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On the uniqueness of (p, h)-gonal automorphisms of Riemann surfaces

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Abstract. Let X be a compact Riemann surface of genus $g \geq 2$. A cyclic subgroup of prime order p of $Aut(X)$ is called properly (p, h)-gonal if it has a fixed point and the quotient surface has genus h. We show that if $p > 6h + 6$, then a properly (p, h)-gonal subgroup of $Aut(X)$ is unique. We also discuss some related results.

1. Introduction. Throughout this paper X will be a compact Riemann surface of genus $g \geq 2$, and $A = Aut(X)$ will denote the full group of (conformal) automorphisms of X. It is a classical result that then A is a finite group whose order is bounded by $84(g - 1)$.

Everywhere in the paper the letters p and q will, without exception, denote prime numbers.

Definition 1.1. A subgroup H of $Aut(X)$ is called (p, h)**-gonal** if H is cyclic of order p and the quotient surface X/H has genus h. An automorphism σ of X is called (p, h)-gonal if $\langle \sigma \rangle$ (the cyclic group generated by σ) is (p, h)-gonal.

The Riemann surface X is called **cyclic** (p, h)**-gonal** if it has a (p, h)-gonal automorphism.

A (p, h)-gonal automorphism of $Aut(X)$ is called **properly** (p, h)**-gonal** if it has at least one fixed point on X. This is automatic for $h \leq 1$.

The terminology generalizes the classical notion of p-gonal, which means $(p, 0)$-gonal. Sometimes the word elliptic-p-gonal is used for $(p, 1)$-gonal.

By the classical Castelnuovo inequality (see Theorem 2.1 below) a (p, h)-gonal subgroup is unique if $g > 2ph + (p - 1)^2$. Controlling the (p, h)-gonal automorphisms when the genus is smaller is a subject of recent interest. We concentrate on elliptic-p-gonal subgroups and omit the case $p = 3$ that requires more case distinctions.

Theorem 1.2. *Fix $p > 3$. Then*

(a) *All $(p, 1)$-gonal subgroups are conjugate in $Aut(X)$.*
(b) *The number of $(p, 1)$-gonal subgroups in $Aut(X)$ is bounded by $6\frac{p-1}{p-6}$ if $p \geq 7$, and by 16 if $p = 5$.*

Proof. (a) [11, Theorem 4.2] or as a special case of [12, Theorem 4.5].
(b) [11, Theorem 5.1] or as a special case of [12, Theorem 5.2].

□

Concerning the (earlier) analogous results on $(p, 0)$-gonal subgroups, see [8, Theorem 1], [9, Theorem 2.1], and [13, Theorem 1] for successively simpler proofs of the conjugacy. Their number is bounded in [10, Theorem 3.1 and Corollary 3.2].

However, it seems to have escaped notice that one can actually obtain a much stronger result than Theorem 1.2, namely uniqueness of the (p, h)-gonal subgroup, provided p is sufficiently big compared to h. See Theorem 2.5 below for elliptic-p-gonal automorphisms, Theorem 4.2 for proper (p, h)-gonal automorphisms, and Theorems 4.3 and 4.4 for even more general results.

Our proofs are very short and use only elementary tools. But this is mainly due to the fact that they heavily rely on Theorem 1.2, respectively its generalization in [12], where the main work has been done.

2. On $(p, 1)$-gonal automorphisms. First, as promised, a special case of the Castelnuovo inequality. See [1, Theorem 3.5] for the general version.

Theorem 2.1. *Let C_1 and C_2 be distinct cyclic subgroups of A of (not necessarily distinct) prime orders p_1 and p_2. If g_i denotes the genus of X/C_i, then*

$$g \leq p_1 g_1 + p_2 g_2 + (p_1 - 1)(p_2 - 1).$$

We recall two more fundamental facts, which we will use frequently.

Theorem 2.2. (a) *If $x \in X$, then its stabilizer $A_x := \{\sigma \in A : \sigma(x) = x\}$ is a cyclic group.*
(b) *An automorphism $\sigma \in A$ of prime order cannot have exactly one fixed point on X.*

Proof. (a) [7, Corollary III.7.7, p. 100].
(b) [7, Theorem V.2.11, p. 266].

□

The following is the key lemma for most of the paper.

