

An optimal odd unimodular lattice in dimension 72

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Abstract. It is shown that if there is an extremal even unimodular lattice in dimension 72, then there is an optimal odd unimodular lattice in that dimension. Hence, the first example of an optimal odd unimodular lattice in dimension 72 is constructed from the extremal even unimodular lattice which has been recently found by G. Nebe.

Mathematics Subject Classification (2010). Primary 11H06;
Secondary 94B05.

Keywords. Optimal unimodular lattice, Odd unimodular lattice,
Theta series.

1. Introduction. A (Euclidean) lattice $L \subset \mathbb{R}^n$ in dimension n is *unimodular* if $L = L^*$, where the dual lattice L^* of L is defined as $\{x \in \mathbb{R}^n \mid (x, y) \in \mathbb{Z} \text{ for all } y \in L\}$ under the standard inner product (x, y) . A unimodular lattice is called *even* if the norm (x, x) of every vector x is even. A unimodular lattice which is not even is called *odd*. An even unimodular lattice in dimension n exists if and only if $n \equiv 0 \pmod{8}$ while an odd unimodular lattice exists for every dimension. Two lattices L and L' are *neighbors* if both lattices contain a sublattice of index 2 in common.

Rains and Sloane [9] showed that the minimum norm $\min(L)$ of a unimodular lattice L in dimension n is bounded by $\min(L) \leq 2\lfloor n/24 \rfloor + 2$ unless $n = 23$ when $\min(L) \leq 3$. We say that a unimodular lattice meeting the upper bound is *extremal*. Gaultier [5] showed that any unimodular lattice in dimension $24k$ meeting the upper bound has to be even, which was conjectured by Rains and Sloane. Hence, an odd unimodular lattice L in dimension $24k$ satisfies $\min(L) \leq 2k + 1$. We say that an odd unimodular lattice L in dimension $24k$ with $\min(L) = 2k + 1$ is *optimal*.

Shadows of odd unimodular lattices appeared in [2] and [3], and shadows play an important role in the study of odd unimodular lattices. For example, shadows are the main tool in [9]. Let L be an odd unimodular lattice, and let L_0 be the subset of vectors of even norm. Then L_0 is a sublattice of L of index 2. The *shadow* of L is defined as $S(L) = L_0^* \setminus L$. We define the *shadow minimum* of L as $\text{smin}(L) = \min\{(x, x) \mid x \in S(L)\}$.

The aim of this note is to show the following:

Theorem 1. *If there is an extremal even unimodular lattice Λ in dimension 72, then there is an optimal odd unimodular lattice L in dimension 72 with $\text{smin}(L) = 2$, which is a neighbor of Λ .*

Recently Nebe [8] has found an extremal even unimodular lattice in dimension 72. It was a long-standing question to determine the existence of such a lattice. As a consequence of Theorem 1, we have the following:

Corollary 2. *There is an optimal odd unimodular lattice L in dimension 72 with $\text{smin}(L) = 2$.*

2. An optimal odd unimodular lattice in dimension 72. The theta series $\theta_L(q)$ of a lattice L is the formal power series $\theta_L(q) = \sum_{x \in L} q^{(x,x)}$. Conway and Sloane [2, 3] showed that when the theta series of an odd unimodular lattice L in dimension n is written as

$$\theta_L(q) = \sum_{j=0}^{\lfloor n/8 \rfloor} a_j \theta_3(q)^{n-8j} \Delta_8(q)^j, \tag{1}$$

the theta series of the shadow $S(L)$ is written as

$$\theta_S(q) = \sum_{j=0}^{\lfloor n/8 \rfloor} \frac{(-1)^j}{16^j} a_j \theta_2(q)^{n-8j} \theta_4(q^2)^{8j} = \sum_i B_i q^i \text{ (say)}, \tag{2}$$

where $\Delta_8(q) = q \prod_{m=1}^{\infty} (1 - q^{2m-1})^8 (1 - q^{4m})^8$ and $\theta_2(q), \theta_3(q)$ and $\theta_4(q)$ are the Jacobi theta series [4]. As the additional conditions, it follows from [2] and [3] that

$$\begin{cases} B_r = 0 & \text{unless } r \not\equiv n/4 \pmod{2}, \\ \text{there is at most one nonzero } B_r & \text{for } r < (\min(L) + 2)/2, \\ B_r = 0 & \text{for } r < \min(L)/4, \\ B_r \leq 2 & \text{for } r < \min(L)/2. \end{cases} \tag{3}$$

Lemma 3. *Let L be an optimal odd unimodular lattice in dimension 72 with $\text{smin}(L) = 2$. Then the theta series of L and $S(L)$ are uniquely determined as*

$$\theta_L(q) = 1 + 27918336q^7 + 3165770864q^8 + \dots, \tag{4}$$

$$\theta_S(q) = 2q^2 + 127800q^6 + \dots, \tag{5}$$

respectively.

