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Martingale inequalities in noncommutative symmetric spaces

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Abstract. We investigate the Burkholder–Gundy inequalities in a noncommutative symmetric space $E(\mathcal{M})$ associated with a von Neumann algebra $\mathcal{M}$ equipped with a faithful normal state. The results extend the Pisier–Xu noncommutative martingale inequalities, and generalize the classical inequalities in the commutative case.

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1. Introduction. In classical martingale theory, the famous Burkholder–Gundy inequalities can be stated as follows: given a probability space $(\Omega, \mathcal{F}, P)$, let $\{\mathcal{F}_n\}_{n\geq 1}$ be a nondecreasing sequence of σ-fields of $\mathcal{F}$ such that $\mathcal{F} = \vee \mathcal{F}_n$. Given $1 < p < \infty$ and an L^p-bounded martingale $f = (f_n)_{n\geq 1}$, we have

$$\|f\|_{L_p} \approx \left\| \left(\sum_{k=1}^{\infty} |df_k|^2 \right)^{1/2} \right\|_{L_p}. \tag{1.1}$$

Note however that in strong contrast with the classical case, the square function $S(f) = (\sum_{k=1}^{\infty} |df_k|^2)^{1/2}$ in the noncommutative (quantum) case can take two different forms so it is very important to formulate the 'right' square functions. This surprising phenomenon was already discovered by Lust-Piquard in [14] while establishing a noncommutative version of the Khintchine inequalities. The square function is defined differently (and it must be changed!)

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according to $p < 2$ or $p \geq 2$. Within this spirit, the noncommutative analogues of the inequalities (1.1) were successfully obtained by Pisier and Xu in [15]. More precisely, for $2 \leq p < \infty$, and any finite noncommutative $L_p(\mathcal{M})$-martingale $x = (x_n)_{n\geq 1}$, (1.1) has the following noncommutative version,

$$\|x\|_{L_p(\mathcal{M})} \approx \max\{\|S_c(x)\|_{L_p(\mathcal{M})}, \|S_r(x)\|_{L_p(\mathcal{M})}\}, \tag{1.2}$$

where $S_c(x)$ and $S_r(x)$ denote column and row versions of square function, see Section 2 for the definitions. Moreover, they obtained a similar inequality for $1 < p < 2$ by duality. Recently, Randrianantoanina [16] proved a weak-type inequality for square functions, which implies Pisier–Xu's noncommutative martingale inequalities by interpolation.

In this paper we consider the Burkholder–Gundy inequalities for noncommutative symmetric space $E(\mathcal{M}), 1 < p_E \leq q_E < \infty$, where p_E and q_E denote the low Boyd index and the upper Boyd index, respectively. We refer to [13] for the detailed discussions. One of our main results can be stated as follows (see Theorem 3.1 for the detailed statement): for $2 < p_E \leq q_E < \infty$, and any bounded $E(\mathcal{M})$-martingale $x = (x_n)_{n\geq 1}$, we have

$$\|x\|_{E(\mathcal{M})} \approx \max\{\|S_c(x)\|_{E(\mathcal{M})}, \|S_r(x)\|_{E(\mathcal{M})}\}. \tag{1.3}$$

Note that if $E = L_p, 2 < p < \infty$, we come back to the inequalities (1.2). We also extend these inequalities to the case $1 < p_E \leq q_E < 2$. Our proof uses a very recent result of Le Merdy–Sukochev in [12] on the Khintchine inequalities in noncommutative symmetric spaces. We should point out that in [1], the inequality (1.3) also be obtained, however, the rearrangement invariant space E is required to be p-convex with $p > 2$ and q-concave with $q < \infty$. Obviously, our conditions are much weaker.

2. Preliminaries. Now we introduce the noncommutative symmetric spaces. Let $(\mathcal{M}, \tau)$ be a tracial noncommutative probability space. Let $L_0(\mathcal{M})$ denote the topological $*$-algebra of all measurable operators with respect to $(\mathcal{M}, \tau)$. For $x \in L_0(\mathcal{M})$, define its generalized singular number by

$$\mu_t(x) = \inf\{\lambda > 0 : \tau(\chi_{(\lambda,\infty)}(|x|)) \leq t\}, \quad t > 0.$$

The function $t \to \mu_t(x)$ from $(0, 1)$ to $[0, \infty)$ is right continuous, nonincreasing and is the inverse of the distribution function $\lambda(x)$, where $\lambda_s(x) = \tau(\chi_{(\lambda,\infty)}(|x|))$, for $s \geq 0$. For a complete study of $\mu(\cdot)$ and $\lambda(\cdot)$, we refer to [8]. For the definition below, we refer to [3] and [13] for the theory of rearrangement invariant function spaces.

