsilon > 0$,

$$S_{\mathbb{F}}(X,Y) = \rho Y + O(X^{\frac{1973}{1358}} Y^{\frac{269}{679}+\epsilon}) + O(X^{\frac{1234823}{737394}} Y^{\frac{205}{679}+\epsilon}) \\ +O(X^{\frac{23917}{21728}} Y^{\frac{8675}{16296}+\epsilon}) + O(X^2 Y^{\epsilon}).$$

Remark. Of course, the result obtained is not completely analogous to (1.3): it fails to contain a second case, corresponding to smaller values of λ. To understand the reason, observe that the deduction of (1.2) ultimately rests on the approximation of the Gauss bracket $[t]$ in the form

$$[t] = t - \frac{1}{2} - \psi(t), \text{ where } \int\limits_I \psi(t)\,\mathrm{d}t = 0 \tag{2.7}$$

for every interval I of unit length. In the case of a quadratic field $\mathbb{F} = \mathbb{Q}(\sqrt{d})$, a weak analogue is known as

$$\sum_{0<N(\mathcal{I})\le t} 1 = \rho_d\, t + \zeta_{\mathbb{F}}(0) + P^*_{\mathbb{F}}(t), \quad \text{where} \quad \int\limits_0^T P^*_{\mathbb{F}}(t)\mathrm{d}t \ll T^{3/4} \tag{2.8}$$

(see Chandrasekharan and Narasimhan [1]). However, $\zeta_{\mathbb{F}}(0) \ne 0$ only if $d < 0$, and even in this case there is little hope to estimate the contribution coming from $P^*_{\mathbb{F}}(t)$ so well that $\zeta_{\mathbb{F}}(0)$ can give a second main term.

3. Preliminaries and classical auxiliary results. It will be important to have at our disposal a tight approximation to the remainder term

$$P_{\mathbb{F}}(t) := \#\{\text{integral ideals } \mathcal{I} \text{ in } \mathcal{O}_{\mathbb{F}}:\ 0 < N(\mathcal{I}) \le t\ \} - \rho_d\, t.$$

Lemma 1. *For any fixed quadratic field $\mathbb{F} = \mathbb{Q}(\sqrt{d})$, large real parameters t, y, and any $\epsilon > 0$, it follows that*

$$P_{\mathbb{F}}(t) = C_D\, t^{1/4} \sum_{0<N(\mathcal{I})\le y} N(\mathcal{I})^{-3/4} \sin\left(\frac{4\pi}{\sqrt{|D|}}\sqrt{t}\sqrt{N(\mathcal{I})} + \frac{\pi}{4}\mathrm{sgn}(d)\right) \\ +O(t^{1/2+\epsilon}\, y^{-1/2}) + O(y^{\epsilon}).$$

Proof. For $\zeta_{\mathbb{F}}(s)$ replaced by $\zeta^2(s)$, an analogue is quite classic and can be found, e.g., in the book of Titchmarsh [14, p. 319], in the context of the Dirichlet divisor problem. The special case of the assertion for $\mathbb{F} = \mathbb{Q}(i)$ has been stated and applied by several authors to deal with the lattice discrepancy of the circular disc. An explicit proof of the present statement is contained in a still more general result by Müller [11, Lemma 3]. By the way, $C_D = \pm|D|^{1/4}(\pi\sqrt{2})^{-1}$. □

Lemma 2. *There exists a one-one correspondence which maps to every ideal class $\mathfrak{C}_j$ in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ a primitive binary quadratic form $f_j(u,v) = a_j u^2 + b_j uv + c_j v^2$ of discriminant D (positive definite if $d < 0$), such that, for every positive integer n,*

$$\#\{I \in \mathfrak{C}_j:\ N(\mathcal{I}) = n\} = \eta_d\, \#(\beta_j(n)),$$

where the following definitions hold:

$$\eta_d := \begin{cases} \frac{1}{w_D} & \text{if } d < 0, \\ 1 & \text{if } d > 0, \end{cases} \qquad \beta_j^*(n) := \{(u,v) \in \mathbb{Z}^2 : \ f_j(u,v) = n\},$$

$$\beta_j(n) := \begin{cases} \beta_j^*(n) \quad \text{if} \ \ d<0, \\ \{(u,v) \in \beta_j^*(n) : \quad u - \theta v > 0, \ 1 < \frac{u-\theta' v}{u-\theta v} \le \epsilon_0^2\} \ \ \text{if} \ \ d > 0, \end{cases}$$

$$\theta := \frac{-b_j + \sqrt{D}}{2a_j}, \qquad \theta' := \frac{-b_j - \sqrt{D}}{2a_j}.$$

