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Groups whose vanishing class sizes are not divisible by a given prime

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Abstract. Let G be a finite group. An element $g \in G$ is a vanishing element of G if there exists an irreducible complex character χ of G such that $\chi(g) = 0$: if this is the case, we say that the conjugacy class of g in G is a *vanishing conjugacy class* of G. In this paper we show that, if the size of every vanishing conjugacy class of G is not divisible by a given prime number p, then G has a normal p-complement and abelian Sylow p-subgroups.

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1. Introduction. A well-established research area in finite group theory consists in exploring the relationship between the structure of a group G and certain sets of positive integers, which are naturally associated to G. One of those sets, denoted by $\mathrm{cs}(G)$, is the set of conjugacy class sizes of the elements of G.

A classical remark concerning the influence of $\mathrm{cs}(G)$ on the group structure of G is the following: *if p is a prime number which does not divide any element of* $\mathrm{cs}(G)$*, then G has a central Sylow p-subgroup* (see [8, Theorem 33.4]). In view of that, one can ask whether particular subsets of $\mathrm{cs}(G)$ still encode non-trivial information on the structure of G. For instance, recall that an element g of G is said to be a *real element* of G if every irreducible complex character of G takes a real value on g. In [4] the following result is proved: *if the sizes of the conjugacy classes of real elements of G are all odd numbers, then G has a normal Sylow 2-subgroup.*

In this spirit, we focus on another subset of $\mathrm{cs}(G)$, also "filtered" by the set $\mathrm{Irr}(G)$ of irreducible complex characters of G. An element $g \in G$ is called a *vanishing element* of G if there exists $\chi \in \mathrm{Irr}(G)$ such that $\chi(g) = 0$. We shall say that the conjugacy class of such an element is a *vanishing conjugacy class* of G. Given a prime number p, we consider the situation in which no vanishing conjugacy class of G has size divisible by p. Since the symmetric group Sym(3) has only vanishing conjugacy classes of size 3, we cannot expect to obtain that G has a normal Sylow p-subgroup. Nevertheless, we can prove the following.

Theorem A. *Let G be a finite group, and p a prime number. If the size of every vanishing conjugacy class of G is not divisible by p, then G has a normal p-complement and abelian Sylow p-subgroups.*

The above result should also be compared with Corollary A of [6]: *if p is a prime number and the order of every vanishing element of G is not divisible by p, then G has a normal Sylow p-subgroup.*

Theorem A will be proved in Section 3, whereas in Section 4 we shall present some examples. It is worth mentioning that our proof of Theorem A relies on the classification of finite simple groups, since Corollary 2 of [7] is used.

2. Preliminaries. Throughout this note every group is assumed to be a finite group. If G is a group, we denote by $\pi(G)$ the set of prime divisors of the order of G, and by g^G the conjugacy class of an element g in G. Also, we denote by $\mathrm{Van}(G)$ the set of vanishing elements of G. Observe that, by a classical result of W. Burnside ([10, 3.15]), $\mathrm{Van}(G) = \emptyset$ if and only if G is abelian.

We start with an easy (and well-known) lemma.

Lemma 2.1. *Let G be a group, N a normal subgroup of G, and x an element of G.*

(a) *If x lies in N and p is a prime divisor of $|x^N|$, then p is a divisor of $|x^G|$.*
(b) *If p is a prime divisor of $|(xN)^{G/N}|$, then p is a divisor of $|x^G|$.*

Proof. We shall prove the contrapositive statements, so, we assume that p does not divide $|x^G|$. Therefore, there exists a Sylow p-subgroup P of G such that $P \leq \mathbf{C}_G(x)$. Now, as N is normal in G, we have $P \cap N \in \mathrm{Syl}_p(N)$ and $PN/N \in \mathrm{Syl}_p(G/N)$. Also, $P \cap N \leq \mathbf{C}_G(x) \cap N = \mathbf{C}_N(x)$ and $PN/N \leq \mathbf{C}_{G/N}(xN)$. As a consequence, if x lies in N then p does not divide $|x^N|$, and in any case p does not divide $|(xN)^{G/N}|$. □

Let q be a prime number, and χ an irreducible character of G. Recall that χ is said to be of q-*defect zero* if q does not divide $|G|/\chi(1)$. By a well-known result of R. Brauer ([10, Theorem 8.17]), if χ is an irreducible character of q-defect zero of G then, for every $g \in G$ such that q divides the order of g, we have $\chi(g) = 0$.