Definition 2.1. Let E be a rearrangement invariant Banach function space on $[0, 1]$. We define the symmetric space $E(\mathcal{M}, \tau)$ of measurable operators by setting

$$E(\mathcal{M}, \tau) = \{x \in L_0(\mathcal{M}) : \mu(x) \in E\}$$

and

$$\|x\|_{E(\mathcal{M},\tau)} = \|\mu(x)\|_E, \quad x \in E(\mathcal{M}, \tau).$$

It is well known that $E(\mathcal{M}, \tau)$ is a Banach space if E is Banach space. The space $E(\mathcal{M}, \tau)$ is often referred to as the noncommutative analogue of the function space E and if $E = L^p[0,1]$ for $1 \leq p \leq \infty$, then $E(\mathcal{M}, \tau)$ coincides with the usual noncommutative L^p-space associated with $(\mathcal{M}, \tau)$. We refer to [4–6, 11] and [17] for more detailed discussions about these spaces. For a finite sequence $a = (a_n)_{n\geq 1}$ in $E(\mathcal{M})$, we define

$$\|a\|_{E(\mathcal{M};\ell_c^2)} = \Big\|\Big(\sum_n |a_n|^2\Big)^{1/2}\Big\|_{E(\mathcal{M})}, \quad \|a\|_{E(\mathcal{M};\ell_r^2)} = \Big\|\Big(\sum_n |a_n^*|^2\Big)^{1/2}\Big\|_{E(\mathcal{M})}.$$

Now, any finite sequence $a = (a_n)$ in $E(\mathcal{M})$ can be regarded as an element in $E(\mathcal{M}\overline{\otimes}B(\ell^2))$. Therefore, $\|\cdot\|_{E(\mathcal{M},\ell_c^2)}$ defines a norm on the family of all finite sequences in $E(\mathcal{M})$. The corresponding completion is a Banach space, denoted by $E(\mathcal{M}, \ell_c^2)$. There are the same arguments for $E(\mathcal{M}, \ell_r^2)$.

We now recall the general setup for noncommutative martingales. Let $(\mathcal{M}_n)_{n\geq 1}$ be an increasing sequence of von Neumann subalgebras of $\mathcal{M}$ such that the union of $\mathcal{M}_n'$s is weak*-dense in $\mathcal{M}$. For each $n \geq 1$, it is well known that there is a unique normal faithful conditional expectation $\mathcal{E}_n$ from $\mathcal{M}$ onto $\mathcal{M}_n$. Since $\mathcal{E}_n$ is trace preserving, it extends to a contractive projection from $L_p(\mathcal{M}, \tau)$ onto $L_p(\mathcal{M}_n, \tau_n)$ for all $1 \leq p \leq \infty$ where τ_n is the restriction of τ on $\mathcal{M}_n$. More generally, a simple interpolation argument would prove that if E is a rearrangement invariant Banach function space on $[0,1]$, then $\mathcal{E}_n$ is a contraction from $E(\mathcal{M}, \tau)$ onto $E(\mathcal{M}_n, \tau_n)$.

Recall that a noncommutative martingale with respect to the filtration $(\mathcal{M}_n)_{n\geq 1}$ is a sequence $x = (x_n)_{n\geq 1}$ in $L_1(\mathcal{M}, \tau)$ such that

$$\mathcal{E}_n(x_{n+1}) = x_n, \quad \forall n \geq 1.$$

If additionally, $x_n \in E(\mathcal{M})$ for $n \geq 1$, then x is called an $E(\mathcal{M})$-martingale. In this case, we set

$$\|x\|_{E(\mathcal{M})} = \sup_{n\geq 1} \|x_n\|_{E(\mathcal{M})}.$$

If $\|x\|_{E(\mathcal{M})} < \infty$, then x is called a bounded $E(\mathcal{M})$-martingale. The difference sequence $dx = (dx_n)_{n\geq 1}$ is defined by $dx_n = x_n - x_{n-1}$ with the usual convention that $x_0 = 0$. We describe the square functions of noncommutative martingales. Following [15], we will consider the following column and row versions of the square function: for a finite martingale $x = (x_n)$, set

$$S_c(x) = \Big(\sum_n |dx_n|^2\Big)^{1/2}, \quad S_r(x) = \Big(\sum_n |dx_n^*|^2\Big)^{1/2}.$$

Define $\mathcal{H}_E^c(\mathcal{M})$, respectively $\mathcal{H}_E^r(\mathcal{M})$, to be the space of all $E(\mathcal{M})$-martingales such that $dx \in E(\mathcal{M}; \ell_c^2)$, respectively $dx \in E(\mathcal{M}; \ell_r^2)$, and set

$$\|x\|_{\mathcal{H}_E^c(\mathcal{M})} = \|dx\|_{E(\mathcal{M};\ell_c^2)}, \quad \|x\|_{\mathcal{H}_E^r(\mathcal{M})} = \|dx\|_{E(\mathcal{M};\ell_r^2)}.$$