Proof. See Zagier [15, pp. 98–99 and 69–71]. □

Combining the results of both of these lemmas and defining, for large real y,

$$\mathcal{B}_j(y) := \bigcup_{0<n\le y} \beta_j(n), \tag{3.1}$$

we obtain the following useful formula:

$$P_{\mathbb{F}}(t) = C_D \eta_d t^{1/4} \sum_{j=1}^{h(D)} \sum_{(u,v)\in\mathcal{B}_j(y)} f_j(u,v)^{-3/4} \sin\left(\frac{4\pi}{\sqrt{|D|}} \sqrt{t f_j(u,v)} + \frac{\pi}{4}\operatorname{sgn}(d) \right)$$
$$+ O(t^{1/2+\epsilon}\, y^{-1/2}) + O(y^{\epsilon}). \tag{3.2}$$

This very last result will make it necessary to have at hand a sharp bound for exponential sums.

Lemma 3. *For positive real parameters $M \ge 1$ and T, suppose that F is a real function on some compact interval I^* of length M, with at least 5 continuous derivatives satisfying throughout*

$$F^{(j)} \asymp \frac{T}{M^j} \quad \text{for} \quad j = 2,\ 3,\ 4. \tag{3.3}$$

Then, with $e(z) := e^{2\pi i z}$, for every interval $I \subseteq I^*$, and each $\epsilon > 0$,

$$\sum_{m\in I} e(F(m)) \ll M^{1/2} T^{32/205+\epsilon} + T^{751/1968+\epsilon} + M^{871/1086} + M T^{-1/2}.$$

Proof. The deep part of this result is due to Huxley [6, Prop. 1, formulae (1.10)–(1.13), and Theorem 1]. Using classic Van der Corput bounds on the ranges where Huxley's conditions are not satisfied, the author established the present statement as a special case of [12, Theorem 1]. □

The next result will help to satisfy the conditions (3.3) in a fairly general situation.

Lemma 4. *For f a real binary quadratic form, let $M_f^+ := \{\mathbf{u} \in \mathbb{R}^2 : \ f(\mathbf{u}) > 0\}$, and $G := f^{-1/2}$ on M_f^+. For $r \in \mathbb{Z}_+$, $\mathbf{a} = (a_1, a_2) \in M_f^+$, and $(0,0) \ne \mathbf{v} \in \mathbb{R}^2$, denote by $G^{(r)}(\mathbf{a};\mathbf{v})$ the directional derivative of order r.*[2] *Then,*

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[2] Explicitly, $G^{(r)}(\mathbf{a};\mathbf{v}) = \frac{\mathrm{d}^r G}{\mathrm{d}\mathbf{v}^r}(\mathbf{a}) = \frac{\mathrm{d}^r}{\mathrm{d}t^r}(G(\mathbf{a}+t\mathbf{v}))\Big|_{t=0}$.

(i) *for each integer* $r \geq 0$, *there exists no point* $\mathbf{a} \in M_f^+$ *so that all partial derivatives of* G *of order* r *vanish at* $\mathbf{a}$.
(ii) *For every set of positive integers* $r_1, \ldots, r_K$ *and every* $\mathbf{a} \in M_f^+$, *there exists a vector* $(0,0) \neq \mathbf{v} \in \mathbb{R}^2$ *for which*

$$\min_{r=r_1,\ldots,r_K} \left| G^{(r)}(\mathbf{a};\mathbf{v}) \right| > 0.$$

Proof. The conclusion from (i) to (ii) is easy, in view of the identity

$$G^{(r)}(\mathbf{a};\mathbf{v}) = \sum_{k=0}^{r} \frac{r!}{k!(r-k)!} \frac{\partial^r G}{\partial u_1^k \partial u_2^{r-k}}(\mathbf{a})\, v_1^k v_2^{r-k}. \tag{3.4}$$

By (i), for every fixed $r \in \mathbb{Z}_+$ and $\mathbf{a} \in M_f^+$, the set of vectors $\mathbf{v}$ for which the right hand side of (3.4) vanishes is a set of measure zero in $\mathbb{R}^2$, and so is the union over $r = r_1, \ldots, r_K$.

