Arch. Math. 94 (2010), 411–418

0003-889X/10/050411-8
published online April 17, 2010
DOI 10.1007/s00013-010-0110-8

Archiv der Mathematik

Groups whose non-linear irreducible characters are rational valued

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Abstract. A finite group G all of whose nonlinear irreducible characters are rational is called a $\mathbb{Q}_1$-group. In this paper, we obtain some results concerning the structure of $\mathbb{Q}_1$-groups.

Mathematics Subject Classification (2010). 20C15.

Keywords. $\mathbb{Q}$-group, $\mathbb{Q}_1$-group, Frobenius group, Solvable group, Nilpotent group, Vanishing-off subgroup, Rational valued character.

1. Introduction. Let G be a finite group. G is called a rational group or a $\mathbb{Q}$-group if every complex irreducible character of G is rational valued. Some recent works about classification of these groups can be found in [2,3,8]. If every nonlinear complex irreducible character of G is rational valued, we say that G is a $\mathbb{Q}_1$-groups. For example, all abelian groups are $\mathbb{Q}_1$-group. The concept of $\mathbb{Q}_1$-group has first been considered in [1]. These groups are a generalization of $\mathbb{Q}$-groups, therefore any $\mathbb{Q}$-group is a $\mathbb{Q}_1$-group, but the converse is false. For example, the alternating group of degree 4 is a $\mathbb{Q}_1$-group which is not a $\mathbb{Q}$-group. There are many examples of nonabelian $\mathbb{Q}_1$-groups which are not $\mathbb{Q}$-groups. Consider, for example, the affine group $Af_1(q)$ of the finite field $F = GF(q)$ where $q > 3$. The group $Af_1(q)$ consists of the mappings $f_{a,b} : F \to F$ where $a, b \in F$, $a \neq 0$, and $f_{a,b}(x) = ax + b$ for all $x \in F$. Therefore $Af_1(q)$ is a group of order $q(q-1)$ and acts 2-transitively on F. In this case we obtain an irreducible character χ of $Af_1(q)$ of degree $q-1$ and $\chi(g) = |\mathrm{Fix}(g)| - 1$, where $\mathrm{Fix}(g)$ is the number of elements of F fixed by $g \in Af_1(q)$. Hence χ is a rational irreducible character of $Af_1(q)$. But $N = \{f_{1,b} | b \in F\}$ is a normal subgroup of $Af_1(q)$ with $\frac{Af_1(q)}{N} \cong F^*$ a cyclic group of order $q-1$. Therefore $Af_1(q)$ has $q-1$ linear characters χ_i, $1 \leq i \leq q-1$ with N in their kernels, and since $(q-1) + (q-1)^2 = q(q-1) = |Af_1(q)|$ the above characters χ_i, $1 \leq i \leq q-1$ and χ are all the irreducible characters of $Af_1(q)$. Since χ is rational-valued, $Af_1(q)$ is a $\mathbb{Q}_1$-group and $q > 3$ implies that $Af_1(q)$ is not a $\mathbb{Q}$-group.

In this paper we study $\mathbb{Q}_1$-groups and find some properties of these groups. We show, among other things, that the concepts of $\mathbb{Q}$-groups and $\mathbb{Q}_1$-groups coincide if G is a nonsolvable group.

We denote the set of all irreducible characters of G, the set of all nonlinear irreducible characters of G, and the set of all linear characters of G by $\mathrm{Irr}(G)$, $\mathrm{nl}(G)$ and $\mathrm{lin}(G)$ respectively. Other notations are standard and one may refer to [5].

2. Preliminaries

Definition 2.1. Let G be a finite group and g be an element of G. Then g is called a rational element if $\chi(g) \in \mathbb{Q}$ for any $\chi \in \mathrm{Irr}(G)$, otherwise g is called a nonrational element. Also $\chi \in \mathrm{Irr}(G)$ is called a rational character if $\chi(g) \in \mathbb{Q}$ for every $g \in G$.

By [5, p. 31], if G is a finite group and $g \in G$, then g is a rational element iff $g \sim g^m$ where $(m, o(g)) = 1$.

Remark 2.2. Let us consider the dihedral and generalized quaternion groups. Let D_{2n} denote the dihedral group of order $2n$. Since every linear character of D_{2n} is rational, D_{2n} is a $\mathbb{Q}_1$-group iff D_{2n} is a $\mathbb{Q}$-group. But D_{2n} is a $\mathbb{Q}$-group iff $n = 1, 2, 3, 4, 6$.