Equipped respectively with the previous norms, $\mathcal{H}_E^c(\mathcal{M})$ and $\mathcal{H}_E^r(\mathcal{M})$ are Banach spaces. We then define the Hardy spaces $\mathcal{H}_E(\mathcal{M})$. For $1 < p_E \leq q_E < 2$,

$$\mathcal{H}_E(\mathcal{M}) = \mathcal{H}_E^c(\mathcal{M}) + \mathcal{H}_E^r(\mathcal{M}),$$

with the norm

$$\|x\|_{\mathcal{H}_E} = \inf\{\|y\|_{\mathcal{H}_E^c} + \|z\|_{\mathcal{H}_E^r} : x = y + z, y \in \mathcal{H}_E^c(\mathcal{M}), z \in \mathcal{H}_E^r(\mathcal{M})\}$$

For $2 < p_E \leq q_E < \infty$,

$$\mathcal{H}_E(\mathcal{M}) = \mathcal{H}_E^c(\mathcal{M}) \cap \mathcal{H}_E^r(\mathcal{M}),$$

with the norm

$$\|x\|_{\mathcal{H}_E} = \max\{\|x\|_{\mathcal{H}_E^c}, \|x\|_{\mathcal{H}_E^r}\}.$$

Throughout the paper the letter C will denote a positive constant, which only depend on E but never on the martingales in consideration, and which may change from line to line. The notation "$\approx$" means norm equivalence.

3. The Burkholder–Gundy inequalities. We now investigate the Burkholder–Gundy inequalities for the noncommutative symmetric space $E(\mathcal{M})$. The principal result of this section is the following

Theorem 3.1. *Let $1 < p_E \leq q_E < \infty$, and let $x = (x_n)_{n\geq 1}$ be a bounded $E(\mathcal{M})$-martingale. Then*

(1) *for $2 < p_E \leq q_E < \infty$*

$$\|x\|_{E(\mathcal{M})} \approx \max\{\|S_c(x)\|_{E(\mathcal{M})}, \|S_r(x)\|_{E(\mathcal{M})}\}\,; \tag{3.1}$$

(2) *for $1 < p_E \leq q_E < 2$*

$$\|x\|_{E(\mathcal{M})} \approx \inf_{x=y+z}\{\|S_c(y)\|_{E(\mathcal{M})} + \|S_r(z)\|_{E(\mathcal{M})}\}, \tag{3.2}$$

where the infimum runs over all decompositions $dx_n = dy_n + dz_n$ with dy_n, dz_n being martingale difference sequences.

The following proposition is the key ingredient of our proof.

Proposition 3.2. *Let $1 < p_E \leq q_E < \infty$ and let (ε_n) be a Rademacher sequence. Then for any bounded $E(\mathcal{M})$-martingale,*

$$\Big\|\sum_n dx_n \otimes \varepsilon_n\Big\|_{E(\mathcal{M}\bar{\otimes}L^\infty(\Omega))} \approx \|x\|_{E(\mathcal{M})}.$$

Proof. Consider the operator

$$T : L^p(\mathcal{M}) \longrightarrow L^p(\mathcal{M}\bar{\otimes}L^\infty(\Omega))$$

by

$$Tx = \sum_n dx_n \otimes \varepsilon_n, \quad \forall x \in L^p(\mathcal{M}) \quad \text{and} \quad x_n = \mathcal{E}_n(x).$$

By Theorem 2.1 in [15],

$$\|x\|_{L^p(\mathcal{M})} = \Big\|\sum_n dx_n\Big\|_{L^p(\mathcal{M})} \approx \Big\|\sum_n dx_n \otimes \varepsilon_n\Big\|_{L^p(\mathcal{M}\bar{\otimes}L^\infty(\Omega))}. \tag{3.3}$$

Then T is bounded in $L^p(\mathcal{M})$ for all $1 < p < \infty$. Choosing r, s to satisfy $1 < r < p_E \leq q_E < s < \infty$. Theorem 2.b.11 in [13] gives that E is an interpolation space for the couple (L_r, L_s). Then by interpolation (see Proposition 2.1 in [12] or Theorem 3.4 in [7]), we obtain

$$\Big\| \sum_n dx_n \otimes \varepsilon_n \Big\|_{E(\mathcal{M}\bar{\otimes}L^\infty(\Omega))} \leq C\|x\|_{E(\mathcal{M})}.$$