To establish (i), we proceed by induction and assume that r is minimal, so that there exists some $\mathbf{a} \in M_f^+$ for which all partial derivatives of G of order $r+1$ vanish at $\mathbf{a}$. By homogeneity, they vanish at any point $\xi\mathbf{a}$, with $\xi > 0$ arbitrary. Denote by g any partial derivative of order r of G. By an appropriate version of (3.4), $\frac{\mathrm{d}}{\mathrm{d}\xi} g(\xi\mathbf{a}) = g'(\xi\mathbf{a};\mathbf{a}) = 0$ identically in $\xi > 0$, hence $g(\xi\mathbf{a})$ is a constant for $\xi > 0$. But g is homogeneous of degree $-1-r$, hence $g(\mathbf{a}) = 0$. Thus *all* partial derivatives of G of order r vanish at $\mathbf{a}$, which contradicts the minimality of r. □

4. Proof of the Theorem. We start from (2.4) and split up the inner sum by a dyadic sequence: Keeping $\mathcal{I}_2$ fixed for the moment, let W run through $(2^{-r}X/N(\mathcal{I}_2))$, where r ranges from 1 to $[\log(X/N(\mathcal{I}_2))/\log 2]$. Then, by Lemma 1, with a certain parameter $U > 0$ at our disposal,

$$\begin{aligned} &\sum_{W<N\leq 2W} N(\mathcal{I}_1)\, P_{\mathbb{F}}(Y/N(\mathcal{I}_1)) \\ &\ll \left| Y^{1/4} \sum_{W<N\leq 2W} N(\mathcal{I}_1)^{3/4} \sum_{1\leq N(\mathcal{I}_3)\leq U} N(\mathcal{I}_3)^{-3/4}\, e\left(\frac{2}{\sqrt{|D|}} \sqrt{Y N(\mathcal{I}_3)/N(\mathcal{I}_1)} \right) \right| \\ &\quad + Y^{1/2+\epsilon} U^{-1/2} \sum_{W<N\leq 2W} N(\mathcal{I}_1)^{1/2} + U^{\epsilon}\, W^2. \end{aligned} \tag{4.1}$$

Interchanging the order of summation, this is (cf. also 4.9 below)

$$\ll Y^{1/4} \sum_{1\leq N(\mathcal{I}_3)\leq U} N(\mathcal{I}_3)^{-3/4}\, |E(W,\mathcal{I}_3)| \; + \; Y^{1/2+\epsilon} U^{-1/2} W^{3/2} + U^{\epsilon} W^2, \tag{4.2}$$

where

$$E(W,\mathcal{I}_3) := \sum_{W<N\leq 2W} N(\mathcal{I}_1)^{3/4} e\left(Z/\sqrt{N(\mathcal{I}_1)} \right), \qquad Z := \frac{2}{\sqrt{|D|}} \sqrt{Y N(\mathcal{I}_3)}. \tag{4.3}$$

Clearly it suffices to restrict this sum to $\mathcal{I}_1$ from a single ideal class $\mathfrak{C}_j$: call this subsum $E_j(W,\mathcal{I}_3)$. Using Lemma 2 and (3.1), we may write this as

$$E_j(W,\mathcal{I}_3) = \eta_d \sum_{(u,v)\in\mathcal{B}_j(2W)\setminus\mathcal{B}_j(W)} f_j(u,v)^{3/4} e\left(Z/\sqrt{f_j(u,v)}\right). \tag{4.4}$$

To deal with this exponential sum, we define[3]

$$\mathcal{D}_0 := \Bigg\{(\xi_1,\xi_2)\in\mathbb{R}^2 :\ 1 < f_j(\xi_1,\xi_2) \le 2, \\ \text{and}\ \ \xi_1-\theta\xi_2 > 0,\ 1 < \frac{\xi_1-\theta'\xi_2}{\xi_1-\theta\xi_2} \le \epsilon_0^2\ \text{ if }\ d>0\Bigg\}, \tag{4.5}$$

with θ,θ' as in Lemma 2. By homogeneity,

$$\mathcal{B}_j(2W)\setminus\mathcal{B}_j(W) = \mathbb{Z}^2 \cap (\sqrt{W}\,\mathcal{D}_0). \tag{4.6}$$

