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The sectional genus of quasi-polarised varieties

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Abstract. T. Fujita conjectured that the sectional genus of a quasi-polarised variety is non-negative. We prove this conjecture. Using the minimal model program we also prove that if the sectional genus is zero the Δ-genus is also zero. This leads to a birational classification of quasi-polarised varieties with sectional genus zero.

1. Introduction. A quasi-polarised variety is a pair (X, A) where X is an integral projective scheme of dimension n over an algebraically closed field of characteristic zero and A a nef and big Cartier divisor on X. For $j \in \{1, \ldots, n\}$, we set

$$t^{[j]} := t(t+1)\ldots(t+j-1)$$

and $t^{[0]} := 1$. Then the Hilbert polynomial of (X, A) can be written as

$$\chi(X, \mathcal{O}_X(tA)) = \sum_{j=0}^{n} \chi_j(X, \mathcal{O}_X(A)) \frac{t^{[j]}}{j!},$$

where the $\chi_j(X, A)$ are integers. By the asymptotic Riemann–Roch theorem one has $\chi_n(X, A) = A^n$. The sectional genus of A is defined as $g(X, A) := 1 - \chi_{n-1}(X, A)$ and plays a prominent role in the classification theory of polarised varieties (cf. [8,9]). T. Fujita conjectured that the sectional genus is always non-negative. In this short note we prove this conjecture:

Theorem 1.1. *Let (X, A) be a quasi-polarised variety. Then $g(X, A) \geq 0$.*

If X is smooth and A is ample, the statement follows from the classification of varieties with long extremal rays by Fujita [7] and Ionescu [16]. Such a classification does not exist in the quasi-polarised setting and our estimate should be seen as a first step towards a more flexible "birational adjunction theory" that includes singular varieties. Indeed we will see that the condition $g(X, A) \leq 0$ yields some heavy restrictions on the global geometry of the variety.

If we work over the complex field $\mathbb{C}$, recent progress on the minimal model program allows us to go further. Recall that if (X, A) is a quasi-polarised pair, the Δ-genus is defined by

$$\Delta(X, A) := \dim X + A^n - h^0(X, \mathcal{O}_X(A)).$$

By [8, 1.1] one always has $\Delta(X, A) \geq 0$. We prove the following

Theorem 1.2. *Let (X, A) be a quasi-polarised normal, complex variety such that $g(X, A) = 0$. Then we have $\Delta(X, A) = 0$.*

Fujita pointed out that this statement would be a consequence of the existence and termination of flips, our work consists in showing that the special termination results of [2] are actually sufficient to conclude.

By [8, 1.1] the condition $\Delta(X, A) = 0$ implies that A is generated by global sections and induces a birational morphism $\mu : X \to X'$ onto a normal projective variety X' endowed with a very ample Cartier divisor A' such that $A \simeq \mu^* A'$ and $\Delta(X', A') = 0$. Polarised pairs (X', A') with Δ-genus zero are completely classified in [8, §5], so our statement determines the birational geometry of quasi-polarised pairs with sectional genus zero. The main step in the proof of Theorem 1.2 is the following statement which should be of independent interest (cf. [15]).

Proposition 1.3. *Let X be a normal, projective complex variety of dimension $n \geq 3$, and let A be a nef and big Cartier divisor on X such that $K_X + (n-1)A$ is not generically nef. Then (X, A) is birationally equivalent to one of the following quasi-polarised pairs*:

1. $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$; *or*
2. $(Q, \mathcal{O}_Q(1))$ *where $Q \subset \mathbb{P}^{n+1}$ is a hyperquadric*; *or*
3. $C_n(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ *a generalised cone over* $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$; *or*
4. *a* $(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$ *bundle over a smooth curve C.*

In particular if $g(X, A) = 0$, then $\Delta(X, A) = 0$.

2. Notation and basic material. We work over an algebraically closed field k of characteristic zero. For general definitions we refer to Hartshorne's book [14], for positivity notions to [19]. We will frequently use standard terminology from Mori's minimal model program, cf. [17] or [6].

A fibration is a proper, surjective morphism $\varphi : X \to Y$ between normal varieties such that $\dim X > \dim Y$ and $\varphi_* \mathcal{O}_X \simeq \mathcal{O}_Y$, that is all the fibres are connected.

Birationally, every projective manifold admits a fibration that separates the rationally connected part and the non-uniruled part: the *MRC-fibration* or *rationally connected quotient*.

