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Gradient estimates for a nonlinear parabolic equation on Riemannian manifolds

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Abstract. Let M be a complete noncompact Riemannian manifold. We consider gradient estimates for the positive solutions to the following nonlinear parabolic equation

$$\frac{\partial u}{\partial t} = \Delta_f u + au\log u + bu$$

on $M \times [0, +\infty)$, where a, b are two real constants, f is a smooth real-valued function on M and $\Delta_f = \Delta - \nabla f \nabla$. Under the assumption that the N-Bakry-Emery Ricci tensor is bounded from below by a negative constant, we obtain a gradient estimate for positive solutions of the above equation. As an application, we obtain a Harnack inequality and a Gaussian lower bound of the heat kernel of such an equation.

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1. Introduction. Let (M, g) be a complete noncompact n-dimensional Riemannian manifold. The f-Laplacian (also called the drifting Laplacian, see [13,14]) is defined by

$$\Delta_f = \Delta - \nabla f \nabla,$$

where f is a smooth real-valued function on M. There is a naturally associated measure $dm = e^{-f} dvol$ for Ricci solitons, which makes the operator Δ_f self-adjoint. The measure dm also plays a key role in Perelman's entropy formulas

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for the Ricci flow [16]. A complete Riemannian manifold M is called a gradient Ricci soliton if

$$\mathrm{Ric}_f := \mathrm{Ric} + \nabla^2 f = \lambda g$$

for some smooth function f and constant λ, where ∇^2 is the Hessian and Ric is the Ricci tensor. We say that M is expanding, steady or shrinking if λ is negative, null or positive respectively. Ricci solitons are generalizations of Einstein metrics and naturally arise in the Ricci flow as special solutions. The N-Bakry-Emery Ricci tensor is

$$\mathrm{Ric}_f^N = \mathrm{Ric}_f - \frac{1}{N} df \otimes df$$

for $0 \le N \le \infty$ and $N = 0$ if and only if $f = 0$. In particular, Ric_f is also called the ∞-Bakry-Emery Ricci tensor.

In this paper, we consider the following evolution equation

$$\frac{\partial u}{\partial t} = \Delta_f u + au\log u + bu \tag{1.1}$$

on $M \times [0, +\infty)$, where a, b are two real constants and f is a smooth real-valued function on M. Here, we do not introduce the importance to study the elliptic equation (1.1), the readers who are interested in it see [4,12,19]. For more research of (1.1) on the Ricci flow, we refer to [3,9]. By replacing u by $e^{\frac{b}{a}}u$, Eq. (1.1) reduces to

$$\frac{\partial u}{\partial t} = \Delta_f u + au\log u. \tag{1.2}$$

Therefore, we only need to consider the Eq. (1.2). Generalizing the results of Ma [12], Yang [19], Chen et al. [4], and Huang et al. [8] studied Eq. (1.2) and obtained the following estimate:

Theorem 1.1. [8] *Let (M, g) be a complete noncompact n-dimensional Riemannian manifold with N-Bakry-Emery Ricci tensor bounded from below by the constant $-K =: -K(2R)$, where $R > 0$ and $K(2R) \ge 0$ in the metric ball $B_{2R}(p)$ with radius $2R$ around $p \in M$. Let u be a positive solution of (1.2) on $M \times [0, +\infty)$. Then*

(1) *if $a \le 0$, we have*

$$\frac{|\nabla u|^2}{u^2} - \beta\frac{u_t}{u} + a\beta\log u \le \frac{(N+n)\beta^2}{2(1-\delta)}\left(\frac{(N+n)c_1^2\beta^2}{\delta(\beta-1)R^2} + A + \frac{1}{t} - \frac{(a(\beta-1)-2K)}{2(\beta-1)}\right); \tag{1.3}$$

(2) *if $a \ge 0$, we have*

$$\frac{|\nabla u|^2}{u^2} - \beta\frac{u_t}{u} + a\beta\log u \le \frac{(N+n)\beta^2}{2(1-\delta)}\left(\frac{(N+n)c_1^2\beta^2}{\delta(\beta-1)R^2} + A + a + \frac{1}{t} + \frac{K}{\beta-1}\right). \tag{1.4}$$

Here c_1, c_2, δ, β *are positive constants with* $0 < \delta < 1$, $\beta > 1$ *and*

$$A = \frac{(n - 1 + \sqrt{nK}R)c_1 + c_2 + 2c_1^2}{R^2}. \tag{1.5}$$

As pointed out by Ruan [18], in order to derive the Gaussian lower bound of the heat kernel for the equation $\frac{\partial u}{\partial t} = \Delta u$, we must assume that $\beta \to 1$. However, when β is close to 1, $\frac{K}{\beta-1}$ will tend to infinity unless $K = 0$. Consequently, the inequalities in Theorem 1.1 do not work if $K > 0$ and β is close to 1. A natural idea is to derive a new estimate which is independent of K. In this paper, by setting $\beta(t)$ is a function of the time t, we can cancel the term of the Ricci curvature. Thus, we obtain the following results for Eq. (1.2):