In order to prove the inverse inequality, recall that for any $1 < p < \infty$, $P \otimes I_{L_p(\mathcal{M})}$ extends to a bounded projection $L_p(\Omega; L_p(\mathcal{M})) \to L_p(\Omega; L_p(\mathcal{M}))$, where $P : L_2(\Omega) \to L_2(\Omega)$ is the orthogonal projection onto the closed subspace generated by the ε_k and $I_{L_p(\mathcal{M})}$ denotes the identity map on $L_p(\mathcal{M})$. Considering $T : L_p(\mathcal{M}\bar{\otimes}L_\infty(\Omega)) \longrightarrow L_p(\mathcal{M})$,

$$T\Big(\sum_n a_n \otimes \varepsilon_n\Big) = \sum_n \mathcal{E}_n(a_n) - \mathcal{E}_{n-1}(a_n), \quad \forall (a_n) \subset L_p(\mathcal{M}).$$

Noting that $(\mathcal{E}_n(a_n) - \mathcal{E}_{n-1}(a_n))$ is a martingale difference sequence, it follows from (3.3) that

$$\begin{aligned}
&\Big\| \sum_n \Big(\mathcal{E}_n(a_n) - \mathcal{E}_{n-1}(a_n)\Big) \Big\|_{L_p(\mathcal{M})} \\
&\quad \leq C \Big\| \sum_n \Big(\mathcal{E}_n(a_n) - \mathcal{E}_{n-1}(a_n)\Big) \otimes \varepsilon_n \Big\|_{L_p\left(\mathcal{M}\bar{\otimes}L_\infty(\Omega)\right)} \\
&\quad \leq C \Big\| \sum_n \mathcal{E}_n(a_n) \otimes \varepsilon_n \Big\|_{L_p\left(\mathcal{M}\bar{\otimes}L_\infty(\Omega)\right)} \\
&\qquad + C \Big\| \sum_n \mathcal{E}_{n-1}(a_n) \otimes \varepsilon_n \Big\|_{L_p\left(\mathcal{M}\bar{\otimes}L_\infty(\Omega)\right)}
\end{aligned}$$

Now we use the noncommutative Khintchine inequality and the noncommutative Stein inequality to estimate the inequality above. For $p \geq 2$,

$$\begin{aligned}
&\Big\| \sum_n \mathcal{E}_n(a_n) \otimes \varepsilon_n \Big\|_{L_p\left(\mathcal{M}\bar{\otimes}L_\infty(\Omega)\right)} \\
&\quad \leq C \max\left\{ \Big\| \Big(\sum_n |\mathcal{E}_n(a_n)|^2\Big)^{\frac{1}{2}} \Big\|_{L_p(\mathcal{M})}, \Big\| \Big(\sum_n |\mathcal{E}_n(a_n)^*|^2\Big)^{\frac{1}{2}} \Big\|_{L_p(\mathcal{M})} \right\} \\
&\quad \leq C \max\left\{ \Big\| \Big(\sum_n |a_n|^2\Big)^{\frac{1}{2}} \Big\|_{L_p(\mathcal{M})}, \Big\| \Big(\sum_n |a_n^*|^2\Big)^{\frac{1}{2}} \Big\|_{L_p(\mathcal{M})} \right\} \\
&\quad \leq C \Big\| \sum_n a_n \otimes \varepsilon_n \Big\|_{L_p\left(\mathcal{M}\bar{\otimes}L_\infty(\Omega)\right)}
\end{aligned}$$

For $1 < p < 2$, let $a_n = b_n + c_n$ be any decomposition with b_n and c_n in $L_p(\mathcal{M})$, then $\mathcal{E}_n(a_n) = \mathcal{E}_n(b_n) + \mathcal{E}_n(c_n)$. Hence

$$\begin{aligned}
&\Big\|\sum_n \mathcal{E}_n(a_n)\otimes\varepsilon_n\Big\|_{L_p\left(\mathcal{M}\bar{\otimes}L_\infty(\Omega)\right)}\\
&\quad\le C\left(\Big\|\Big(\sum_n|\mathcal{E}_n(b_n)|^2\Big)^{\frac12}\Big\|_{L_p(\mathcal{M})}+\Big\|\Big(\sum_n|\mathcal{E}_n(c_n)^*|^2\Big)^{\frac12}\Big\|_{L_p(\mathcal{M})}\right)\\
&\quad\le C\left(\Big\|\Big(\sum_n|b_n|^2\Big)^{\frac12}\Big\|_{L_p(\mathcal{M})}+\Big\|\Big(\sum_n|c_n^*|^2\Big)^{\frac12}\Big\|_{L_p(\mathcal{M})}\right)
\end{aligned}$$

Now taking infimum over all the decompositions above, we get

$$\Big\|\sum_n \mathcal{E}_n(a_n)\otimes\varepsilon_n\Big\|_{L_p\left(\mathcal{M}\bar{\otimes}L_\infty(\Omega)\right)}\le\Big\|\sum_n a_n\otimes\varepsilon_n\Big\|_{L_p\left(\mathcal{M}\bar{\otimes}L_\infty(\Omega)\right)}.$$