Lemma 2.3. *Let H be a subgroup of A such that X/H has genus 1. Let σ be an automorphism of X that has a fixed point on X. Assume that $H \cap \langle \sigma \rangle = \{id\}$ and that σ normalizes H. Then the order of σ is 1, 2, 3, 4 or 6.*

Proof. Under those conditions σ induces an automorphism $\widetilde{\sigma}$ of the same order on X/H. Obviously, $\widetilde{\sigma}$ inherits the fixed point from σ. It is well known that an automorphism of a torus that fixes a point can only have one of the listed orders. $\qquad\square$

We state the next result in more generality than we need, as it might also be useful when investigating certain p-Sylow subgroups of A.

Proposition 2.4. *Let $\sigma \in A$ be a $(p, 1)$-gonal automorphism with $p > 3$, and let $C \subseteq A$ be a cyclic group of order p^e with $\sigma \in C$. Then the number of $(p, 1)$-gonal subgroups of A is congruent to 1 modulo p^e.*

In particular, for $p > 3$ the number of $(p, 1)$-gonal subgroups of A (if there are any) is congruent to 1 modulo p.

Proof. Consider the action of C by conjugation on the set of all $(p, 1)$-gonal subgroups of A. Obviously, $\langle \sigma \rangle$ is fixed. We claim that all other orbits have length p^e. If not, then $\langle \sigma \rangle$ normalizes another $(p, 1)$-gonal subgroup H. But then Lemma 2.3 contradicts the condition that σ has order $p > 3$. $\qquad\square$

Now we are ready to state the first main result of this paper.

Theorem 2.5. *Let X be cyclic $(p, 1)$-gonal.*
 (a) *If $p > 11$, then the $(p, 1)$-gonal subgroup is unique (and hence normal) in $Aut(X)$.*
 (b) *For $p = 11$ the possible numbers of $(p, 1)$-gonal subgroups are 1 and 12; for $p = 7$ they are 1, 8, 15, 22, 29 and 36; and for $p = 5$ they are 1, 6, 11 or 16.*

Proof. This follows from combining Theorem 1.2 and Proposition 2.4. $\qquad\square$

What happens if we allow different primes at the same time?

Proposition 2.6. *Suppose that $Aut(X)$ has $(p, 1)$-gonal and $(q, 1)$-gonal automorphisms for primes $p < q$. Then $p \leq 3$, $q \leq 7$ and $g \leq 10$.*

Proof. First let's assume $p > 3$. Then the $(q, 1)$-gonal subgroup cannot be unique, as this would contradict Lemma 2.3. Thus $q \leq 11$ by Theorem 2.5. Moreover, again by Lemma 2.3, the number of $(q, 1)$-gonal subgroups must be divisible by p, and the number of $(p, 1)$-gonal subgroups must be divisible by q. By Theorem 2.5 this excludes the remaining possibilities.

So we have shown $p \leq 3$. If $p = 2$ and $q = 3$, the Castelnuovo inequality shows $g \leq 7$. If $q \geq 5$, once again by Lemma 2.3, the $(p, 1)$-gonal subgroup cannot be unique, and hence the Castelnuovo inequality implies $g \leq 10$. This in term implies $q \leq 7$ by the Hurwitz formula. $\qquad\square$

One of the results in [5], namely Theorem 7, says that if a genus 3 surface is cyclic $(3, 0)$-gonal, then the $(3, 0)$-gonal subgroup is unique. The following proposition might be considered as a generalization to other primes.

Proposition 2.7. *If the genus of X is a prime $p > 7$ and $Aut(X)$ has a subgroup of order p, then this subgroup is unique.*

Proof. Let P be such a subgroup. From the Hurwitz formula it is clear that P has exactly 2 fixed points and is $(p, 1)$-gonal. So for $p > 11$ everything is already proved by Theorem 2.5.

Now let $p = 11$. If P is not unique, then by Theorem 2.5 there are exactly 12 such subgroups. Since by Theorem 1.2 they are all conjugate, we have $|A : N_A(P)| = 12$ where $N_A(P)$ denotes the normalizer of P in A. So $11 \times 12 = 132$ divides the order of A. Actually, 132 is the order by the bound $\#A \leq 240$ for $g = 11$ from [3, Table 13, p. 91].

