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Top local cohomology modules and Gorenstein injectivity with respect to a semidualizing module

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Abstract. Let $(R, \mathfrak{m})$ be a commutative Noetherian local ring of Krull dimension d, and let C be a semidualizing R-module. In this paper, it is shown that if R is complete, then C is a dualizing module if and only if the top local cohomology module of $R, \mathrm{H}^d_{\mathfrak{m}}(R)$, has finite G_C-injective dimension. This generalizes a recent result due to Yoshizawa, where the ring is assumed to be complete Cohen-Macaulay.

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1. Introduction. Throughout this paper, R is a commutative Noetherian ring and all modules are unital. In [3, Corollary 9.5.13] it is shown that a local ring $(R, \mathfrak{m})$ is Gorenstein if and only if R is a Cohen-Macaulay ring and the top local cohomology module of $R, \mathrm{H}^{\dim R}_{\mathfrak{m}}(R)$, is isomorphic to $\mathrm{E}(R/\mathfrak{m})$. In [9] Sazeedeh showed that over a Gorenstein local ring of Krull dimension at most two the top local cohomology module $\mathrm{H}^{\dim R}_{J}(R)$ is a Gorenstein injective R-module for any ideal J of R. Later, in [10] it is shown that over a complete Cohen-Macaulay local ring $(R, \mathfrak{m})$ of Krull dimension d the top local cohomology module $\mathrm{H}^d_{\mathfrak{m}}(R)$ is a strongly cotorsion module. Recently, in [15] Yoshizawa generalized Sazeedeh's results, using the fact that over a Gorenstein ring with finite Krull dimension strongly cotorsion modules are precisely Gorenstein injective modules. Recall that the class of strongly cotorsion modules, which contains the class of injective modules, has been introduced by Xu in [14]. Indeed, Yoshizawa showed that over a complete Cohen-Macaulay local ring $(R, \mathfrak{m})$ of Krull dimension d the following conditions are equivalent:

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(i) R is Gorenstein.
(ii) $\mathrm{H}^d_{\mathfrak{m}}(R)$ is an injective R-module.
(iii) $\mathrm{H}^d_{\mathfrak{m}}(R)$ is a Gorenstein injective R-module.

Over a commutative local ring, semidualizing modules provide a common generalization of a dualizing (canonical) module and a free module of rank one. In this paper, we extend the recent result of Yoshizawa in the relative setting with respect to a semidualizing module C. Indeed, we give a characterization of a dualizing module C in term of G_C-injectivity of local cohomology modules. More precisely in Theorem 3.1, we show that:
Let $(R, \mathfrak{m})$ be a complete local ring of Krull dimension d and let C be a semidualizing R-module. Then the following statements are equivalent.

(i) C is a dualizing R-module.
(ii) $\mathrm{H}^d_{\mathfrak{m}}(R)$ is a C-injective R-module.
(iii) $\mathrm{H}^d_{\mathfrak{m}}(R)$ is a G_C-injective R-module.
(iv) $\mathrm{H}^d_{\mathfrak{m}}(R)$ has finite G_C-injective dimension.

2. Preliminaries. Throughout this paper R is a commutative Noetherian ring and $\mathcal{M}(R)$ denotes the category of R-modules. We use the term "subcategory" to mean a "full, additive subcategory $\mathcal{X} \subseteq \mathcal{M}(R)$ such that, for all R-modules M and N, if $M \cong N$ and $M \in \mathcal{X}$, then $N \in \mathcal{X}$". Write $\mathcal{P}(R), \mathcal{F}(R)$ and $\mathcal{I}(R)$ for the subcategories of all projective, flat and injective R-modules, respectively.

An R-complex is a sequence

$$X = \cdots \xrightarrow{\partial^X_{n+1}} X_n \xrightarrow{\partial^X_n} X_{n-1} \xrightarrow{\partial^X_{n-1}} \cdots$$

of R-modules and R-homomorphisms such that $\partial^X_{n-1}\partial^X_n = 0$ for each integer n.