Similarly,

$$\Big\|\sum_n \mathcal{E}_{n-1}(a_n)\otimes\varepsilon_n\Big\|_{L_p\left(\mathcal{M}\bar{\otimes}L_\infty(\Omega)\right)}\le\Big\|\sum_n a_n\otimes\varepsilon_n\Big\|_{L_p\left(\mathcal{M}\bar{\otimes}L_\infty(\Omega)\right)}.$$

Therefore, for all $1<p<\infty, T: L_p(\mathcal{M}\bar{\otimes}L_\infty(\Omega))\to L_p(\mathcal{M})$ is bounded. Noting that $P\otimes I_{L_p(\mathcal{M})}$ extends to a bounded projection $L_p(\mathcal{M}\bar{\otimes}L_\infty(\Omega))\to L_p(\mathcal{M}\bar{\otimes}L_\infty(\Omega))$, by interpolation again, we have

$$\Big\|\sum_n\Big(\mathcal{E}_n(a_n)-\mathcal{E}_{n-1}(a_n)\Big)\Big\|_{E(\mathcal{M})}\le C\Big\|\sum_n a_n\otimes\varepsilon_n\Big\|_{E(\mathcal{M})},\quad\forall(a_n)\subset L_p(\mathcal{M}).$$

We get the desired inequality by taking $(a_n)=(dx_n)$. The proof is complete. □

It is easy to recognize that the following lemma is the Stein inequality for noncommutative symmetric spaces.

Lemma 3.3. *Let $1<p_E\le q_E<\infty$ and let $a=(a_n)_{n\ge1}$ be a finite sequence in $E(\mathcal{M})$. Then there exists a constant C such that*

$$\Big\|\Big(\sum_n|\mathcal{E}_n(a_n)|^2\Big)^{\frac12}\Big\|_{E(\mathcal{M})}\le C\Big\|\Big(\sum_n|a_n|^2\Big)^{\frac12}\Big\|_{E(\mathcal{M})}.$$

Proof. Let us consider the von Neumann algebra tensor product $\mathcal{M}\bar{\otimes}\mathcal{B}(\ell^2)$ with the product trace $\tau\otimes\mathrm{tr}$. Then $\tau\otimes\mathrm{tr}$ is a semi-finite normal faithful trace. Let $E(\mathcal{M}\bar{\otimes}\mathcal{B}(\ell^2))$ be the associated noncommutative symmetric space. Then $E(\mathcal{M}\bar{\otimes}\mathcal{B}(\ell^2))$ is an interpolation space for the couple $(L_p(\mathcal{M}\bar{\otimes}\mathcal{B}(\ell^2)), L_q(\mathcal{M}\bar{\otimes}\mathcal{B}(\ell^2)))$, where $1<p<p_\Phi\le q_\Phi<q<\infty$. Recalling that the column subspace of $L_p(\mathcal{M}\bar{\otimes}\mathcal{B}(\ell^2))$ is a 1-complemented subspace, we define

$$T: L_p(\mathcal{M}\bar{\otimes}\mathcal{B}(\ell^2))+L_q(\mathcal{M}\bar{\otimes}\mathcal{B}(\ell^2))\to L_p(\mathcal{M}\bar{\otimes}\mathcal{B}(\ell^2))+L_q(\mathcal{M}\bar{\otimes}\mathcal{B}(\ell^2)),$$

by

$$T\begin{pmatrix}a_1&0&\dots&0\\a_2&0&\dots&0\\\vdots&\vdots&&\vdots\\a_n&0&\dots&0\end{pmatrix}=\begin{pmatrix}\mathcal{E}_1(a_1)&0&\dots&0\\\mathcal{E}_2(a_2)&0&\dots&0\\\vdots&\vdots&&\vdots\\\mathcal{E}_n(a_n)&0&\dots&0\end{pmatrix}.$$

It follows from Theorem 2.3 in [15] that T is a bounded linear operator on both $L_p(\mathcal{M}\bar{\otimes}\mathcal{B}(\ell^2))$ and $L_q(\mathcal{M}\bar{\otimes}\mathcal{B}(\ell^2))$. By interpolation again, we obtain the desired result. $\square$