Theorem 2.1. ([5,13,18]) *Let X be a projective manifold. Then there exists a projective manifold X', a birational morphism $\mu : X' \to X$ and a fibration $\varphi : X' \to Y$ onto a projective manifold Y such that the general fibre is rationally connected and the variety Y is not uniruled.*

Remark 2.2. We call Y the base of the MRC-fibration. This is a slight abuse of language since the MRC-fibration is only unique up to birational equivalence of fibrations. Since the dimension of Y does not depend on the birational model, it still makes sense to speak of the dimension of the base of the MRC-fibration.

Definition 2.3. Two quasi-polarised varieties (X_1, A_1) and (X_2, A_2) are said to be birationally equivalent if there exists a birational map $\psi : X_1 \dashrightarrow X_2$ such that $p_1^* A_1 \simeq p_2^* A_2$, where $p_i : \Gamma \to X_i$ denotes the natural morphism from the graph of ψ onto X_i.

Remark 2.4. It is straightforward to see that (X_1, A_1) and (X_2, A_2) are birationally equivalent and X_1 and X_2 are normal, their sectional and Δ-genus are the same.

Definition 2.5. ([20]) Let X be a projective manifold of dimension n, and let A be a Cartier divisor on X. We say that A is generically nef if

$$A \cdot H_1 \cdot \cdots \cdot H_{n-1} \geq 0$$

for $H_1, \ldots, H_{n-1}$ any collection of ample Cartier divisors on X.

Remarks 2.6.
1. Since the closure of the ample cone is the nef cone, the divisor A is generically nef if and only if

$$A \cdot H_1 \cdot \cdots \cdot H_{n-1} \geq 0$$

   for $H_1, \ldots, H_{n-1}$ any collection of *nef* Cartier divisors on X.
2. An effective divisor is generically nef.

Lemma 2.7. *Let X be a projective manifold of dimension n, and let $X \dashrightarrow Y$ be a rational fibration onto a non-uniruled variety of dimension m. Let A be a nef and big Cartier divisor on X. Then $K_X + (n - m + 1)A$ is generically nef.*

Proof. Let $\mu : X' \to X$ and $\varphi : X' \to Y$ be a holomorphic model of the fibration $X \dashrightarrow Y$. It is easy to see that $K_X + (n - m + 1)A$ is generically nef if $K_{X'} + (n - m + 1)\mu^* A$ is generically nef. In order to simplify the notation we suppose that $X = X'$.

Let us recall briefly why there exists a $j \in \{1, \ldots, n - m + 1\}$ such that

$$H^0(F, \mathcal{O}_F(K_F + jA|_F)) \neq 0.$$

Indeed the general fibre F has dimension $n - m$ and $A|_F$ is nef and big, so by the Kawamata–Viehweg vanishing theorem

$$h^0(F, \mathcal{O}_F(K_F + jA|_F)) = \chi(F, \mathcal{O}_F(K_F + jA|_F)) \qquad \forall j \in \mathbb{N}^*.$$

Since the Hilbert polynomial $\chi(F, \mathcal{O}_F(K_F + tA|_F))$ has at most $n-m$ roots, we obtain the non-vanishing. Thus the direct image sheaf $V := \varphi_* \mathcal{O}_X(K_{X/Y} + jA)$ is not zero for some $j \leq n - m + 1$ and direct image techniques [22, Chapter 2] show that V is weakly positive in the sense of Viehweg, i.e., there exists a non-empty Zariski open set $U \subset Y$ and an ample Cartier divisor H on Y such that for every $a \in \mathbb{N}^*$ there exists some $b \in \mathbb{N}^*$ such that $S^{ab} V \otimes H^{\otimes b}$ is globally generated over U, that is the evaluation map of global sections

$$H^0(Y, S^{ab} V \otimes H^b) \otimes \mathcal{O}_Y \to S^{ab} V \otimes H^b$$

is surjective over U. The canonical morphism

$$\varphi^* V = \varphi^* \varphi_* \mathcal{O}_X(K_{X/Y} + jA) \to \mathcal{O}_X(K_{X/Y} + jA)$$

is generically surjective, so $\mathcal{O}_X(K_{X/Y} + jA)$ is also weakly positive. Since $\mathcal{O}_X(K_{X/Y} + jA)$ has rank one this simply means that for every $a \in \mathbb{N}^*$ there exists some $b \in \mathbb{N}^*$ such that $\mathcal{O}_X(ab(K_{X/Y} + jA) + b\varphi^* H)$ has a global section. Hence we get

$$(ab(K_{X/Y} + jA) + b\varphi^* H) \cdot H_1 \cdot \dots \cdot H_{n-1} \geq 0$$

for $H_1, \dots, H_{n-1}$ any collection of ample Cartier divisors on X. Taking a arbitrarily large and dividing by b we see that the Cartier divisor $K_{X/Y} + jA$ is generically nef.