The following lemma, combined with Proposition 2.3, will be useful when handling nonabelian simple groups which *do not* have irreducible characters of q-defect zero for some prime q.

Lemma 2.2. *Let S be a nonabelian simple group, and assume that there exists a prime q such that S does not have any irreducible character of q-defect zero. Then there exist $\theta \in \mathrm{Irr}(S)$ and a conjugacy class C of S such that θ allows an extension to $\mathrm{Aut}(S)$, the size of C is divisible by every prime in $\pi(S)$, and θ vanishes on C.*

Proof. By Corollary 2 in [7], the prime q is 2 or 3, and S is isomorphic either to a sporadic simple group among M_{12}, $M_{22}, M_{24}, J_2, HS, Suz, Ru, Co_1, Co_3$, BM, or to some alternating group $\mathrm{Alt}(n)$ with $n \geq 7$.

Assume first that S is a sporadic simple group. For each possibility, we list a character and a conjugacy class satisfying the required condition (the notation is that of [3]).

Group	Character	Class
M_{12}	χ_7	$3B$
M_{22}	χ_3	$6A$
M_{24}	χ_3	$6A$
J_2	χ_6	$3B$
HS	χ_7	$5C$
Suz	χ_3	$8B$
Ru	χ_2	$6A$
Co_1	χ_2	$6H$
Co_3	χ_9	$6E$
BM	χ_2	$10D$

Next, we consider the case $S \simeq \mathrm{Alt}(n)$. Although (in view of [7, Corollary 2]) our assumptions yield some restrictions on n, we shall prove the desired conclusion for every $n \geq 7$. Observe that, for such an n, we have $\mathrm{Aut}(S) \simeq \mathrm{Sym}(n)$ (and we shall actually identify $\mathrm{Aut}(S)$ with $\mathrm{Sym}(n)$).

Recall that every irreducible character of $\mathrm{Sym}(n)$ corresponds naturally to a partition of n (if σ is such a partition, we shall denote by χ_σ the corresponding irreducible character), and that the restriction of χ_σ to $\mathrm{Alt}(n)$ is irreducible if and only if the Young diagram corresponding to σ is not symmetric (see [12]).

Now, if n is an odd number which is not a prime, choose x to be an n-cycle in S (thus $|x^S| = (n-1)!/2$), and choose the partition $\sigma = (n-3, 2, 1)$.

If n is a prime number, then choose x to be of type $(n-3, 2, 1)$ (thus $|x^S| = n!/2(n-3)$), and $\sigma = (n-2, 2)$.

If n is even and $n-1$ is not a prime, then choose x to be of type $(n-1, 1)$ (thus $|x^S| = n!/2(n-1)$), and $\sigma = (n-1, 1)$.

Finally, if n is even and $n-1$ is a prime, then choose x to be an element of type $(n-2, 2)$ (thus $|x^S| = n!/2(n-2)$), and $\sigma = (n-3, 2, 1)$.

In all cases, we get that every prime in $\pi(S)$ is a divisor of $|x^S|$ (see [1, p. 413]), the partition σ is not symmetric (therefore the restriction of χ_σ to S is irreducible), and $\chi_\sigma(x) = 0$. The latter condition can be easily checked using the Murnaghan-Nakayama formula ([12, 2.4.7]). □

Proposition 2.3 ([2, Lemma 5]). *Let G be a group, and $M = S_1 \times \cdots \times S_k$ a minimal normal subgroup of G, where every S_i is isomorphic to a nonabelian simple group S. If $\theta \in \mathrm{Irr}(S)$ extends to $\mathrm{Aut}(S)$, then $\theta \times \cdots \times \theta \in \mathrm{Irr}(M)$ extends to G.*

We close this preliminary section recalling the following statement.