Definition 2.1. Let $\mathcal{X}$ be a class of R-modules and let M be an R-module. An $\mathcal{X}$-*resolution* of M is a complex of R-modules in $\mathcal{X}$ of the form

$$X = \cdots \xrightarrow{\partial^X_2} X_1 \xrightarrow{\partial^X_1} X_0 \longrightarrow 0$$

such that $\mathrm{H}_0(X) \cong M$ and $\mathrm{H}_n(X) = 0$ for $n \geqslant 1$. The $\mathcal{X}$-*projective dimension* of M is the quantity

$$\mathcal{X}\text{-}\operatorname{pd}_R(M) = \inf\{\sup\{n | X_n \neq 0\} | X \text{ is an } \mathcal{X}\text{-resolution of } M\}.$$

In particular, $\mathcal{X}\text{-}\operatorname{pd}_R(0) = -\infty$. The modules of $\mathcal{X}$-projective dimension zero are the non-zero modules in $\mathcal{X}$.
Dually, an $\mathcal{X}$-*coresolution* of M is a complex of R-modules in $\mathcal{X}$ of the form

$$X = 0 \longrightarrow X_0 \xrightarrow{\partial^X_0} X_{-1} \xrightarrow{\partial^X_{-1}} \cdots$$

such that $\mathrm{H}_0(X) \cong M$ and $\mathrm{H}_n(X) = 0$ for $n \leqslant -1$. The $\mathcal{X}$-*injective dimension* of M is the quantity

$$\mathcal{X}\text{-}\operatorname{id}_R(M) = \inf\{\sup\{n | X_n \neq 0\} | X \text{ is an } \mathcal{X}\text{-coresolution of } M\}.$$

In particular, $\mathcal{X}\text{-}\operatorname{id}_R(0) = -\infty$. The modules of $\mathcal{X}$-injective dimension zero are the non-zero modules in $\mathcal{X}$.
When $\mathcal{X}$ is the class of projective R-module we write $\operatorname{pd}_R(M)$ for the associated homological dimension and call it the projective dimension of M. Similarly, the flat and injective dimensions of M are denoted $\operatorname{fd}_R(M)$ and $\operatorname{id}_R(M)$.

The notion of semidualizing modules, defined next, goes back at least to Vasconcelos [12], but was rediscovered by others. For more details about semidualizing module the reader may consult [1] and [6–8].

Definition 2.2. A finitely generated R-module C is called *semidualizing* if the natural homothety morphism $R \to \operatorname{Hom}_R(C, C)$ is an isomorphism and $\operatorname{Ext}_R^{\geqslant 1}(C, C) = 0$. An R-module D is called *dualizing* if it is semidualizing and has finite injective dimension.

For a semidualizing R-module C, we set

$$\mathcal{P}_C(R) = \{P \otimes_R C \mid P \text{ is a projective } R\text{-module}\},$$
$$\mathcal{F}_C(R) = \{F \otimes_R C \mid F \text{ is a flat } R\text{-module}\},$$
$$\mathcal{I}_C(R) = \{\operatorname{Hom}_R(C, I) \mid I \text{ is an injective } R\text{-module}\},$$

The R-modules in $\mathcal{P}_C(R), \mathcal{F}_C(R)$ and $\mathcal{I}_C(R)$ are called *C-projective*, *C-flat* and *C-injective*, respectively.

The next definition is due to Holm and Jørgensen [4].

Definition 2.3. Let C be a semidualizing R-module. A *complete $\mathcal{I}_C\mathcal{I}$-resolution* is a complex Y of R-modules satisfying the following:

(i) Y is exact and $\operatorname{Hom}_R(I, Y)$ is exact for each $I \in \mathcal{I}_C(R)$, and
(ii) $Y_i \in \mathcal{I}_C(R)$ for all $i \geqslant 0$ and Y_i is injective for all $i < 0$.

An R-module M is *G_C-injective* if there exists a complete $\mathcal{I}_C\mathcal{I}$-resolution Y such that $M \cong \operatorname{Coker}(\partial_1^Y)$; in this case Y is a *complete $\mathcal{I}_C\mathcal{I}$-resolution* of M.

A *complete $\mathcal{P}\mathcal{P}_C$-resolution* is a complex X of R-modules such that:

(i) X is exact and $\operatorname{Hom}_R(X, P)$ is exact for each $P \in \mathcal{P}_C(R)$, and
(ii) X_i is projective for all $i \geqslant 0$ and $X_i \in \mathcal{P}_C(R)$ for all $i < 0$.

An R-module M is *G_C-projective* if there exists a complete $\mathcal{P}\mathcal{P}_C$-resolution X such that $M \cong \operatorname{Coker}(\partial_1^X)$; in this case X is a *complete $\mathcal{P}\mathcal{P}_C$-resolution* of M.