Thus $N_A(P) = P$. On the other hand it is known (see [4, p. 575] or [6, Corollary 3.2]) that in such a situation $N_A(P)$ contains a dihedral group D_{11} of order 22, which finishes the proof by contradiction.

Alternatively, by a simple group theoretic argument we can avoid using the last fact. Counting shows that A has exactly 12 elements whose orders are different from 11. So if the 3-Sylow subgroup is not normal, then the 2-Sylow subgroup must be normal. Together they generate a subgroup B of order 12, which for lack of other elements is normal in A. But B cannot contain 11 elements of the same order. So the action of P on B by conjugation is trivial, which implies that P is normal in A. $\square$

Remark 2.8. The uniqueness in Theorem 2.5 and Proposition 2.7 does not hold for $p = 7$, as there exist Riemann surfaces of genus 7 whose automorphism group is the simple group $PSL_2(\mathbb{F}_8)$ (of order 504).

3. Interaction with $(p, 0)$-gonal automorphisms. In this section we show that a Riemann surface X with $g(X) \geq 2$ cannot be cyclic $(p, 1)$-gonal and cyclic $(q, 0)$-gonal when both primes are bigger than 3.

Proposition 3.1. *If X has a $(p, 1)$-gonal automorphism and a $(p, 0)$-gonal automorphism, then $p \leq 3$ and $g(X) \leq 7$.*

Proof. We could argue as in Section 2. Fix a $(p, 0)$-gonal automorphism σ, and let $\langle \sigma \rangle$ act by conjugation on the set of all $(p, 1)$-gonal subgroups of A. Assuming $p > 3$, Lemma 2.3 implies that all orbits have length p, in contradiction to Proposition 2.4.

But a completely elementary argument also works. From the Hurwitz formula we see that $p - 1$ divides $2g - 2$ and $2g - 2 + 2p$, so $(p-1)|2p$.

Then $g \leq 7$ follows from $p \leq 3$ by Theorem 2.1. $\square$

Proposition 3.2. *If X has a $(p, 1)$-gonal subgroup and a $(q, 0)$-gonal subgroup with $p < q$, then $p \leq 3$ and $g \leq 10$ and $q \leq 19$.*

Proof. Assume $p > 3$. Then $q \geq 7$ and by Lemma 2.3 the number r of $(p, 1)$-gonal subgroups must be divisible by q. By Theorem 2.5 the only possibilities are $(p, q, r) = (7, 11, 22)$, $(7, 29, 29)$ and $(5, 11, 11)$. By [14, Theorem 8.1] in these cases the $(q, 0)$-gonal subgroup $\langle \sigma \rangle$ must be normal. (Here we are using

that by [8, Theorem 1] or [9, Theorem 2.1] all $(q, 0)$-gonal subgroups are conjugate; so a cyclic $(q, 0)$-gonal surface is either normal $(q, 0)$-gonal or non-normal $(q, 0)$-gonal, but not both.)

Fix a $(p, 1)$-gonal subgroup $H = \langle \tau \rangle$. Then $\sigma\tau = \tau\sigma^n$ for some $n < q$, which shows that τ acts on the fixed points of σ.

If $(p, q) = (7, 11)$, necessarily $n = 1$, contradicting Lemma 2.3.

If $(p, q) = (7, 29)$, we have $g \leq 50$ since H is not unique. Thus by the Hurwitz formula σ has at most 5 fixed points. So H and σ have a common fixed point, and hence by Theorem 2.2 (a) they commute, contradicting Lemma 2.3.

We are left with the case $(p, q) = (5, 11)$. Then $g \leq 26$, and actually $g = 15$ since the number of fixed points of σ must be divisible by 5. Since τ induces a nontrivial automorphism of the genus 0 surface $X/\langle\sigma\rangle$, some of the 7 fixed points of τ on X must fall together on $X/\langle\sigma\rangle$. So assume that $\tau(x) = x$ and that $\sigma(x)$ is also a fixed point of τ. Then $\tau\sigma^n(x) = \sigma\tau(x) = \sigma(x) = \tau\sigma(x)$. So $n = 1$ or x is also a fixed point of σ, either one a contradiction. Finally we have proved $p \leq 3$.