Lemma 2.4 ([5, Lemma 2.6]). *Let G be a solvable group, and let F be the Fitting subgroup of G. If G/F is abelian, then $G \backslash F \subseteq \mathrm{Van}(G)$.*

3. A proof of Theorem A. The aim of this section is to prove Theorem A, which was stated in the Section 1.

Proof of Theorem A. We start by remarking that the assumptions of the theorem are inherited by factor groups. In fact, consider a normal subgroup N of G. Since every irreducible character of G/N can be regarded (by inflation) as an irreducible character of G which is constant on N-cosets, if xN is in $\mathrm{Van}(G/N)$ then every element in xN is in $\mathrm{Van}(G)$. Now, taking into account Lemma 2.1(b), it is clear that no vanishing conjugacy class of G/N can have size divisible by p.

In view of that, if G has a normal p-complement K, then it follows immediately that a Sylow p-subgroup P of G is abelian. In fact we have $P \simeq G/K$, whence P does not have any noncentral vanishing conjugacy class. On the other hand, no element $x \in \mathbf{Z}(P)$ is vanishing in P, since $|\chi(x)| = \chi(1) > 0$ for each $\chi \in \mathrm{Irr}(P)$. So, $\mathrm{Van}(P) = \emptyset$ and hence P is abelian.

It will then be enough to show that G has a normal p-complement, and we shall argue by induction on the order of the group.

If $\mathbf{O}_{p'}(G) \neq 1$ then, by the inductive hypothesis, $G/\mathbf{O}_{p'}(G)$ has a normal p-complement and we are done. Therefore we can assume $\mathbf{O}_{p'}(G) = 1$, and we shall aim to prove that G is a p-group. Let M be a minimal normal subgroup of G, and observe that p is a prime divisor of $|M|$.

Assume first that M is nonabelian. Therefore we have $M = S_1 \times \cdots \times S_k$, where every S_i is isomorphic to a nonabelian simple group S.

We claim that, if q is a prime and S has an irreducible character of q-defect zero, then every element of M of order divisible by q is a vanishing element of G. In fact, take $\theta \in \mathrm{Irr}(S)$ of q-defect zero, and set $\psi = \theta \times \cdots \times \theta$. It is clear that ψ is an irreducible character of q-defect zero of M. Choose now $\chi \in \mathrm{Irr}(G)$ lying over ψ. By Clifford's theorem, χ_M is a sum of G-conjugates ψ_i of ψ. As $\psi_i(1) = \psi(1)$, every ψ_i is an irreducible character of q-defect zero of M. Hence every ψ_i vanishes on every element of M of order divisible by q. If now x is such an element, we get $\chi(x) = \sum_i \psi_i(x) = 0$, whence $x \in \mathrm{Van}(G)$.

Next, if S has an irreducible character of q-defect zero for every prime q, then by the paragraph above every nontrivial element of M lies in $\mathrm{Van}(G)$. Since M does not have a central Sylow p-subgroup, there certainly exists $x \in M$ such that p divides $|x^M|$. By Lemma 2.1(a), p divides $|x^G|$ as well, a contradiction. On the other hand, let us assume that there exists a prime q such that S does not have any irreducible character of q-defect zero, thus Lemma 2.2 applies. As a consequence, there exist $x \in S$ with p dividing $|x^S|$, and $\theta \in \mathrm{Irr}(S)$ which extends to $\mathrm{Aut}(S)$, such that $\theta(x) = 0$. Now, by Proposition 2.3, the irreducible character $\theta \times \cdots \times \theta$ of M allows an extension χ to G. We clearly have $\chi(x) = 0$, so that $x \in M \cap \mathrm{Van}(G)$. Also, by Lemma 2.1(a) (applied twice) we get that p is a divisor of $|x^G|$, again a contradiction.