A *complete $\mathcal{F}\mathcal{F}_C$-resolution* is a complex Z of R-modules such that:

(i) Z is exact and $Z \otimes_R I$ is exact for each $I \in \mathcal{I}_C(R)$, and
(ii) Z_i is flat for all $i \geqslant 0$ and $Z_i \in \mathcal{F}_C(R)$ for all $i < 0$.

An R-module M is *G_C-flat* if there exists a complete $\mathcal{F}\mathcal{F}_C$-resolution Z such that $M \cong \operatorname{Coker}(\partial_1^Z)$; in this case Z is a *complete $\mathcal{F}\mathcal{F}_C$-resolution* of M.

Definition 2.4. Let $(R, \mathfrak{m})$ be a local ring. A finitely generated R-module K is said to be a *canonical module* of R if $\operatorname{Hom}_R(\mathrm{H}_{\mathfrak{m}}^d, \mathrm{E}(R/\mathfrak{m})) \cong \widehat{K}$, where $\mathrm{E}(R/\mathfrak{m})$ is the injective hull of $R/\mathfrak{m}$.

Note that if R is a complete local ring, then $\mathrm{Hom}_R(\mathrm{H}^d_{\mathfrak{m}}, \mathrm{E}(R/\mathfrak{m}))$ is a canonical module of R. If R is a Cohen-Macaulay ring, then K is a canonical module of R if and only if K is a dualizing module of R (see [6, Corollary 2.2.13]).

Definition 2.5. Let $(R, \mathfrak{m})$ be a local ring and let M be an R-module. Then the R-module $M^\vee = \mathrm{Hom}_R(M, \mathrm{E}(R/\mathfrak{m}))$ is called the Matlis dual of M.

3. Main result. The purpose of this section is to extend the recent result of Yoshizawa [15, Theorem 2.6] in the relative setting with respect to the semidualizing module C. Indeed, we give a characterization of the dualizing module C by G_C-injectivity of top local cohomology modules.

First we recall the notion of trivial extension of the ring R by an R-module. If C is an R-module, then the direct sum $R \oplus C$ can be equipped with the product:

$$(a, c)(a', c') = (aa', ac' + a'c),$$

where $a, a' \in R$ and $c, c' \in C$. This turns $R \oplus C$ into a ring which is called the trivial extension of R by C and denoted $R \ltimes C$. There are canonical ring homomorphism $R \rightleftarrows R \ltimes C$, which enable us to view R-modules as $(R \ltimes C)$-modules and vice versa.

Theorem 3.1. *Let $(R, \mathfrak{m})$ be a complete local ring of Krull dimension d and let C be a semidualizing R-module. Then the following statements are equivalent.*

(i) *C is a dualizing R-module.*
(ii) *$\mathrm{H}^d_{\mathfrak{m}}(R)$ is a C-injective R-module.*
(iii) *$\mathrm{H}^d_{\mathfrak{m}}(R)$ is a G_C-injective R-module.*
(iv) *$\mathrm{H}^d_{\mathfrak{m}}(R)$ has finite G_C-injective dimension.*

Proof. (i) $\Rightarrow$ (ii) Let C be a dualizing R-module. Since C is a non-zero finitely generated R-module of finite injective dimension, a corollary of the new intersection theorem implies that R is Cohen-Macaulay. By [3, Remark 9.5.18] we have

$$\mathrm{Hom}_R(\mathrm{H}^d_{\mathfrak{m}}(R), \mathrm{E}(R/\mathfrak{m})) \cong \widehat{C}.$$

Therefore C is a canonical module of R. Using the Local Duality Theorem (see [3, Theorem 9.5.17]) we have

$$\mathrm{H}^d_{\mathfrak{m}}(R) \cong \mathrm{Hom}_R(\mathrm{Hom}_R(R, C), \mathrm{E}(R/\mathfrak{m})) \cong \mathrm{Hom}_R(C, \mathrm{E}(R/\mathfrak{m})).$$

Thus $\mathrm{H}^d_{\mathfrak{m}}(R)$ is a C-injective R-module.

(ii) $\Rightarrow$ (iii) In [13, Proposition 2.6] White showed that projective R-modules and C-projective R-modules are G_C-projective. Similarly, one can show that injective R-modules and C-injective R-modules are G_C-injective.