Also let Q_{2^n} denote generalized quaternion group of order 2^n. Since every linear character of Q_{2^n} is rational, Q_{2^n} is a $\mathbb{Q}_1$-group iff Q_{2^n} is a $\mathbb{Q}$-group. But Q_{2^n} is a $\mathbb{Q}$-group iff $n = 3$.

Theorem 2.3. [7, Proposition 8] *Let G be a finite group.*

(1) *A quotient of a $\mathbb{Q}$-group is a $\mathbb{Q}$-group.*
(2) *The direct product of a finite number of $\mathbb{Q}$-groups is a $\mathbb{Q}$-group.*

The vanishing-off subgroup of G is denoted by $V(G)$ and is defined as follows:
$V(G)$ is the subgroup generated by the elements of G where χ is not 0 for some $\chi \in \mathrm{nl}(G)$:

$$V(G) = \langle g \in G | \text{ there exists } \chi \in \mathrm{nl}(G) \text{ such that } \chi(g) \neq 0\rangle$$

This subgroup is a normal subgroup of G and $V(G)$ is the smallest subgroup such that χ vanishes on $G \setminus V(G)$ for every $\chi \in \mathrm{nl}(G)$ [6].

The concept of this subgroup is very useful for the study of $\mathbb{Q}_1$-groups.

Lemma 2.4. [6] *Let G be a nonsolvable group. Then $G = V(G)$.*

Lemma 2.5. [6, Lemma 3.3] *Let N be a normal subgroup of G so that G/N is nonabelian. Then $N \leq V(G)$ and $V(G/N) \leq V(G)/N$.*

Lemma 2.6. [4, Corollary B.1] *Let G be a noncyclic simple group. Then G is a rational group if and only if $G \cong \mathrm{Sp}_6(2)$ or $Q_8^+(2)'$.*

Theorem 2.7. [5, Theorem 12.3] *Let G be solvable and assume that G' is the unique minimal normal subgroup of G. Then all nonlinear irreducible characters of G have equal degree f and one of the following situations holds:*

(a) *G is a p-group, $Z(G)$ is cyclic and $G/Z(G)$ is elementary abelian of order f^2.*

(b) *G is a Frobenius group with abelian Frobenius complement of order f. Also, G' is the Frobenius kernel and is an elementary abelian p-group.*

3. Main results. In this section we will prove some properties of $\mathbb{Q}_1$-groups. In particular we will prove that if G is a $\mathbb{Q}_1$-group which is not a $\mathbb{Q}$-group, then G is solvable.

Lemma 3.1. *Let G be a $\mathbb{Q}_1$-group. If N is a normal subgroup of G, then G/N is a $\mathbb{Q}_1$-group.*

Proof. Let χ be an arbitary nonlinear character of G/N. Then the induced character of χ, i.e., $\hat{\chi}$ with definition $\hat{\chi}(g) = \chi(gN)$ is a nonlinear irreducible character of G. Since G is a $\mathbb{Q}_1$-group, we have that every nonlinear irreducible character of G is rational, hence $\hat{\chi}$ is rational, and therefore χ is rational. Thus every $\chi \in \mathrm{nl}(G/N)$ is rational implying G/N is a $\mathbb{Q}_1$-group. □

Lemma 3.2. *The direct product of a finite number of nonabelian groups $G_1, \ldots,$ $G_n (n \geq 2)$, is a $\mathbb{Q}_1$-group iff each G_i is a $\mathbb{Q}$-group.*

Proof. Let G_i be a $\mathbb{Q}$-group for each $1 \leq i \leq n$, then the direct product $G = G_1 \times \cdots \times G_n$ is a $\mathbb{Q}$-group by Theorem 2.3, therefore G is a $\mathbb{Q}_1$-group. Conversely, suppose that $G = G_1 \times \cdots \times G_n$ is a $\mathbb{Q}_1$-group, $n \geq 2$. By Lemma 3.1, each G_i is a $\mathbb{Q}_1$-group. If G_i is not a $\mathbb{Q}$-group for some i, $1 \leq i \leq n$, then there exists an irreducible linear character ψ of G_i such that ψ is nonrational. Hence there exists an element $g_i \in G_i$ such that $\psi(g_i) \notin \mathbb{Q}$. Since $n \geq 2$ we can choose $\chi \in \mathrm{nl}(G_j)(j \neq i)$, and let 1_{G_k} be the principal character of $G_k (k \neq i, j)$. Then $\theta = \psi\chi 1_{G_{k_1}} \cdots 1_{G_{k_t}} \in \mathrm{nl}(G)$ and $\theta(g_i) \notin \mathbb{Q}$, and hence θ is a nonrational irreducible character of G, and this is impossible, because we have assumed that G is a $\mathbb{Q}_1$-group. Therefore each G_i must be a $\mathbb{Q}$-group. □