We conclude that M is a p-group, so that $N := \mathbf{O}_p(G) \neq 1$. The inductive hypothesis yields that G/N, whence G, is p-separable. As $\mathbf{O}_{p'}(G)$ is trivial, Hall-Higman Lemma 1.2.3 ([9, VI.6.5]) yields $\mathbf{C}_G(N) \leq N$. Thus, if $g \in G$ centralizes a Sylow p-subgroup of G, then g lies in N. In other words, for every element in $G \setminus N$, we get that p divides $|g^G|$ and so, by our assumptions, no element in $G \setminus N$ lies in $\mathrm{Van}(G)$. In particular we get $\mathrm{Van}(G/N) = \emptyset$, hence G/N is abelian. Now G is solvable, N is the Fitting subgroup of G (since $\mathbf{O}_{p'}(G) = 1$) and, by Lemma 2.4, every element of $G \setminus N$ lies in $\mathrm{Van}(G)$. We conclude that $G \setminus N = \emptyset$, hence $G = N$ is a p-group, and the proof is complete. □

4. Examples. Let p be a prime number and P an abelian p-group. It is clear that, for every choice of a p'-group K, the group $G = P \times K$ satisfies the assumptions of Theorem A. A group is of this type if and only if it has a central Sylow p-subgroup, and in this case *every* conjugacy class has size coprime to p. It is tempting to conjecture that some further structural information can be derived for groups as in Theorem A which are not of this type. The following example shows that, in any case, we cannot expect solvability.

Example 4.1. Let p and q be prime numbers such that $p \geq 7$ and $q \equiv 1 \mod 5p$ (such a q certainly exists for every choice of p, by Dirichlet's theorem on primes in an arithmetic progression). Let V be the additive group of the field $\mathrm{GF}(q^2)$. If λ is an element of order p in the multiplicative group of $\mathrm{GF}(q)$, then the scalar matrix Λ with eigenvalue λ yields a fixed-point-free automorphism of order p of V. Furthermore, since $q^2 \equiv 1 \mod 5$, there exists a subgroup K of $\mathrm{Aut}(V) \simeq \mathrm{GL}(2, q)$ acting fixed-point freely on V and such that $K \simeq \mathrm{SL}(2, 5)$ (see [9, V.8.8 b)]). Now, set $H = \langle \Lambda \rangle K$, and let G be the semidirect product $V \rtimes H$ formed according to the natural action. As $[\langle \Lambda \rangle, K] = 1$ and $(|\langle \Lambda \rangle|, |K|) = 1$, the group G is a Frobenius group with kernel V.

Since V is an abelian Sylow q-subgroup of G, Theorem A of [11] yields $\mathrm{Van}(G) \cap V = \emptyset$. But every element of $G \setminus V$ lies in a Frobenius complement, hence it centralizes a Sylow p-subgroup of G. We conclude that p does not divide the size of any vanishing conjugacy class of G.

Another natural question is whether, for a group G as in Theorem A, $G/\mathbf{Z}(G)$ is either a p'-group or a Frobenius group. Also in this case, the answer is negative.

Example 4.2. Denote by C_n a cyclic group of order n and set $H := C_5 \times \mathrm{Sym}(3)$. Consider an action of H on C_{11} whose kernel K is the Sylow 3-subgroup of H, and let $G = C_{11} \rtimes H$ be the corresponding semidirect product. Since G/K is a Frobenius group, for every element $x \in G$ lying in a conjugacy class of size divisible by 5 we get $o(x) \in \{11, 33\}$; in particular, every such x lies in the abelian normal Hall $\{3, 11\}$-subgroup of G. Consider an irreducible character χ of G, and observe that the degree d of χ is a divisor of 10. Now, $\chi(x)$ is a sum of d 33th roots of unity, therefore $\chi(x) \neq 0$ by the Main Theorem of [13]. It follows that 5 does not divide the size of any vanishing conjugacy class of G. (In fact, it can be checked that the sizes of the vanishing conjugacy classes of G are 11, 22 and 33.) Anyway, the centre $\mathbf{Z}(G)$ is trivial and G is not a Frobenius group.

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