Arch. Math. 96 (2011), 1–7

0003-889X/11/010001-7
published online January 11, 2011
DOI 10.1007/s00013-010-0215-0

Archiv der Mathematik

On p-nilpotence and solubility of groups

Qianlu Li and Xiuyun Guo

Abstract. Recall a result due to O. J. Schmidt that a finite group whose proper subgroups are nilpotent is soluble. The present note extends this result and shows that if all non-normal maximal subgroups of a finite group are nilpotent, then (i) it is soluble; (ii) it is p-nilpotent for some prime p; (iii) if it is not nilpotent, then the number of prime divisors contained in its order is between 2 and $k + 2$, where k is the number of normal maximal subgroups which are not nilpotent.

Mathematics Subject Classification (2010). 20D20; 20F19.

Keywords. p-nilpotence, Solubility, Normality.

1. Introduction. Let $|G|$ denote the order of a finite group G and $\pi(G)$ the set of prime divisors contained in $|G|$. Let $H < \cdot G$ denote that H is a maximal subgroup of G. Recall that a finite group is called a minimal non-nilpotent group, if every proper subgroup is nilpotent but the group itself not. Many people investigated the p-nilpotence and solubility of finite groups via the nilpotence of maximal subgroups. O. J. Schmidt showed that

Lemma 1.1 ([4, Theorem 9.1.9]). *Let G be a minimal non-nilpotent group. Then*

(i) *G is soluble,*
(ii) *$|G| = p^m q^n$ where p and q are unequal primes,*
(iii) *there is a unique Sylow p-subgroup P and a Sylow q-subgroup Q is cyclic. Hence $G = PQ$ and $P \trianglelefteq G$.*

This work was supported by the National Natural Science Foundation of China (Grant No. 10771132), the Natural Science Foundation of the Ministry of Education of China for the Returned Overseas Scholars (Grant No. 2008101) and the Natural Science Foundation of Shanxi for the Returned Overseas Scholars (Grant No. 200799).

Thompson gave a notable improvement of Schmidt's result: a finite group is soluble, if it possesses a nilpotent maximal subgroup of odd order (for details see [4, Theorem 10.4.2]). Suppose that p is a prime and P a finite p-group. Write $\Omega_i(P) = \langle x \in P/|x| = p^i\rangle$ and $\Omega(P) = \Omega_1(P)$, if $p > 2$; $\Omega(P) = \langle\Omega_1(P), \Omega_2(P)\rangle$, if $p = 2$. Deskins and Janko showed that Thompson's result remains true if the group has a nilpotent maximal subgroup in which the Sylow 2-subgroup has class at most two ([3, IV, Satz 7.4]). In 2000, Ballester-Bolinches and Guo [1] extending Thompson's result obtained: a finite group G with a nilpotent maximal subgroup M is soluble, if M has a Sylow 2-subgroup P such that $\Omega(P \cap G')$ is contained in the center of P. Very recently J. Shi, W. Shi, and C. Zhang proved that, if the P above is quaternion-free and one of the following conditions is true, then G is soluble: (i) $\Omega(P \cap O^2(G))$ is contained in the center of $Z(P)$; (ii) $\Omega_1(P \cap O^2(G))$ is contained in the center of $Z(O^2(G))$; (iii) $\Omega_1(P \cap G')$ is contained in the center of $Z(G')$. And they showed that the quaternion-free hypothesis is necessary [6].

We extend the results of O. J. Schmidt in another direction, i.e., we investigate the finite group with some non-nilpotent subgroups whose union covers the whole non-nilpotent part of the group and establish the following criterion for p-nilpotence and solubility:

Theorem 1.2 (Main Theorem). *Let G be a finite group. If all non-normal maximal subgroups of G are nilpotent, then*

I. *G is soluble*
II. *G is p-nilpotent for some prime p*
III. *If G is not nilpotent, then $2 \leq |\pi(G)| \leq k + 2$, where k is the number of normal maximal subgroups of G which are not nilpotent.*

2. Preliminaries. In this section we consider some characterizations of finite groups whose non-normal maximal subgroups are nilpotent and show that these groups contain minimal non-nilpotent groups.

Theorem 2.1. *Let G be a finite group. Then all of its non-normal maximal subgroups are nilpotent if and only if it is a nilpotent group or a minimal non-nilpotent group or it possesses some proper normal subgroups $H_1, \ldots, H_k$ which are not nilpotent and $H_i \not\leq H_j$ when $i \neq j$ such that any proper subgroup $S \not\leq H_i, i = 1, \ldots, k$, is nilpotent.*

Proof. Suppose that G is neither a nilpotent group nor a minimal non-nilpotent group but its non-normal maximal subgroups are nilpotent. Then we may suppose that $H_1, \ldots, H_k$ are all of the non-nilpotent maximal subgroups of G. It follows that H_i are normal. For any $S \leq G, S \not\leq H_i$, assume that $S \leq M < \cdot G$. Then $M \neq H_i$ and thus M is nilpotent so is S.