Lemma 3.3. *Let G be a nonabelian $\mathbb{Q}_1$-group and g be a nonrational element of G. Then $\psi(g) = 0$ for every $\psi \in \mathrm{nl}(G)$.*

Proof. Since g is a nonrational element of G, there exists an irreducible character $\chi \in \mathrm{Irr}(G)$ such that $\chi(g) \notin \mathbb{Q}$. Since G is a $\mathbb{Q}_1$-group, χ should be linear. Suppose that $\psi \in \mathrm{nl}(G)$. Since $\chi \in \mathrm{lin}(G)$, we have $\chi\psi \in \mathrm{nl}(G)$ and therefore $\chi\psi(g) \in \mathbb{Q}$ for all $g \in G$. But $\chi\psi(g) = \chi(g)\psi(g)$ and $\chi(g) \notin \mathbb{Q}$, hence we must have $\psi(g) = 0$ and the proof is completed. □

Since every abelian group is a $\mathbb{Q}_1$-group, in the rest of this paper, we assume that all the $\mathbb{Q}_1$-groups are nonabelian. In [7], it is shown that the order of a $\mathbb{Q}$-group is even. We prove this fact about $\mathbb{Q}_1$-groups.

Theorem 3.4. *Let G be a $\mathbb{Q}_1$-group. Then $|G|$ is even.*

Proof. Assume that G has exactly r conjugacy classes with representatives $1, x_2, \ldots, x_r$. We claim that there is an element of G which is a rational element. Assume that for every i, $1 \leq i \leq r$, x_i is a nonrational element. Since G is a $\mathbb{Q}_1$-group, it follows from Lemma 3.3 that $\psi(x_i) = 0$ for every $\psi \in \mathrm{nl}(G)$. Now from $1 = [\psi, \psi] = (1/|G|) \sum_{g \in G} \psi(g)\overline{\psi(g)} = (1/|G|)\psi(1)^2$ it follows that $|G| = \psi(1)^2$, which is impossible. Therefore some x_i is rational, and hence our claim is proved. From the rationality of x_i it follows that $x_i \sim x_i^{-1}$, which means that $2||G|$. □

Corollary 3.5. *Let G be a p-group. If G is a $\mathbb{Q}_1$-group, then $p = 2$.*

Theorem 3.6. *Let G be a $\mathbb{Q}_1$-group. Then $Z(G)$ is an elementary abelian 2-group.*

Proof. If $Z(G) = 1$, then there is nothing to prove. Assume that $Z(G) \neq 1$ and $1 \neq z \in Z(G)$. We claim that z is a rational element of G. If z is a nonrational element, then by Lemma 3.3, $\psi(z) = 0$ for every $\psi \in \mathrm{nl}(G)$. Since $\psi \in \mathrm{Irr}(G)$, we know that $\psi_{Z(G)} = \psi(1)\lambda$ for some character $\lambda \in \mathrm{Irr}(Z(G))$. As λ is linear, we know that $\lambda(z) \neq 0$. This implies that $\psi(z) = \psi(1)\lambda(z) \neq 0$, which is a contradiction. Therefore z is a rational element and hence z is conjugate to z^{-1}. Since $z \in Z(G)$, we have $z^{-1} = z$ which means z is of order 2. Therefore $Z(G)$ is an elementary abelian 2-group. □

Lemma 3.7. *Suppose that G is a $\mathbb{Q}_1$-group. If $G = H \times K$ where H is abelian and K is non-abelian, then H is an elementary abelian 2-group.*

Proof. Since K is nonabelian, we can find $\gamma \in \mathrm{Irr}(K)$ with $\gamma(1) > 1$. Observe that $1_H \times \gamma \in \mathrm{Irr}(G)$, so γ is rational. If $\alpha \in \mathrm{Irr}(H)$, then we also have that $\alpha \times \gamma \in \mathrm{Irr}(G)$ and is nonlinear, so $\alpha \times \gamma$ is rational. It follows that α is rational, and hence H is a $\mathbb{Q}$-group. But an abelian group is a $\mathbb{Q}$-group iff it is an elementary abelian 2-groups. Therefore H is an elementary abelian 2-group. □

Theorem 3.8. *Suppose that G is a nonabelian $\mathbb{Q}_1$-group. If G is nilpotent, then G is a 2-group.*