(iii) $\Rightarrow$ (iv) It is trivial.

(iv) $\Rightarrow$ (i) Suppose that $\mathrm{H}^d_{\mathfrak{m}}(R)$ has finite G_C-injective dimension. Since $\mathrm{H}^d_{\mathfrak{m}}(R)$ is an Artinian R-module we have $\mathrm{H}^d_{\mathfrak{m}}(R)^\vee$ is a finitely generated R-module (see [3, Theorem 3.4.7]). By [4, Theorem 2.16], $\mathrm{Gid}_{R \ltimes C}(\mathrm{H}^d_{\mathfrak{m}}(R)) < \infty$. Since R is complete, $\mathrm{Gfd}_{R \ltimes C}(\mathrm{H}^d_{\mathfrak{m}}(R)^\vee) < \infty$ by [2, Theorem 4.25]. So

$\mathrm{Gpd}_{R\ltimes C}(\mathrm{H}^d_{\mathfrak{m}}(R)^\vee)<\infty$, by [2, Theorem 4.23]. In addition, $\mathcal{GP}_C\text{-}\,\mathrm{pd}_R(\mathrm{H}^d_{\mathfrak{m}}(R)^\vee) = \mathrm{Gpd}_{R\ltimes C}(\mathrm{H}^d_{\mathfrak{m}}(R)^\vee) < \infty$, by [4, Theorem 2.16]. The Local Duality Theorem implies that $\mathrm{H}^d_{\mathfrak{m}}(R)^\vee \cong \Omega$, where Ω is a canonical module of R. By definition, this module is non-zero and finitely generated, so it has finite depth, and it has finite injective dimension. Since it has finite G_C-projective dimension, the fact that C is dualizing follows from [8, Corollary 2.9]. □

Remark 3.2. Let C be a semidualizing R-module. The Auslander class with respect to C is the class $\mathcal{A}_C(R)$ of R-modules M such that:

(i) $\mathrm{Tor}^R_i(C,M)=0=\mathrm{Ext}^i_R(C,C\otimes_R M)$ for all $i\geqslant 1$, and
(ii) the natural map $M\to \mathrm{Hom}_R(C,C\otimes_R M)$ is an isomorphism.

The Bass class with respect to C is the class $\mathcal{B}_C(R)$ of R-modules M such that:

(i) $\mathrm{Ext}^i_R(C,M)=0=\mathrm{Tor}^R_i(C,\mathrm{Hom}_R(C,M))$ for all $i\geqslant 1$, and
(ii) the natural evaluation map $C\otimes_R \mathrm{Hom}_R(C,M)\to M$ is an isomorphism.

The class $\mathcal{A}_C(R)$ contains all R-modules of finite flat dimension and the class $\mathcal{B}_C(R)$ contains all R-modules of finite injective dimension (see [11, 1.9]). Also, Takahashi and White in [11, Corollary 2.9] showed that if $\mathcal{P}_C-\mathrm{pd}_R(M)<\infty$ (resp. $\mathcal{I}_C-\mathrm{id}_R(M)<\infty$), then $M\in\mathcal{B}_C(R)$ (resp. $M\in\mathcal{A}_C(R)$).

Now it is natural to ask what can we say about C when the top local cohomology module of local ring R is in $\mathcal{A}_C(R)$ or $\mathcal{B}_C(R)$? What happens if the top local cohomology module of C is C-injective?

These authors do not know the general answer yet. But we have the following partial answer.

Proposition 3.3. *Let $(R,\mathfrak{m})$ be a complete local ring of Krull dimension d, and let C be a semidualizing R-module. Assume that R has a dualizing module D, and set $C'=\mathrm{Hom}_R(C,D)$. Then $C'\cong R$ if and only if $\mathrm{H}^d_{\mathfrak{m}}(R)\in\mathcal{B}_{C'}(R)$.*

Proof. The forward implication is clear, because $\mathcal{B}_R(R)=\mathcal{M}(R)$. For the reverse implication, assume that $\mathrm{H}^d_{\mathfrak{m}}(R)\in\mathcal{B}_{C'}(R)$. Then $\mathrm{H}^d_{\mathfrak{m}}(R)$ has finite G_C-injective dimension by [4, Theorem 4.6]. By Theorem 3.1, C is dualizing. So we conclude that $C'\cong R$. □

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