By [21] the canonical divisor of a non-uniruled variety is generically nef, so $K_X + jA = K_{X/Y} + jA + \varphi^* K_Y$ is generically nef. □

The following proposition is the key point in the proof of Theorem 1.1:

Proposition 2.8. *Let X be a projective manifold of dimension n, and let A be a nef and big Cartier divisor on X. If $K_X + (n-1)A$ is not generically nef, then*

$$\frac{1}{2}(K_X + (n-1)A) \cdot A^{n-1} + \chi(X, \mathcal{O}_X) = 0. \qquad (1)$$

Proof. Since A is nef and big, the hypothesis implies that $K_X + jA$ is not effective for $j \in \{1, \dots, n-1\}$. By the Kawamata–Viehweg vanishing theorem this implies that $1, \dots, n-1$ are roots of the Hilbert polynomial $\chi(X, \mathcal{O}_X(K_X + tA))$. The statement follows by comparing coefficients in the Riemann–Roch formula (cf. [15, Lemma 4.1] for details). □

3. Non-negativity of the sectional genus.

Proof of Theorem 1.1. Reduction to the smooth case. Let $\nu : X' \to X$ be the normalisation of X. Since ν is finite, the higher direct images $R^j \nu_* \mathcal{O}_{X'}$ vanish. Thus by the projection formula

$$\chi(X, \nu_* \mathcal{O}_{X'} \otimes \mathcal{O}_X(tA)) = \chi(X', \mathcal{O}_{X'}(t\nu^* A)).$$

Since X is integral we have an exact sequence

$$0 \to \mathcal{O}_X \to \nu_* \mathcal{O}_{X'} \to \mathcal{F} \to 0,$$

where $\mathcal{F}$ is a coherent sheaf supported on a finite union of proper subvarieties $D \subset X$. By the additivity of the Euler characteristic this implies

$$\chi(X', \mathcal{O}_{X'}(t\nu^* A)) = \chi(X, \mathcal{O}_X(tA)) + \chi(D, \mathcal{F} \otimes \mathcal{O}_D(tA)) \qquad \forall\, t \in \mathbb{Z},$$

hence

$$\chi_{n-1}(X', \mathcal{O}_{X'}(\nu^* A)) = \chi_{n-1}(X, \mathcal{O}_X(A)) + \chi_{n-1}(D, \mathcal{F} \otimes \mathcal{O}_D(tA)).$$

Since the irreducible components of D have dimension at most $n-1$ and $A|_D$ is nef, the asymptotic Riemann–Roch theorem [6, Proposition 1.31] implies that $\chi_{n-1}(D, \mathcal{F} \otimes \mathcal{O}_D(tA)) = \mathrm{rk}\mathcal{F} \cdot (A|_D)^{n-1} \geq 0$. Hence $g(X', \nu^* A) \leq g(X, A)$, so it is sufficient to show the conjecture for normal varieties. Moreover by

[8, Lemma 1.8] the sectional genus is a birational invariant of normal quasi-polarised varieties, so by resolution of singularities we can replace X' by a smooth variety. In order to simplify the notation we will suppose from now on that X is smooth. □

The computation. By [8] one has

$$g(X, A) = 1 + \frac{1}{2}(K_X + (n-1)A) \cdot A^{n-1} \tag{2}$$

If $K_X + (n-1)A$ is generically nef, the intersection $(K_X + (n-1)A) \cdot A^{n-1}$ is non-negative and the statement is trivial. Thus we can suppose without loss of generality that $K_X + (n-1)A$ is not generically nef. By Lemma 2.7 this implies that the base of the MRC-fibration has dimension at most one.

First case. X is rationally connected. In this case $\chi(X, \mathcal{O}_X) = 1$, so Theorem 1.1 follows from Equation (1) and Formula (2).

Second case. The base of the MRC-fibration has dimension one.