If $p = 2$ and $q = 3$, then $g \leq 4$ by Theorem 2.1. In all other cases q is bigger than 3; then the $(p, 1)$-gonal subgroup cannot be unique, which by Theorem 2.1 implies $g \leq 10$. This forces $q \leq 19$ by the Hurwitz formula.   □

Proposition 3.3. *If X has a $(p, 1)$-gonal subgroup and a $(q, 0)$-gonal subgroup with $p > q$, then $q \leq 3$. Moreover, if X is not hyperelliptic, then the $(3, 0)$-gonal subgroup is unique.*

Proof. Assume $q > 3$. Then, as several times before, the number of $(p, 1)$-gonal subgroups must be divisible by q. This leaves only the possibility $p = 7$, $q = 5$.

Now fix a $(7, 1)$-gonal subgroup H. If the $(5, 0)$-gonal subgroup $\langle\sigma\rangle$ were normal, H would act trivially on it, so they would commute element-wise. Hence σ would normalize H, in contradiction to Lemma 2.3. Thus A has a non-normal $(5, 0)$-gonal subgroup and 35 divides the order of A. But by [14, Theorem 8.1] no such A exists. So we have shown $q \leq 3$.

Now assume that A has more than one $(3, 0)$-gonal subgroup. Then $g \leq 4$ and hence $p \leq 3$, so $q \leq 2$, which means that X is hyperelliptic.   □

Remark 3.4. In Proposition 3.3 we can neither bound p nor the genus of X.

In fact, for every prime $p > 7$ there exist uncountably many hyperelliptic surfaces of genus p with a $(p, 1)$-gonal automorphism, for example

$$Y^2 = X(X^p - 1)(X^p - \lambda)$$

with $\lambda \in \mathbb{C}$ different from 0 and 1. The obvious automorphism group of order p is unique by Proposition 2.7. Its quotient is the genus 1 surface

$$U^2 = W(W - 1)(W - \lambda)$$

with $W = X^p$ and $U = X^{\frac{p-1}{2}}Y$. So if two surfaces of genus p as above are isomorphic, the corresponding genus 1 surfaces must also be isomorphic. But for any given λ_0 there are at most 5 further values λ for which this happens.

Also, for every prime $p \geq 5$ there exist cyclic trigonal surfaces of genus p with a $(p, 1)$-gonal automorphism, for example

$$Y^3 = X(X^p - 1) \ \text{ if } p \equiv 1 \ mod \ 3,$$

$$Y^3 = X^2(X^p - 1) \ \text{ if } p \equiv 2 \ mod \ 3.$$

We finish this section with another result in the spirit of the previous proofs.

Lemma 3.5. *Let $p > 7$ and let A have a non-normal $(p, 0)$-gonal subgroup. Then p^2 divides the order of A.*

Proof. Let P be such a non-normal $(p, 0)$-gonal subgroup. Then there exists a conjugate P^α of P in A with $P^\alpha \neq P$. By [10, Corollary 3.2] the orbit of P^α under conjugation with elements from P is bounded by $6\frac{p-2}{p-6} < p$ for $p \geq 11$. Hence P normalizes P^α. Consequently, P and P^α commute element-wise and generate a group of order p^2. □

This lemma might seem a bit aimless, but actually it offers an alternative proof to the arguments in Section 7 of [14] that for non-normal cyclic $(p, 0)$-gonal X with $p > 7$ the case $p^2 \nmid \#A$ does not occur.

4. Properly (p, h)-gonal automorphisms with $h \geq 2$. Theorem 1.2 has been generalized to properly (p, h)-gonal subgroups in [12] (Theorems 4.5 and 5.2). We reproduce only the part that we need, in slightly modified form.

Lemma 4.1. *Let $h \geq 2$ and $p > 2h + 1$. Then the size of a conjugacy class of properly (p, h)-gonal subgroups in A is bounded by $6(h + \frac{6h-1}{p-6})$.*

Proof. Under these conditions the size was bounded in [12, Theorem 5.2] by $6\frac{(p-1)(g-1)}{(p-6)(g-1-p(h-1))}$. We rewrite this as $6\frac{p-1}{p-6}(1 + \frac{p(h-1)}{g-1-p(h-1)})$. Since the subgroup acts properly, by the Hurwitz formula and Theorem 2.2 (b) we have $g - 1 - p(h - 1) \geq p - 1$. So we can bound the size of the conjugacy class by $6\frac{p-1}{p-6}(1 + \frac{p(h-1)}{p-1}) = 6\frac{ph-1}{p-6} = 6(h + \frac{6h-1}{p-6})$. □