Conversely, it is trivial if G is a nilpotent group or a minimal non-nilpotent group, or if all of its maximal subgroups are normal. Thus we may suppose that G possesses k proper non-nilpotent normal subgroups $H_1, \ldots, H_k, H_i \not\leq H_j$ when $i \neq j$ such that any proper subgroup $S \not\leq H_i$ for $i = 1, \ldots, k$ is nilpotent. Suppose that M is a non-normal maximal subgroup of G. Then $M \neq H_i, i = 1, \ldots, k$. It follows that M is nilpotent, as required. □

Remark 2.2. Groups with all non-normal maximal subgroups nilpotent extend nilpotent groups and minimal non-nilpotent groups.

Let us see some examples.

Example 2.3. The symmetric group S_3 is a minimal non-nilpotent group.

Proof. Clearly S_3 is non-nilpotent but its proper subgroups are nilpotent since their possible orders are $1, 2$ or 3. □

Example 2.4. Let $G = S_3 \times <x>, x^5 = 1$, an external direct product of S_3 and $<x>$. Then all non-normal maximal subgroups of G are nilpotent but it is not a minimal non-nilpotent group.

Proof. Taking $H = S_3 \times 1$, then note that H is a maximal normal subgroup of G and it is a minimal non-nilpotent group by Example 2.3. Assuming that $y \in G,\ y \notin H$, then $y = (a, x^r)$, where $a \in S_3, x^r \neq 1$. Choosing an integer k and $b \in S_3$, then $(b, 1)y^k = (ba^k, x^{kr})$ and x^{kr}, ba^k respectively run through $<x>$ and S_3. Thus $H<y>= G$.

Let $L < G,\ L \nleq H$ and $y \in L\backslash H$. Then $1\times <x>\leq L$ but $H \nleq L$ since $H<y>= G$. Thus there exists $L_1 < S_3$ so that $L = L_1\times <x>$. By Example 2.3, L_1 is nilpotent, so is L. By Theorem 2.1 all non-normal maximal subgroups of G are nilpotent. □

Theorem 2.5. *Assume that G possesses k proper non-nilpotent normal subgroups $H_1, \ldots, H_k, H_i \nleq H_j$ when $i \neq j$ such that any proper subgroup $S \nleq H_i$ for $i = 1, \ldots, k$ is nilpotent. Then $H_i < \cdot G$ and $\bigcap_{i=1}^{k} H_i \neq 1$.*

Proof. First show that $H_i < \cdot G$. For $k = 1$, if there exists a subgroup L of G such that $H_1 < L < G$, then $L \nleq H_1$ following that L is nilpotent so is H_1, a contradiction. Hence $H_1 < \cdot G$.

Then assume that $k > 1$. It suffices to show that $H_1 < \cdot G$. Similarly, if $H_1 < L < G$, then for some $i > 1$, $L \leq H_i$ since otherwise, $L \nleq H_j$, $j = 1, \ldots, k$ following that L is nilpotent, so is H_1, a contradiction. Hence $L \leq H_i$ for some i and thus $H_1 < H_i$, a contradiction. So $H_1 < \cdot G$.

Now show that $\bigcap_{i=1}^{k} H_i \neq 1$. Writing $|G : H_i| = p_i$, then p_i are prime. If there are H_i, H_j such that $H_i \cap H_j = 1$, then $G = H_i \times H_j$ and thus $|G| = p_i p_j$. This follows that G is a nilpotent group, a contrary to G non-nilpotent. Hence any $H_i \cap H_j \neq 1$. By induction we can assume that $L_i =: \bigcap_{1\leq j\leq s,\ j\neq i} H_j \neq\ 1, i = 1, \ldots, k$. If $\bigcap_{j=1}^{k} H_j = 1$, then $L_i \nleq H_i$ for $i = 1, 2, \ldots, k$. It follows that $G = L_i \times H_i$ and thus $|L_i| = p_i$. On the other hand, note that $|G| = |G : H_1||H_1 : H_1 \cap H_2| \cdots |H_1 \cap \cdots \cap H_{k-2} : H_1 \cap \cdots \cap H_{k-1}||H_1 \cap \cdots \cap H_{k-1}| = p_1 p_2 \cdots p_k$ and $L_i \cap \prod_{j\neq i} L_j \leq L_i \cap H_i = 1$. Hence $G = L_1 \times \cdots \times L_k$ is nilpotent, a contradiction, following the result. □