Proof of Theorem. 3.1. Recall that if $1 < p_E \leq q_E < 2$, then $E(\mathcal{M})$ is an interpolation space for the couple $(L_1(\mathcal{M}), L_2(\mathcal{M}))$, and if $2 < p_E \leq q_E < \infty$, then $E(\mathcal{M})$ is an interpolation space for the couple $(L_2(\mathcal{M}), L_q(\mathcal{M}))$ for some finite q. Thus by Corollary (4.1) or Corollary (4.2) in [12], we have for $2 < p_E \leq q_E < \infty$,

$$\Big\| \sum_n dx_n \otimes \varepsilon_n \Big\|_{E(\mathcal{M}\bar{\otimes}L^\infty(\Omega))} \approx \max\Big\{ \|S_c(x)\|_{E(\mathcal{M})},\ \|S_r(x)\|_{E(\mathcal{M})} \Big\};$$

and for $1 < p_E \leq q_E < 2$,

$$\Big\| \sum_n dx_n \otimes \varepsilon_n \Big\|_{E(\mathcal{M}\bar{\otimes}L^\infty(\Omega))} \approx \inf\Bigg\{ \Big\| \Big(\sum_n |a_n|^2 \Big)^{1/2} \Big\|_{E(\mathcal{M})} + \Big\| \Big(\sum_n |b_n|^2 \Big)^{1/2} \Big\|_{E(\mathcal{M})} \Bigg\},$$

where $dx_n = a_n + b_n$ and a_n, b_n belong to $E(\mathcal{M}_n)$. Then by Proposition 3.2, we immediately obtain the desired equivalence (3.1). To complete the proof, it is enough to set, for $n \geq 1$,

$$dy_n = a_n - \mathcal{E}_{n-1}(a_n), \quad dz_n = b_n - \mathcal{E}_{n-1}(b_n).$$

Then $(dy_n)_{n\geq 1}$ and $(dz_n)_{n\geq 1}$ are martingale difference sequences with $dx_n = dy_n + dz_n$. We will write e_{ij} for the usual matrix units of $M_n(\mathbb{C})$. According to Lemma 3.3,

$$\begin{aligned}
\Big\| \Big(\sum_{k=1}^n |dy_k|^2 \Big)^{1/2} \Big\|_{E(\mathcal{M})} &= \Big\| \sum_{k=1}^n dy_k \otimes e_{k1} \Big\|_{E(\mathcal{M}\bar{\otimes}\mathcal{B}(\ell^2))} \\
&= \Big\| \sum_{k=1}^n \big(a_k - \mathcal{E}_{k-1}(a_k)\big) \otimes e_{k1} \Big\|_{E(\mathcal{M}\bar{\otimes}\mathcal{B}(\ell^2))} \\
&\leq \Big\| \sum_{k=1}^n a_k \otimes e_{k1} \Big\|_{E(\mathcal{M}\bar{\otimes}\mathcal{B}(\ell^2))} \\
&\quad + \Big\| \sum_{k=1}^n \mathcal{E}_{k-1}(a_k) \otimes e_{k1} \Big\|_{E(\mathcal{M}\bar{\otimes}\mathcal{B}(\ell^2))} \\
&= \Big\| \Big(\sum_{k=1}^n |a_k|^2 \Big)^{1/2} \Big\|_{E(\mathcal{M})} + \Big\| \Big(\sum_{k=1}^n |\mathcal{E}_{k-1}(a_k)|^2 \Big)^{1/2} \Big\|_{E(\mathcal{M})} \\
&\leq C \Big\| \Big(\sum_{k=1}^n |a_k|^2 \Big)^{1/2} \Big\|_{E(\mathcal{M})}.
\end{aligned}$$

The same arguments are applied to $\|(\sum_{k=1}^n |dz_k|^2)^{1/2}\|_{E(\mathcal{M})}$, we then deduce

$$\begin{aligned}
&\left\|\left(\sum_{k=1}^n |dy_k|^2\right)^{1/2}\right\|_{E(\mathcal{M})} + \left\|\left(\sum_{k=1}^n |dz_k|^2\right)^{1/2}\right\|_{E(\mathcal{M})} \\
&\leq C\left\|\left(\sum_{k=1}^n |a_k|^2\right)^{1/2}\right\|_{E(\mathcal{M})} + C\left\|\left(\sum_{k=1}^n |b_k|^2\right)^{1/2}\right\|_{E(\mathcal{M})} \\
&\leq C\left\|\sum_n dx_n \otimes \varepsilon_n\right\|_{E(\mathcal{M}\bar{\otimes}L^\infty(\Omega))} \\
&\leq C\|x\|_{E(\mathcal{M})}.
\end{aligned}$$

We get the desired inequalities (3.2). The proof is complete. □

We now can restate Theorem 3.1 as follows.

Theorem 3.4. *Let $x = (x_n)_{n\geq 1}$ be any finite $E(\mathcal{M})$-martingale, $1 < p_E \leq q_E < 2$ or $2 < p_E \leq q_E < \infty$. Then x is bounded in $E(\mathcal{M})$ iff x belongs to $\mathcal{H}_E(\mathcal{M})$. Moreover, if this is the case,*

$$\|x\|_{E(\mathcal{M})} \approx \|x\|_{\mathcal{H}_E(\mathcal{M})}.$$

Consequently, $E(\mathcal{M}) = \mathcal{H}_E(\mathcal{M})$ with equivalent norm.