Proof. Let $p_1 = 2, p_2, \ldots, p_r$ be distinct prime numbers which divide the order of G. Since G is nilpotent, it follows that $G = S_{p_1} \times \cdots \times S_{p_r}$, where S_{p_i} denote the Sylow p_i-subgroup of G. Now $G = S_2 \times A$ where $A = S_{p_2} \times \cdots \times S_{p_r}$. Since $A \cong G/S_2$ and quotients of $\mathbb{Q}_1$-groups are $\mathbb{Q}_1$-groups, we have that A is a $\mathbb{Q}_1$ group. If A is nonabelian, then by Theorem 3.4 we have $|A|$ is even, which is impossible. Hence A is abelian and by Lemma 3.7 must be an elementary abelian 2-group. Since $2 \nmid |A|$, A must be trivial and therefore G is a 2-group. □

The importance of vanishing-off subgroup in studying $\mathbb{Q}_1$-groups is clear by the following theorem:

Theorem 3.9. *Let G be a nonabelian finite group. Then G is a $\mathbb{Q}_1$-group iff every element of $V(G)$ is a rational element.*

Proof. Assume that G is $\mathbb{Q}_1$-group. Let g be a generator of $V(G)$. If g is a nonrational element of G, then by Lemma 3.3, for every $\psi \in \mathrm{nl}(G)$ we have $\psi(g) = 0$, which is impossible. Thus every generator of $V(G)$ is a rational element. Now let x be an arbitary element of $V(G)$. Then

$$x = g_1 g_2 \cdots g_m,$$

where $g_i (1 \leq i \leq m)$ are generators of $V(G)$. We claim that x is a rational element, otherwise there is an irreducible character $\chi \in \mathrm{Irr}(G)$ such that $\chi(x) \notin \mathbb{Q}$. Since G is a $\mathbb{Q}_1$-group, χ must be linear and hence

$$\chi(x) = \chi(g_1 g_2 \cdots g_m) = \chi(g_1)\chi(g_2) \cdots \chi(g_m).$$

By assumption we have $\chi(g_i) \in \mathbb{Q}$, hence $\chi(x) \in \mathbb{Q}$, which is a contradiction. Therefore x is a rational element. Hence every element of $V(G)$ is rational element.

Conversely, suppose that every element of $V(G)$ is rational. Let $\psi \in \mathrm{nl}(G)$ and x be an arbitary element of G. If $x \in V(G)$, then by hypothesis x is a rational element, therefore $\psi(x) \in \mathbb{Q}$. If $x \notin V(G)$, then by definition every nonlinear irreducible character vanishes on $G \setminus V(G)$, hence $\psi(x) = 0 \in \mathbb{Q}$, therefore $\psi(x) \in \mathbb{Q}$ for every $x \in G$ and ψ is a rational character. Now every $\psi \in \mathrm{nl}(G)$ is rational, hence G is a $\mathbb{Q}_1$-group. □

With respect to this theorem, a finite $\mathbb{Q}_1$-group G is a $\mathbb{Q}$-group iff $V(G) = G$. By definition of $\mathbb{Q}_1$-groups, every $\mathbb{Q}$-group is a $\mathbb{Q}_1$-group, but the converse is false. We show that if G is nonsolvable group, then these two concepts are the same.

Theorem 3.10. *Let G be a nonsolvable group. Then G is a $\mathbb{Q}_1$-group iff G is a $\mathbb{Q}$-group.*

Proof. It is sufficient to prove that if G is a nonsolvable $\mathbb{Q}_1$-group, then G is a $\mathbb{Q}$-group. Assume that G is nonsolvable, then by Lemma 2.4, $G = V(G)$ and by Theorem 3.9 every element of $V(G)$ is rational, hence G is a $\mathbb{Q}$-group. □

Corollary 3.11. *Let G be a noncyclic simple group. Then G is a $\mathbb{Q}_1$-group iff $G \cong \mathrm{Sp}_6(2)$ or $O_8^+(2)'$.*

Proof. Since G is a noncyclic simple group, it follows that G is nonsolvable. Therefore by Theorem 3.10, G is a $\mathbb{Q}_1$-group iff G is a $\mathbb{Q}$-group. Using Lemma 2.6, it follows that G is a $\mathbb{Q}_1$-group iff $G \cong \mathrm{Sp}_6(2)$ or $O_8^+(2)'$. □

Since every nonsolvable $\mathbb{Q}_1$-group is a $\mathbb{Q}$-group, for the reminder of this paper we focus on solvable $\mathbb{Q}_1$-groups.