Let $\mu : X' \to X$ and $\varphi : X' \to Y$ be the MRC-fibration of X. Since the sectional genus is a birational invariant of normal varieties [8, Lemma 1.8] we can suppose without loss of generality that $X' = X$. We conclude with the following

Proposition 3.1. *Let X be a projective manifold of dimension n that admits a fibration $\varphi : X \to Y$ onto a smooth curve Y such that the general fibre is rationally connected. Let A be a nef and big divisor on X such that $K_X + (n-1)A$ is not generically nef. Then we have*

$$g(X, A) = h^1(X, \mathcal{O}_X) \geq 0.$$

Proof. By Proposition 2.8 we have

$$-\chi(X, \mathcal{O}_X) = \frac{1}{2}(K_X + (n-1)A) \cdot A^{n-1}.$$

Since the general φ-fibre is rationally connected, we have

$$h^i(X, \mathcal{O}_X) = h^0(X, \Omega_X^i) = 0 \qquad \forall\, i > 1.$$

So we get

$$g(X, A) = 1 + \frac{1}{2}(K_X + (n-1)A) \cdot A^{n-1} = 1 - \chi(X, \mathcal{O}_X) = h^1(X, \mathcal{O}_X).$$

□

Remark 3.2. T. Fujita also conjectured that the sectional genus satisfies the stronger estimate $g(X, A) \geq h^1(X, \mathcal{O}_X)$. Together with [11, Theorem 1.6] the proposition proves this conjecture in the case where the base of the MRC-fibration has dimension one.

4. The $\mathbf{\Delta}$-genus. In this whole paragraph we will work over the complex field $\mathbb{C}$.

We denote by $N^1(X)_{\mathbb{R}} := N^1(X) \otimes_{\mathbb{Z}} \mathbb{R}$ the vector space of $\mathbb{R}$-Cartier divisors modulo numerical equivalence, and by $N_1(X)_{\mathbb{R}}$ its dual, the space of 1-cycles modulo numerical equivalence. A divisor class $\alpha \in N^1(X)_{\mathbb{R}}$ is pseudoeffective if it is in the closure of the cone of effective divisors in $N^1(X)_{\mathbb{R}}$. By [3] this is equivalent to

$$\alpha \cdot C \geq 0$$

for every C a member of a covering family of curves for X. In particular a Cartier divisor that is not generically nef is not pseudoeffective.

For the proof of Proposition 1.3 we will need the following lemma.

Lemma 4.1. ([7, Lemma 2.5], [1, Theorem 2.1]) *Let X be a normal, projective variety of dimension n with at most terminal singularities. Let $\mu : X \to X'$ be an elementary contraction of birational type contracting a K_X-negative extremal ray Γ. Let $\mu^{-1}(y)$ be a fibre of dimension $r > 0$.*

If A is a nef and big Cartier divisor on X such that $A \cdot \Gamma > 0$, then

$$(K_X + rA) \cdot \Gamma \geq 0.$$

Proof of Proposition 1.3. Let $\nu : X' \to X$ be a desingularisation. Since $K_X + (n-1)A$ is not generically nef, $K_{X'} + (n-1)\nu^* A$ is not generically nef by [15, Lemma 2.9.c)]. In order to simplify the notation we will suppose from now on that X is smooth.

Step 1. The $K_X + (n-1)A$-MMP. Since $K_X + (n-1)A$ is not generically nef, it is not pseudoeffective. We will construct a birationally equivalent pair by using an appropriate MMP: since A is nef and big, there exists an effective $\mathbb{Q}$-divisor D on X such that $D \sim_{\mathbb{Q}} (n-1)A$ and the pair (X, D) is klt. Since $K_X + D \sim_{\mathbb{Q}} K_X + (n-1)A$ is not pseudoeffective we know by [2, Corollary 1.2.3] that we can run a $K_X + D$-MMP with scaling

$$(X_0, D_0) := (X, D) \stackrel{\mu_0}{\dashrightarrow} (X_1, D_1) \stackrel{\mu_1}{\dashrightarrow} \cdots \stackrel{\mu_s}{\dashrightarrow} (X_s, D_s)$$

such that (X_s, D_s) is a Mori fibre space, i.e., admits a $K_{X_s} + D_s$-negative contraction of fibre type $\varphi : X_s \to Y$. We claim that if $\mu_i : (X_i, D_i) \dashrightarrow (X_{i+1}, D_{i+1})$ is an elementary contraction contracting an extremal ray Γ_i in this MMP, then $D_i \cdot \Gamma_i = 0$. Moreover one has $D_{i+1} \sim_{\mathbb{Q}} (n-1)A_{i+1}$ with A_{i+1} a nef and big Cartier divisor such that $A_i = \mu_i^* A_{i+1}$. In particular the $K_{X_i} + D_i$-negative contraction is K_{X_i}-negative, so X_{i+1} has at most terminal singularities [6, Proposition 7.44]. □

Proof of the claim. We proceed by induction on i.