We end this section with one open problem, which is related to the noncommutative Burkholder inequality proved by Junge and Xu in [10], and extended to the frame of Lorentz spaces in [9]. At the time of this writing, it is still unknown if the conditional version of Theorem 3.1 is true.

Problem 3.5. Let $x = (x_n)_{n\geq 1}$ be a bounded $E(\mathcal{M})$-martingale.

(1) If $1 < p_E \leq q_E < 2$, then

$$\begin{aligned}
\|x\|_{E(\mathcal{M})} \approx \inf\Big\{&\Big\|\Big(\sum_k \mathcal{E}_{k-1}|dy_k|^2\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})} \\
&+ \Big\|\Big(\sum_k \mathcal{E}_{k-1}|dz_k^*|^2\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})} + \Big\|\sum_k dw_k \otimes e_n\Big\|_{E(\mathcal{M}\bar{\otimes}\ell^\infty)}\Big\}
\end{aligned}$$

where the infimum runs over all decompositions $dx_k = dy_k + dz_k + dw_k$ where (dy_k), (dz_k) and (dw_k) are all martingale difference sequences, and (e_n) denotes the canonical unit of ℓ^∞.

(2) If $2 < p_E \leq q_E < \infty$, then

$$\begin{aligned}
\|x\|_{E(\mathcal{M})} \approx \max\Big\{&\Big\|\Big(\sum_k \mathcal{E}_{k-1}|dx_k|^2\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})}, \\
&\Big\|\Big(\sum_k \mathcal{E}_{k-1}|dx_k^*|^2\Big)^{\frac{1}{2}}\Big\|_{E(\mathcal{M})}, \Big\|\sum_k dx_k \otimes e_n\Big\|_{E(\mathcal{M}\bar{\otimes}\ell^\infty)}\Big\}.
\end{aligned}$$

4. Examples. In this section we apply the main result to some concrete examples.

First consider the noncommutative weak L_p space $wL_p(\mathcal{M})$, $0 < p < \infty$, which is defined as the space of all measurable operator x such that

$$\|x\|_{wL_p(\mathcal{M})} := t^{\frac{1}{p}}\mu_t(x) < \infty.$$

Equipped with $\|x\|_{wL_p(\mathcal{M})}$, $wL_p(\mathcal{M})$ is a quasi-Banach space. However, for $p > 1$, $wL_p(\mathcal{M})$ can be renormed as a Banach space. Now take $E = wL_r(\mathcal{M})$ with $1 < r < \infty$. Then $p_E = q_E = r$ but E is q-concave for no finite q. Consequently, we can not obtain the Burkholder–Gundy inequality by the result in [1]. But by Theorem 3.1 in this paper, it is easy to deduce the following

Corollary 4.1. *Let $1 < r < \infty$, and $x = (x_n)_{n\geq 1}$ be a bounded $wL_r(\mathcal{M})$-martingale. Then*

(1) *for $2 < r < \infty$*

$$\|x\|_{wL_r(\mathcal{M})} \approx \max\{\|S_c(x)\|_{wL_r(\mathcal{M})}, \|S_r(x)\|_{wL_r(\mathcal{M})}\}\,; \tag{4.1}$$

(2) *for $1 < r < 2$*

$$\|x\|_{wL_r(\mathcal{M})} \approx \inf_{x=y+z}\{\|S_c(y)\|_{wL_r(\mathcal{M})} + \|S_r(z)\|_{wL_r(\mathcal{M})}\}, \tag{4.2}$$

where the infimum runs over all decompositions $dx_n = dy_n + dz_n$ with dy_n, dz_n being martingale difference sequences.

More generally, we consider the noncommutative weak Orlicz space $wL_\Phi(\mathcal{M})$. Let Φ be an Orlicz function on $[0,\infty)$, i.e., a continuous increasing and convex function on with $\Phi(0) = 0$ and $\lim_{t\to\infty}\Phi(t) = \infty$. Since Φ is convex, $\Phi'(t)$ is defined as the right derivative for each $t > 0$ except for a countable set. Two standard indices associated to an Orlicz function Φ are defined as follows,

$$a_\Phi =: \inf_{t>0}\frac{t\Phi'(t)}{\Phi(t)}, \quad b_\Phi =: \sup_{t>0}\frac{t\Phi'(t)}{\Phi(t)}.$$

It is known that $1 \leq a_\Phi \leq b_\Phi \leq \infty$. For an Orlicz function Φ, define $wL_\Phi(\mathcal{M})$ as the set of all measurable operators x such that $\sup_{t>0} t\Phi(\frac{\mu_t(x)}{c}) < \infty$ for some $c > 0$. Equipped with

$$\|x\|_{wL_\Phi(\mathcal{M})} := \inf\Big\{c > 0 : \sup_{t>0} t\Phi\Big(\frac{\mu_t(x)}{c}\Big) < 1\Big\},$$