Remark 2.6. For $k = 1, 2$, the normal hypothesis of H_i can be omitted. In fact, when $k = 1$, if there is $g \in G$ such that $H_1^g \neq H_1$, then H_1^g is nilpotent so is H_1, a contradiction. Hence $H_1 \trianglelefteq G$; when $k = 2$, if there is $g \in G$ such that $H_1^g \neq H_1$, then $H_1^g = H_2$ since otherwise, then H_1^g is nilpotent so is

H_1, a contradiction. Hence $H_1^g = H_2, i.e., |G : N_G(H_1)| = 2$. By Theorem 2.5 $N_G(H_1) = H_1$. Thus $H_1 \trianglelefteq G$, and similarly $H_2 \trianglelefteq G$.

Theorem 2.7. *Let G be a finite group with all non-normal maximal subgroups nilpotent. Let $A \trianglelefteq G$ and $A \leq \Phi(G)$. Then all non-normal maximal subgroups of G/A are nilpotent.*

Proof. It is trivial if G is a nilpotent group or a minimal non-nilpotent group. So by Theorem 2.1 we can suppose that $H_1, \ldots, H_k, H_i \not\leq H_j$ when $i \neq j$ are proper non-nilpotent normal subgroups of G such that any proper subgroup $S \not\leq H_i$ for $i = 1, \ldots, k$ is nilpotent. By Theorem 2.5, $H_i < \cdot G$. Thus $A \leq H_i, i = 1, \ldots, k$. For any proper subgroup $\overline{B} = B/A$ of G/A, if $\overline{B} \not\leq H_i/A$ for $i = 1, \ldots, k$, then $B \not\leq H_i$ and thus B is nilpotent, so is $\overline{B}$.

Now remain to show that H_i/A, $i = 1, \ldots, k$, are non-nilpotent by Theorem 2.1. It suffices to prove that H_1/A does. For any $p||H_1|$, assuming that $P \in \mathrm{Syl}_p H_1$, then $PA/A \in \mathrm{Syl}_p(H_1/A)$. If H_1/A is nilpotent, then PA/A char $H_1/A \trianglelefteq G/A$ and thus $PA/A \trianglelefteq G/A$. By Frattini argument $G = PAN_G(P) = AN_G(P) = N_G(P)$. Hence $P \trianglelefteq H_1$ and thus H_1 is nilpotent, a contradiction. Hence H_1/A is non-nilpotent, completing the proof. □

3. Proof of Theorem 1.2. Let p be a prime and P a p-group. Denote by $J(P)$ the subgroup generated by all abelian subgroups of P with maximal rank. The proof of the theorem rely heavily on the following fundamental criterion for p-nilpotence due to Thompson.

Lemma 3.1 ([4, Theorem 10.4.1]). *Let G be a finite group, p an odd prime and P a Sylow p-subgroup of G. Then G is p-nilpotent if and only if $N_G(J(P))$ and $C_G(Z(P))$ are p-nilpotent.*

Proof of Main Theorem.

I. Assume that the statement of (I) is false and choose for G a counterexample of smallest order. Then G is neither a nilpotent group nor a minimal non-nilpotent group by Lemma 1.1. It follows from Theorem 2.1 that G has k proper non-nilpotent normal subgroups $H_1, \ldots, H_k, H_i \not\leq H_j$ when $i \neq j$ such that any proper subgroup $S \not\leq H_i$ for $i = 1, \ldots, k$ is nilpotent. The proof of (I) is in four steps.

(1) Let A be a proper normal subgroup of G. Then for some $i, A \leq H_i$.

Assuming that $A \not\leq H_i, i = 1, \ldots, k$, then A is nilpotent. First show that $A \cap H_i = 1$. Since otherwise, we may suppose that $B =: A \cap H_1 > 1$. If $B = H_1$, then by the maximality of H_1 we have $H_1 = A$, a contradiction. Hence $B < H_1$. Then renumber the H_i such that B is properly contained in H_i for $i = 1, \ldots, r$, but B is not contained in H_i for $i > r$. For any $1 \leq t \leq r$, note that $B = A \cap H_1 \cap H_t \leq A \cap H_t$. If $B = A \cap H_t$, then $H_t/B = H_t/A \cap H_t \cong H_tA/A = G/A$ is not soluble; if $B < A \cap H_t$, then $G/A \cong H_t/A \cap H_t \cong H_t/B/(A \cap H_t)/B$. It follows that H_t/B is not soluble since $(A \cap H_t)/B$ is soluble but G/A not. Let L be a subgroup of G containing B such that $L/B \neq G/B$ and $L/B \not\leq H_t/B$ for $1 \leq t \leq r$. Then $L \not\leq H_t$ and $L \neq G$. On the other hand, for $i > r$ since $B \not\leq H_i$, then $L \not\leq H_i$. Hence L is nilpotent, so is L/B. By Theorem 2.1 G/B