We now apply Lemma 4, along with an obvious continuity argument:[4] For every point $(\xi_1,\xi_2)\in\overline{\mathcal{D}_0}$ (the bar denoting the topological closure), there exist a *rational* vector $\mathbf{v}^* = \mathbf{v}^*(\xi_1,\xi_2) \ne (0,0)$ and a positive number $\delta(\xi_1,\xi_2)$ such that (with $G = f_j^{-1/2}$ as in Lemma 4)

$$\min_{r=2,3,4} \left| G^{(r)}((\xi_1',\xi_2');\mathbf{v}^*(\xi_1,\xi_2))\right| > 0$$

for all $(\xi_1',\xi_2')\in\mathbb{R}^2$ with $\|(\xi_1',\xi_2')-(\xi_1,\xi_2)\|_2 < 2\delta(\xi_1,\xi_2)$. Let

$$\mathcal{D}^*(\xi_1,\xi_2) := \{(\xi_1',\xi_2')\in\mathbb{R}^2 :\ \|(\xi_1',\xi_2')-(\xi_1,\xi_2)\|_2 < \delta(\xi_1,\xi_2)\},$$

then $\overline{\mathcal{D}_0}$ is covered by the union of all open discs $\mathcal{D}^*(\xi_1,\xi_2)$ with $(\xi_1,\xi_2)\in\overline{\mathcal{D}_0}$, and, by compactness, also by finitely many of them, say

$$\overline{\mathcal{D}_0} \subseteq \bigcup_{k=1}^{K} \mathcal{D}^*(\xi_1^{(k)},\xi_2^{(k)}).$$

We thus define sets $\mathcal{D}_k \subseteq \mathcal{D}^*(\xi_1^{(k)},\xi_2^{(k)})$, $k=1,\dots,K$, by the recursion formula

$$\mathcal{D}_k := \mathcal{D}_0 \cap \left(\mathcal{D}^*(\xi_1^{(k)},\xi_2^{(k)}) \setminus \bigcup_{1\le\kappa<k} \mathcal{D}^*(\xi_1^{(\kappa)},\xi_2^{(\kappa)})\right).$$

These sets are pairwise disjoint, their union is $\mathcal{D}_0$, and any straight line intersects every $\mathcal{D}_k$ in at most $O(1)$ line segments. Further, for each $k=1,\dots,K$, we find a vector $\mathbf{v}^{(k)}\in\mathbb{Z}^2$, with coprime coordinates, which is collinear to $\mathbf{v}^*(\xi_1^{(k)},\xi_2^{(k)})$. By construction, it is immediate that

$$\min_{k=1,\dots,K}\ \min_{r=2,3,4}\ \inf_{(\xi_1,\xi_2)\in\mathcal{D}_k} \left|G^{(r)}((\xi_1,\xi_2);\mathbf{v}^{(k)})\right| > 0. \tag{4.7}$$

[3] To visualize the domain $\mathcal{D}_0$ for $d>0$, the figure in Zagier [15, p. 71] is helpful.

[4] This part of the proof uses ideas which can be traced back to the author's earlier work, jointly with Krätzel [10].

Moreover, for each $\mathbf{v}^{(k)}$ there exists a vector $\mathbf{v}^{(k,2)} \in \mathbb{Z}^2$ so that $\{\mathbf{v}^{(k)}, \mathbf{v}^{(k,2)}\}$ forms a basis of $\mathbb{Z}^2$. Hence every lattice point $(u,v) \in \mathbb{Z}^2 \cap (\sqrt{W}\,\mathcal{D}_k)$ has a unique representation

$$(u,v) = m_1\mathbf{v}^{(k)} + m_2\mathbf{v}^{(k,2)}, \qquad (m_1,m_2) \in \mathbb{Z}^2,$$

and it is clear that $m_1, m_2 \ll \sqrt{W}$. We keep m_2 fixed and denote by $\mathcal{U}$ any of the $O(1)$ intervals which form the range for m_1, such that

$$m_1\mathbf{v}^{(k)} + m_2\mathbf{v}^{(k,2)} \in \mathbb{Z}^2 \cap (\sqrt{W}\,\mathcal{D}_k).$$

With (4.4) and Lemma 3 in the back of mind, we write

$$F(t) := Z\,G(t\mathbf{v}^{(k)} + m_2\mathbf{v}^{(k,2)}),$$

then it is immediate by (4.7) and the homogeneity of G and its partial derivatives that

$$F^{(r)}(t) \asymp Z\,W^{-(r+1)/2}, \quad r = 2,3,4,$$

as long as $t\mathbf{v}^{(k)} + m_2\mathbf{v}^{(k,2)} \in \sqrt{W}\,\mathcal{D}_k$. For every subinterval $\mathcal{U}'$ of $\mathcal{U}$ we may thus apply Lemma 3, with $M \asymp \sqrt{W}, T \asymp Z/\sqrt{W}$, to deduce that[5]

$$\sum_{m_1 \in \mathcal{U}'} e(F(m_1)) \ll_{[\epsilon]} W^{141/820}Z^{32/205} + \frac{Z^{751/1968}}{W^{751/3936}} + W^{871/2172} + \frac{W^{3/4}}{\sqrt{Z}}.$$