Theorem 4.2. *Fix $h \geq 1$ and $p > 6h + 6$. Then for every properly cyclic (p, h)-gonal Riemann surface X the properly (p, h)-gonal subgroup is unique (and hence normal) in $Aut(X)$.*

Proof. For $h = 1$ this is part of Theorem 2.5.
So let $h \geq 2$. Assume that there are two distinct properly (p, h)-gonal subgroups P_1 and P_2. By Lemma 4.1 the length of the orbit of P_1 under conjugation with elements from P_2 is bounded by $6(h + \frac{6h-1}{p-6})$, which for $p > 6h + 6$ is smaller than $6(h + \frac{6h-1}{6h}) \leq p - \frac{1}{h} < p$. Hence P_2 normalizes P_1 and induces an automorphism of order p on the genus h surface X/P_1. But this is not possible for $p > 2h + 1$. □

We can even go one step further.

Theorem 4.3. *Suppose that $Aut(X)$ has a proper (p,h)-gonal subgroup P with $h \geq 2$ and $p > 6h + 6$. Then $Aut(X)$ has no other subgroups at all of prime order q with $q \geq p$. Actually, p is the biggest prime divisor of $\#Aut(X)$, every other prime divisor is smaller than $\frac{p}{3}$, and p^2 does not divide the order of $Aut(X)$. Moreover, $\#Aut(X)$ is smaller than $14p(p-12)$, and P is the unique p-Sylow subgroup of $Aut(X)$ and hence normal.*

Proof. Let Q be any subgroup of prime order $q \geq p$ of $Aut(X)$. Note that we do not make any assumptions on the genus of X/Q; and we also do not require that Q is proper.

Then Q normalizes P, because as in the proof of Theorem 4.2 the length of the orbit of P under conjugation with Q is smaller than $p \leq q$.

If Q is different from P, then it induces an automorphism of order q on the genus h surface X/P. But this is not possible since $q \geq p > 2h + 1$.

Similarly, P cannot be contained in a cyclic group of order p^2, because that would also induce an automorphism of order p on X/P. As there also are no other subgroups of order p, we see that P is a p-Sylow subgroup of $Aut(X)$, unique, and hence normal.

Thus $Aut(X)/P$ is a subgroup of $Aut(X/P)$. But the order of $Aut(X/P)$ is bounded by $84(h-1) < 14(p-12)$, and its prime divisors are bounded by $2h+1 < \frac{p}{3}$.   □

One implication is that Riemann surfaces satisfying the condition of Theorem 4.3 are presumably rare.

On the other hand, starting with a Riemann surface R of genus $h \geq 2$, one can, for every prime $p > 6h + 6$, easily construct cyclic coverings $X \to R$ of degree p of arbitrarily large genus. Theorem 4.3 says that the automorphism groups of such coverings are subject to severe restrictions.

It was already proved in [12, Corollary 4.6] that p^2 does not divide $\#Aut(X)$ for (p,h)-gonal X with $p > 2h + 1$ and $h \geq 2$. By the Sylow theorems, then all (p,h)-gonal P are conjugate [12, Theorem 4.5]. But the bound on the size of the conjugacy class in [12, Theorem 5.2] only makes sense for properly (p,h)-gonal P. However, at the price of making p much bigger than h we can get a version of Theorem 4.3 without this condition.

Theorem 4.4. *Suppose that $P \subseteq Aut(X)$ is (p,h)-gonal with $p > 84(h-1)$ and $h \geq 2$. Then all the conclusions of Theorem 4.3 hold. Moreover, $q < \frac{p}{42} + 3$ for all prime divisors $q \neq p$ of $\#Aut(X)$.*

Proof. By Theorem 4.3 we can suppose that P has no fixed points. Then $g = p(h-1) + 1$ and $\#Aut(X) \leq 84(g-1) < p^2$. So the uniqueness of P is a consequence of the Sylow theorems. The rest follows as in the proof of Theorem 4.3.   □

For $(p,2)$-gonal automorphisms without fixed points Theorem 4.3 remains true, and actually much more precise results can be read off from [2, Theorem 1] in combination with [2, Section 6].

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