is a group with all non-normal maximal subgroups nilpotent and thus by the minimal hypothesis G/B is soluble, so is G, a contradiction. Hence $A \cap H_i = 1$ and thus $G = A \times H_i$ for $i = 1, \ldots, k$. Fix i and suppose that $R < H_i$. Since $AR \not\leq H_j, j = 1, \ldots, k$ and $A \times R \neq G$, then AR is nilpotent, so is R. Hence H_i is a minimal non-nilpotent group and thus soluble by Lemma 1.1. This follows that G is soluble, a contradiction. Hence $A \leq H_i$ for some i.

(2) Let A be a nontrivial proper normal subgroup of G. Then A is not nilpotent.

Assuming that A is nilpotent, then $A \neq H_i, i = 1, \ldots, k$. By (1) we may renumber the H_i such that $A < H_i$ for $i = 1, \ldots, r$, but $A \not\leq H_i$ for $i > r$. Note that G is not soluble, neither is G/A. For any $H/A < \cdot G/A$ but $H/A \neq H_t/A, t = 1, \ldots, r$, obviously $H \neq H_j$ for $j > r$ and thus H is nilpotent. Therefore if each H_t/A is not nilpotent, then G/A with $H_1/A, \ldots, H_r/A$ forms a group with type of G. By the minimal hypothesis G/A is soluble, a contradiction. Hence some H_j/A is nilpotent. Thus H_j is soluble, so is G, another contradiction. Hence A is not nilpotent.

(3) G is a q-nilpotent group for some odd prime q.

By Theorem 2.5, $\bigcap_{i=1}^k H_i \neq 1$. Supposing that N is a minimal normal subgroup of G contained in $\bigcap_{i=1}^k H_i$ and B a minimal supplement of N in G, then $G = BN$ with $B \cap N \leq \Phi(B)$. Hence B is nilpotent since $B \not\leq H_i$ for $i = 1, \ldots, k$. Because $G/N \cong B/B \cap N$, then N is not soluble and thus $|\pi(N)| > 2$. Choose an odd prime divisor q of $|N|$ and suppose that Q_1 is a Sylow q-subgroup of N. By Frattini argument $G = NN_G(Q_1)$ and $N_G(Q_1) < G$ by the minimal normality of N. Suppose that $N_G(Q_1) \leq M < \cdot G$. Then M is nilpotent since $N_G(Q_1) \not\leq H_i$. By (2) the core of M in G must be 1. Supposing that $Q_2 \in \mathrm{Syl}_q M$ such that $Q_1 \leq Q_2$, then $N_G(Q_2) < G$ and thus $N_G(Q_2) = M$. On the other hand, there exists subgroup $Q \in \mathrm{Syl}_q G$ so that $Q_2 \leq Q$. Note that $N_Q(Q_2) = Q \cap M = Q_2$. By the normalizer condition $Q = Q_2$. Since $J(Q)$ char $Q \trianglelefteq M$, then $J(Q) \trianglelefteq M$. But $J(Q)$ cannot be normal in G, it follows that $N_G(J(Q)) = M$. For similar arguments above $C_G(Z(Q)) = M$. Thus by Lemma 3.1 G is q-nilpotent.

(4) Last contradiction.

By (3) $G = QO_{q'}(G)$ and all q'-elements of G belong to $O_{q'}(G)$. Obviously $N \not\leq O_{q'}(G)$ since $q||N|$. It follows that $N \cap O_{q'}(G) = 1$ by the minimal normality of N. Hence N is a q-group, a contrary to $|\pi(N)| > 2$. This completes the proof of (I).