In view of (4.4), we have to take into account the factors

$$f_j(u,v)^{3/4} = f_j(m_1\mathbf{v}^{(k)} + m_2\mathbf{v}^{(k,2)})^{3/4}.$$

Since the first partial derivatives of $f_j^{3/4}$ are homogeneous of order $\frac{1}{2}$, it follows that

$$f_j(\xi\mathbf{v}^{(k)} + m_2\mathbf{v}^{(k,2)})^{3/4} \ll W^{3/4}, \quad \frac{\mathrm{d}}{\mathrm{d}\xi}(f_j(\xi\mathbf{v}^{(k)} + m_2\mathbf{v}^{(k,2)})^{3/4}) \ll W^{1/4},$$

uniformly in $\xi \in \mathcal{U}$. Therefore, summation by parts gives

$$\sum_{m_1 \in \mathcal{U}} f_j(m_1\mathbf{v}^{(k)} + m_2\mathbf{v}^{(k,2)})^{3/4} e(F(m_1))$$
$$\ll_{[\epsilon]} W^{189/205}Z^{32/205} + W^{2201/3936}Z^{751/1968} + W^{625/543} + \frac{W^{3/2}}{\sqrt{Z}}.$$

Trivial summation over m_2 gives a factor $\sqrt{W}$, while summing over all domains $\mathcal{D}_k$ and all ideal classes $\mathfrak{C}_j$ just multiplies the bound by $O(1)$. Hence, returning to (4.3), we arrive at

$$E(W,\mathcal{I}_3) \ll_{[\epsilon]} W^{583/410}Z^{32/205} + W^{4169/3936}Z^{751/1968} + W^{1793/1086} + \frac{W^2}{\sqrt{Z}}. \tag{4.8}$$

[5] In what follows, we will use the notation $A_1 \ll_{[\epsilon]} A_2$ to mean that certain expressions A_1 and $A_2 > 0$, which may be ultimately bounded in terms of the large parameter Y, satisfy $A_1 \ll A_2 Y^{\epsilon}$, for every $\epsilon > 0$.

We recall the well-known fact that, for U large,

$$\sum_{1\le N(\mathcal{I})\le U} N(\mathcal{I})^{\tau} \ll \begin{cases} U^{\tau+1} & \text{for each fixed } \tau > -1, \\ \log U & \text{for } \tau = -1, \\ 1 & \text{for each fixed } \tau < -1. \end{cases} \tag{4.9}$$

Using this along with (4.8) and the fact that $Z \asymp \sqrt{YN(\mathcal{I}_3)}$ in (4.2), we compute that

$$\sum_{W<N\le 2W} N(\mathcal{I}_1)\, P_{\mathbb{F}}(Y/N(\mathcal{I}_1))$$
$$\ll_{[\epsilon]} W^2 + (UY)^{1/4}W^{1793/1086} + (UY)^{269/820}W^{583/410} + (UY)^{1735/3936}W^{4169/3936} + U^{-1/2}W^{3/2}Y^{1/2}.$$

To optimize the estimate, we balance the third term at the right hand side against the last one, to get $U \asymp W^{64/679}Y^{141/679}$, and thus

$$\sum_{W<N\le 2W} N(\mathcal{I}_1)\, P_{\mathbb{F}}(Y/N(\mathcal{I}_1))$$
$$\ll_{[\epsilon]} W^{\frac{1973}{1358}}Y^{\frac{269}{679}} + W^{\frac{1234823}{737394}}Y^{\frac{205}{679}} + W^{\frac{23917}{21728}}Y^{\frac{8675}{16296}} + W^2.$$

Summation over $W = 2^{-r}X/N(\mathcal{I}_2), r = 1, \ldots, [\log(X/N(\mathcal{I}_2))/\log 2]$, replaces just W by $X/N(\mathcal{I}_2)$ at the right hand side. Finally, going back to (2.4) and summing over $\mathcal{I}_2$, we arrive at

$$\mathcal{R} \ll_{[\epsilon]} X^{\frac{1973}{1358}}Y^{\frac{269}{679}} + X^{\frac{1234823}{737394}}Y^{\frac{205}{679}} + X^{\frac{23917}{21728}}Y^{\frac{8675}{16296}} + X^2.$$

In view of (2.3), this completes the proof of our Theorem.

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