Since the contraction μ_i is $K_{X_i} + D_i$-negative, we have

$$(K_{X_i} + (n-1)A_i) \cdot \Gamma_i = (K_{X_i} + D_i) \cdot \Gamma_i < 0.$$

Since μ_i is birational Lemma 4.1 shows that $A_i \cdot \Gamma_i = 0$.

(a) If the contraction is divisorial, then μ_i is a morphism and $K_{X_i} = \mu_i^* K_{X_{i+1}} + E_i$ with E_i an effective $\mathbb{Q}$-divisor. Since $A_i \cdot \Gamma_i = 0$ there exists a nef and big Cartier divisor A_{i+1} on X_{i+1} such that $A_i = \mu_i^* A_{i+1}$. Thus we have $\Delta_{i+1} \sim_{\mathbb{Q}} (n-1)A_{i+1}$.

(b) If the contraction is small, denote by $\nu : X_i \to X'$ and $\nu_+ : X_{i+1} \to X'$ the birational morphisms defining the flip. Since $A_i \cdot \Gamma_i = 0$ there exists a nef and big Cartier divisor A' on X_s such that $A_i = \nu^* A'$. Thus $A_{i+1} := \nu_+^* A'$ is a nef and big Cartier divisor such that $\Delta_{i+1} \sim_{\mathbb{Q}} (n-1)A_{i+1}$.

Step 2. Classification. We have an elementary contraction $\varphi : X_s \to Y$ such that

$$K_{X_s} + (n-1)A_s \sim_{\mathbb{Q}} K_{X_s} + D_s$$

is φ-antiample. Let now M be an ample line bundle on Y such that $L := A_s \otimes \varphi^* M$ is ample. Then

$$K_{X_s} + (n-1)L$$

is φ-antiample, so the nefvalue [4, Definition 1.5.3] of the pair (X, L) is strictly larger than $n-1$. Thus by [4, Table 7.1] one of the following holds:

1. $(X, L) \simeq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$; or
2. $(X, L) \simeq (Q, \mathcal{O}_Q(1))$ where $Q \subset \mathbb{P}^{n+1}$ is a hyperquadric; or
3. $(X, L) \simeq C_n(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$ is a generalised cone over $(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$; or
4. (X, L) is a $(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(1))$ bundle over a smooth curve C.

Let us now see that if $g(X, A) = 0$, then $\Delta(X, A) = 0$. Since these are birational invariants of normal quasi-polarised varieties it is sufficient to show this statement for the pair (X_s, A_s). Note that in the cases 1)-3), the Picard number of X is one, so the elementary contraction φ maps to a point. In particular we have $L = A_s$ and one easily checks that $\Delta(X_s, A_s) = 0$.

In the fourth case we know by Proposition 3.1 that $g(X_s, A_s) = 0$ implies $Y \simeq \mathbb{P}^1$. Since

$$X \simeq \mathbb{P}(\varphi_* \mathcal{O}_{X_s}(L)) \simeq \mathbb{P}(\varphi_* \mathcal{O}_{X_s}(A_s)),$$

we have $A_s \simeq \mathcal{O}_{\mathbb{P}(\varphi_* \mathcal{O}_{X_s}(A_s))}(1)$. Since A_s is nef and big, we see that

$$\varphi_* \mathcal{O}_{X_s}(A_s) \simeq \oplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(b_i)$$

with $b_i \geq 0$ and $\sum_{i=1}^n b_i > 0$. Conclude with an elementary computation. □

Proof of Theorem 1.2. The statement is well-known if $\dim X \leq 2$, so we will suppose without loss of generality that $\dim X \geq 3$.

By Formula (2) we see that that $g(X, A) = 0$ implies that $K_X + (n-1)A$ is not generically nef. Conclude with Proposition 1.3. □

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