II. It is trivial for any nilpotent group and minimal non-nilpotent group. So suppose that G is neither nilpotent nor minimal non-nilpotent. First consider

Case $\Phi(G) = 1$. Let N be a minimal normal subgroup of G contained in $\bigcap_{i=1}^k H_i$ and B a minimal supplement of N in G. Then $G = BN$ and B is nilpotent since $B \not\leq H_i, i = 1, \ldots, k$. By (I) N is soluble and thus $|N| = q^l$, where q is a prime. Further noting that $\Phi(G) = 1$, then $B \cap N = 1$. Suppose that $B = B_1 \times \cdots \times B_m$, where $B_i \in \mathrm{Syl}_{p_i}(B)$. Since G is non-nilpotent,

$|\pi(G)| > 1$. Thus there exists $p_i \neq q$, say $p_t \neq q$. Note that $B_1 \cdots B_{m-1} \trianglelefteq B$ and thus $NB_1 \cdots B_{m-1} \trianglelefteq G$ following that G is p_m-nilpotent.

Case $\Phi(G) > 1$. By Theorem 2.7 $\overline{G} = G/\Phi(G)$ has the same type of G. Noting that $\Phi(\overline{G}) = 1$, by the first case $\overline{G}$ is p-nilpotent for some prime p. That is, $\overline{G} = \overline{P}O_{p'}(\overline{G})$, where $\overline{P} = P/\Phi(G) \in \mathrm{Syl}_p(\overline{G})$. Note that $O_{p'}(\overline{G}) = O_{p'}(G)/\Phi(G)$. Then $G = PO_{p'}(G)$, as required.

III. Suppose that G is not nilpotent and thus $|\pi(G)| > 1$. Let N, B be as in (II). Then $G = BN$, where B is nilpotent and $|N| = q^l$. Suppose that $|G : H_i| = p_i$ for $i = 1, \dots, k$. Assume that $p_1, \dots, p_k$ are distinct and $|B| = p_1^{l_1} \cdots p_t^{l_t}$, where $t \geq k$ and each p_j is prime. Thus $B = B_1 \times \cdots \times B_t$, where B_j denotes the Sylow p_j-subgroup of B. If for some $i \leq k, NB_i$ is not nilpotent, then $G = NB_1 \cdots B_k$ since $NB_1 \cdots B_k \not\leq H_i$. Thus $|\pi(G)| \leq k+1$.

Now assume that $NB_i, i = 1, \dots, k$ are nilpotent. Then B_i char $NB_i \trianglelefteq G$ following that $B_i \trianglelefteq G$. Hence $t > k$ since G is not nilpotent.

(i) $q \neq p_j, j = k+1, \dots, t$.

If $NB_j, j = k+1, \dots, t$ are nilpotent, then $B_j \trianglelefteq G$, a contrary to G non-nilpotent. Hence some NB_j is not nilpotent. Therefore $G = NB_1 \cdots B_k B_j$ and thus $|\pi(G)| \leq k+2$.

(ii) q equals some $p_j, j > k$.

Assuming that $q = p_{k+1}$, then NB_{k+1} is the unique Sylow q-subgroup of G. Similar arguments to know that some $NB_j, j > k+1$, is not nilpotent. Then $G = NB_1 \cdots B_k B_{k+1} B_j$ and thus $|\pi(G)| \leq k+2$, completing the proof of the theorem. □

Corollary 3.2. *If the number of non-nilpotent maximal subgroups of a finite group is less 3, then it is soluble and p-nilpotent for some prime p.*

Proof. It directly follows from Theorem 1.2 and Remark 2.6. □

Acknowledgement. We are grateful to the referee for numerous suggestions for improvement and for pointing out several linguistic inaccuracies. We thank Professor F. Point for her valuable suggestions.

References

[1] A. Ballester-Bolinches and X. Guo, Some results on p-nilpotence and solubility of finite groups, J. Algebra **228** (2000), 491–496.

[2] X. Guo and K. P. Shum, p-nilpotence of finite groups and minimal subgroups, J. Algebra **207** (2003), 459–470.

[3] B. Huppert, Endliche Gruppen I, Springer-Verlag, Berlin/Heidelberg/New York, 1983.

[4] D. J. Robinson, A course in the theory of groups, 2nd ed., Springer-Verlag, New York/Heidelberg/Berlin, 1996.

[5] J. Shi, A note on p-nilpotence of finite groups, J. Algebra **241** (2001), 435–436.

[6] J. Shi, W. Shi, and C. Zhang, A note on p-nilpotence and solvability of finite groups, J. Algebra **321** (2009), 1555–1560.

Qianlu Li
Department of Mathematics,
Shanxi Datong University,
Datong, Shanxi 037009,
People's Republic of China
e-mail: qianluli@126.com

Xiuyun Guo
Department of Mathematics,
Shanghai University,
Shanghai 200444,
People's Republic of China
e-mail: xyguo@shu.edu.cn

Received: 15 September 2010