Theorem 3.12. *Let G be a nonabelian solvable $\mathbb{Q}_1$-group. Then there exists a normal subgroup K of G such that $K \leq V(G)$ and one of the following occurs:*

(a) *G is a $\mathbb{Q}$-group.*
(b) *G/K is a nonabelian 2-group.*
(c) *G/K is a Frobenius group with cyclic complement.*

Proof. If G is not a $\mathbb{Q}$-group, then $V(G) < G$. Let K be the maximal normal subgroup of G such that $\overline{G} = G/K$ is nonabelian. Then $(G/K)'$ is the unique minimal normal subgroup of $\overline{G}$ and by Theorem 12.3 of [5] all nonlinear irreducible characters of G/K have equal degree f and one of the following cases occurs:
Case 1. $\overline{G}$ is a p-group, $Z(\overline{G})$ is cyclic and $\overline{G}/Z(\overline{G})$ is elementary abelian of order f^2.
Case 2. $\overline{G}$ is a Frobenius group with an abelian Frobenius complement of order f. Also $(G/K)'$ is the Frobenius kernel and is an elementary abelian p-group. Since G/K is nonabelian, by Lemma 2.5 we have $K \leq V(G)$. Assume that case (1) occurs. Then $\overline{G}$ is the quotient of a $\mathbb{Q}_1$-group, hence by Lemma 3.3, G is a $\mathbb{Q}_1$-group. Since the order of nonabelian $\mathbb{Q}_1$-groups are even, $|\overline{G}|$ is even and $\overline{G}$ must be a 2-group.

If case (2) occurs it leads to the case (c) and since Frobenius complement is abelian, it follows that it is cyclic. □

Let G be a nonabelian solvable $\mathbb{Q}_1$-group. According to Theorem 3.12, we call G a $\mathbb{Q}_1$-group of type 1 and type 2 if the cases (b) and (c) of above theorem occurs respectively.

Corollary 3.13. *Let G be a nonabelian solvable $\mathbb{Q}_1$-group. If G' is the unique minimal normal subgroup of G, then one of the following situations occurs:*

(1) *G is an extra-special 2-group.*
(2) *G is a Frobenius group with kernel G' and cyclic complement.*

Proof. Since G' is the unique minimal normal subgroup of G, conditions of Theorem 12.3 of [5] holds and one of the following cases occurs:
Case 1. G is a 2-group, $Z(G)$ is cyclic and $G/Z(G)$ is an elementary abelian group. Since $Z(G)$ is cyclic, by Theorem 3.6, $Z(G)$ is of order 2. Also G' as the unique minimal normal subgroup of G with $Z(G)$ cyclic, leads to $G' = Z(G)$. In addition $G/Z(G)$ is an elementary abelian 2-group, hence G is an extra-special 2-group.
Case 2. G is a Frobenius group with kernel G' and abelian complement. Also G' is an elementary abelian p-group for some p. □

Corollary 3.14. *Let G be a $\mathbb{Q}_1$-group of type 1 and p be an odd prime divisor of $|G|$. Then $p - 1||G|$.*

Proof. Let $p||G|$ and p be an odd prime. Since G/K is a 2-group, it follows that $p||K|$. By Theorem 3.12, $K \leq V(G)$ and by Theorem 3.10, each element of $V(G)$ is a rational element, thus each element of K is rational. If x is an element of order p, then rationality of x leads to $N_G(x)/C_G(x) \cong \operatorname{Aut}(x)$, therefore $|N_G(x)/C_G(x)| = |\operatorname{Aut}(x)| = p - 1$ which implies $p - 1||G|$. □

Corollary 3.15. *Let G be a $\mathbb{Q}_1$-group of type 1. Then Sylow 2-subgroups of G are nonabelian.*

Proof. This follows immediately from Theorem 3.12. □

Corollary 3.16. *Let G be a $\mathbb{Q}_1$-group of type 1. Then G has a normal 2-complement.*

Proof. By Theorem 3.12, $K \leq V(G)$, hence $|G : V|||G : K|$ and G/K is a 2-group, therefore $|G : V(G)| = 2^m$ for some m. Since $2||G : V(G)|$, it follows that $2|\chi(1)$ for every $\chi \in \mathrm{nl}(G)$. Using Theorem 12.2 of [5] we conclude that G has a normal 2-complement L. Also $G/L \cong S_2$ is nonabelian, therefore by Lemma 2.5, we have $L \leq V(G)$ and the proof is completed. □

Corollary 3.17. *Let G be a $\mathbb{Q}_1$-group of type 1. Then the Sylow 2-subgroup of G is a $\mathbb{Q}_1$-groups.*

Proof. If S_2 is a Sylow 2-subgroup of G, then by combining Corollary 3.16 with Lemma 3.1, we have that S_2 is a $\mathbb{Q}_1$-group. □

Acknowledgement. The authors would like to thank the referee for carefully reading the earlier version of this paper and making useful suggestions.

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Received: 8 August 2009