Proof. In (1) and (2), it follows from $\min(L) = 7$ that

$$\begin{aligned} a_0 &= 1, a_1 = -144, a_2 = 7056, a_3 = -136704, \\ a_4 &= 928656, a_5 = -1518336, a_6 = 136704. \end{aligned}$$

Since $S(L)$ does not have $\mathbf{0}$, $a_9 = 0$. Hence, we have the following possible theta series:

$$\theta_L(q) = 1 + (28901376 + a_7)q^7 + (3108623472 + a_8 - 24a_7)q^8 + \dots, \quad (6)$$

$$\theta_S(q) = \frac{a_8}{2^{24}}q^2 + \left(\frac{-15a_8}{2^{21}} - \frac{a_7}{2^{12}}\right)q^4 + \left(136704 + \frac{1767a_8}{2^{22}} + \frac{3a_7}{2^7}\right)q^6 + \dots, \quad (7)$$

If $x \in S(L)$ with $(x, x) = 2$ then $-x \in S(L)$. It follows from (3) that $B_2 = 2$ and $B_4 = 0$. Hence, we have that

$$a_7 = -15 \cdot 2^{16}, a_8 = 2^{25}.$$

Therefore, the theta series of L and $S(L)$ are uniquely determined. □

Now we start on the proof of Theorem 1. Let Λ be an extremal even unimodular lattice in dimension 72. Since Λ has minimum norm 8, there exists a vector $x \in \Lambda$ with $(x, x) = 8$. Fix such a vector x . Put

$$\Lambda_x^+ = \{v \in \Lambda \mid (x, v) \equiv 0 \pmod{2}\}.$$

If (x, y) is even for all vectors $y \in \Lambda$ then $\frac{1}{2}x \in \Lambda^* = \Lambda$ and $(\frac{1}{2}x, \frac{1}{2}x) = 2 < \min(\Lambda)$, which is a contradiction. Thus, Λ_x^+ is a sublattice of Λ of index 2, and there exists a vector $y \in \Lambda$ such that (x, y) is odd. Fix such a vector y . Define the lattice

$$\Gamma_{x,y} = \Lambda_x^+ \cup \left(\frac{1}{2}x + y\right) + \Lambda_x^+.$$

It is easy to see that $\Gamma_{x,y}$ is an odd unimodular lattice, which is a neighbor of Λ .

We show that $\Gamma_{x,y}$ has minimum norm 7. Since $\min(\Lambda_x^+) \geq 8$, it suffices to show that $(u, u) \geq 7$ for all vectors $u \in (\frac{1}{2}x + y) + \Lambda_x^+$. Let $u = \frac{1}{2}x + y + \alpha$ ($\alpha \in \Lambda_x^+$). Then we have

$$\left(u, \frac{1}{2}x\right) = \left(\frac{1}{2}x, \frac{1}{2}x\right) + \left(y, \frac{1}{2}x\right) + \left(\alpha, \frac{1}{2}x\right) \in \frac{1}{2} + \mathbb{Z}. \quad (8)$$

Here, we may assume without loss of generality that $(u, \frac{1}{2}x) \leq -\frac{1}{2}$. Then

$$\left(u + \frac{1}{2}x, u + \frac{1}{2}x\right) = (u, u) + 2 + 2\left(u, \frac{1}{2}x\right) \leq (u, u) + 1.$$

If $u + \frac{1}{2}x$ is the zero vector $\mathbf{0}$ then $(u, \frac{1}{2}x) = -(\frac{1}{2}x, \frac{1}{2}x) = -2$, which contradicts (8). Hence, $u + \frac{1}{2}x$ is a nonzero vector in Λ . Then we obtain $8 \leq (u, u) + 1$. Therefore, $\Gamma_{x,y}$ is an odd unimodular lattice with minimum norm 7, which is a neighbor of Λ .

It follows that $(\Gamma_{x,y})_0 = \Lambda_x^+$. For any vector $\alpha \in \Lambda_x^+$, $(\frac{1}{2}x, \alpha) = \frac{1}{2}(x, \alpha) \in \mathbb{Z}$. Hence, $\frac{1}{2}x$ is a vector of norm 2 in $S(\Gamma_{x,y})$. Therefore, we have Theorem 1. □

Remark 4. A similar argument can be found in [6] for dimension 48.

By Lemma 3, the theta series of $\Gamma_{x,y}$ and $S(\Gamma_{x,y})$ are uniquely determined as (4) and (5), respectively.

Remark 5. The extremal even unimodular lattice in [8], which we denote by N_{72} , contains a sublattice $\{(x, \mathbf{0}, \mathbf{0}), (\mathbf{0}, y, \mathbf{0}), (\mathbf{0}, \mathbf{0}, z) \mid x, y, z \in L_{24}\}$, where L_{24} is isomorphic to $\sqrt{2}\Lambda_{24}$ and Λ_{24} is the Leech lattice. Since Λ_{24} contains many 4-frames, N_{72} contains many 8-frames (see e.g. [1, 7] for undefined terms in this remark). Take one of the vectors of an 8-frame \mathcal{F} as x in the construction of $\Gamma_{x,y}$. It follows that $\Gamma_{x,y} \supset \Lambda_x^+ \supset \mathcal{F}$. Therefore, there is a self-dual \mathbb{Z}_8 -code C_{72} of length 72 and minimum Euclidean weight 56 such that $\Gamma_{x,y}$ is isomorphic to the lattice obtained from C_{72} by Construction A. A generator matrix of C_{72} can be obtained electronically from <http://sci.kj.yamagata-u.ac.jp/~mharada/Paper/z8-72-I.txt>.

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Received: 22 August 2011