$wL_\Phi(\mathcal{M})$ is called a noncommutative weak Orlicz space. Taking $\Phi(t) = t^p$, $wL_\Phi(\mathcal{M}) = wL_p(\mathcal{M})$. If Φ is an Orlicz function with $1 < a_\Phi \leq b_\Phi < \infty$, then $wL_\Phi(\mathcal{M})$ can be renormed as a Banach space. Consequently $wL_\Phi(\mathcal{M})$ can be regarded as a noncommutative symmetric space; see Remark 3.2 and Corollary 4.2 in [2] for detailed discussions. Noting that

$$a_\Phi \leq p_E \leq q_E \leq b_\Phi,$$

where p_E and q_E denote respectively the lower and upper Boyd indices of $E = wL_\Phi(\mathcal{M})$ (see Corollary 4.3 in [2]), we have the following inequalities for noncommutative weak Orlicz spaces.

Corollary 4.2. *Let* $1 < a_\Phi \leq b_\Phi < \infty$, *and let* $x = (x_n)_{n\geq 1}$ *be a bounded* $wL_\Phi(\mathcal{M})$*-martingale. Then*

(1) *for* $2 < a_\Phi \leq b_\Phi < \infty$

$$\|x\|_{wL_\Phi(\mathcal{M})} \approx \max\{\|S_c(x)\|_{wL_\Phi(\mathcal{M})}, \|S_r(x)\|_{wL_\Phi(\mathcal{M})}\}\,; \tag{4.3}$$

(2) *for* $1 < a_\Phi \leq b_\Phi < 2$

$$\|x\|_{wL_\Phi(\mathcal{M})} \approx \inf_{x=y+z}\{\|S_c(y)\|_{wL_\Phi(\mathcal{M})} + \|S_r(z)\|_{wL_\Phi(\mathcal{M})}\}, \tag{4.4}$$

where the infimum runs over all decompositions $dx_n = dy_n + dz_n$ *with* dy_n, dz_n *being martingale difference sequences.*

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References

[1] T. N. Bekjan, Φ-inequalities of noncommutative martingales, Rocky Mountain J. Math., **36** (2006), 401–412.

[2] T. N. Bekjan, et al. Noncommutative weak Orlicz spaces and martingales inequalities, Studa Math., **204** (2011), 195–212.

[3] C. Bennett and R. Sharply, Interpolation of operator. Academic press, 1988.

[4] V. I. Chilin and F. A. Sukochev, Symmetric spaces over semifinite von Neumann algebras, Dokl. Akad. Nauk SSSR, **313** (1990), 811–815.

[5] P. G. Dodds, T. K. Dodds, and B. de Pagter, Noncommutative Banach function spaces, Math. Z., **201** (1989), 583–579.

[6] P. G. Dodds, T. K. Dodds, and B. de Pagter, Noncommutative Kothe duality, Trans. Amer. Math. Soc., **339** (1993), 717–750.

[7] P. G. Dodds, T. K. Dodds, and B. de Pagter, Fully symmetric operators spaces, Integral Equations Operator Theory, **15** (1992), 942–972.

[8] T. Fack and H. Kosaki, Generalized s-number of τ-measure operators, Pacific Math. J., **123** (1986), 269–300.

[9] Y. Jiao, Burkholder's inequality in noncommutative Lorentz spaces, Proc. Amer. Math. Soc., **138** (2010), 2431–2441.

[10] M. Junge and Q. Xu, Noncommutative Burkholder/Rosenthal inequalities. Ann. Probab, **31** (2003), 948–995.

[11] N. Kalton and F. Sukochev, Symmetric norms and spaces of operators, J. Reine Angew. Math., **621** (2008), 81–121.

[12] C. Le Merdy and F. Sukochev, Radermacher averages on noncommutative symmetric spaces. J. Funct. Anal, **255** (2008), 3329–3355.

[13] J. Lindenstrauss and L. Tzafriri, Classical Banach spaces II, Springer-Verlag, Berlin, 1979.

[14] F. Lust-Piquard and G. Pisier, Noncommutative Khintchine and Paley inequalities. Ark. Mat., **29** (1991), 241–260.

[15] G. Pisier and Q. Xu, Noncommutative martingale inequalities. Comm. Math. Phys, **189** (1997), 667–698.

[16] N. Randrianatoanina, A weak type inequality for noncommutative martingales and applications. Proc. London Math. Soc., **91** (2005), 509–544.

[17] Q. Xu, Analytic functons with values in lattices and symmetric spaces of measurable operators, Math. Proc. Cambridge Philos. Soc., **109** (1991), 541–563.

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