Theorem 1.2. *Let* (M, g) *be a complete noncompact* n*-dimensional Riemannian manifold with* N*-Bakry-Emery Ricci tensor bounded from below by the constant* $-K =: -K(2R)$*, where* $R > 0$ *and* $K(2R) > 0$ *in the metric ball* $B_{2R}(p)$ *with radius* $2R$ *around* $p \in M$*. Let* u *be a positive solution of* (1.2) *on* $M \times [0, +\infty)$*. Then*

(1) *if* $a \le 0$*, we have*

$$\beta\frac{|\nabla u|^2}{u^2} + a\log u - \frac{u_t}{u} \le \frac{N+n}{2(1-\delta)\beta}\left(\frac{(N+n)c_1^2}{4\delta\beta(1-\beta)R^2} - \frac{a}{2} + A + \frac{1}{t}\right);$$

(2) *if* $a \ge 0$*, we have*

$$\beta\frac{|\nabla u|^2}{u^2} + a\log u - \frac{u_t}{u} \le \frac{N+n}{2(1-\delta)\beta}\left(\frac{(N+n)c_1^2}{4\delta\beta(1-\beta)R^2} + a + A + \frac{1}{t}\right),$$

where A *is given by* (1.5)*,* c_1, c_2, δ *are positive constants with* $0 < \delta < 1$ *and* $\beta = e^{-2Kt}$*.*

Letting $R \to \infty$, we can obtain the following global gradient estimates for the nonlinear parabolic equation (1.2):

Corollary 1.3. *Let* (M, g) *be a complete noncompact* n*-dimensional Riemannian manifold with* N*-Bakry-Emery Ricci tensor bounded from below by the constant* $-K$*, where* $K > 0$*. Let* u *be a positive solution of* (1.2) *on* $M \times [0, +\infty)$*. Then*

(1) *if* $a \le 0$*, we have*

$$\beta\frac{|\nabla u|^2}{u^2} + a\log u - \frac{u_t}{u} \le \frac{N+n}{2(1-\delta)\beta}\left(-\frac{a}{2} + \frac{1}{t}\right); \tag{1.6}$$

(2) *if* $a \ge 0$*, we have*

$$\beta\frac{|\nabla u|^2}{u^2} + a\log u - \frac{u_t}{u} \le \frac{N+n}{2(1-\delta)\beta}\left(a + \frac{1}{t}\right).$$

Here $0 < \delta < 1$ *and* $\beta = e^{-2Kt}$*.*

Corollary 1.4. *Let* (M, g) *be a complete noncompact* n*-dimensional Riemannian manifold with* N*-Bakry-Emery Ricci tensor bounded from below by the* *constant $-K$, where $K > 0$. Let $u(x,t)$ be a positive smooth solution to the equation*

$$\frac{\partial u}{\partial t} = \Delta_f u \tag{1.7}$$

on $M \times [0, +\infty)$. Then

(1) *the positive solution u satisfies*

$$\frac{u_t}{u} - e^{-2Kt}\frac{|\nabla u|^2}{u^2} + e^{2Kt}\frac{N+n}{2t} \geq 0; \tag{1.8}$$

(2) *for any points (x_1, t_1) and (x_2, t_2) on $M \times [0, +\infty)$ with $0 < t_1 < t_2$, we have the following Harnack inequality:*

$$u(x_1, t_1) \leq u(x_2, t_2) \left(\frac{t_2}{t_1}\right)^{\frac{N+n}{2}} e^{\phi(x_1, x_2, t_1, t_2) + B}, \tag{1.9}$$

where

$$\phi(x_1, x_2, t_1, t_2) = \inf_{\gamma} \int_{t_1}^{t_2} 4e^{2Kt}|\dot{\gamma}|^2 \, \mathrm{d}t, \quad B = \frac{N+n}{2}(e^{2Kt_2} - e^{2Kt_1});$$

(3) *suppose that $\lim_{t\to 0}(4\pi t)^{\frac{N+n}{2}} H(x, x, t) \geq C$, then the heat kernel $H(x, y, t)$ of the Eq. (1.7) satisfies*

$$H(x, y, t) \geq C(4\pi t)^{-\frac{N+n}{2}} e^{-\phi(x,y,t) - B'},$$

where

$$\phi(x, y, t) = \inf_{\gamma} \int_{0}^{t} 4e^{2Kt}|\dot{\gamma}|^2 \, \mathrm{d}t, \quad B = \frac{N+n}{2}(e^{2Kt} - 1).$$

Remark 1.5. In [7], Hamilton proved a gradient estimate for the heat equation $\frac{\partial u}{\partial t} = \Delta u$. Moreover, he obtain a Harnack inequality and a Gaussian lower bound of the heat kernel of $\frac{\partial u}{\partial t} = \Delta u$. His results are included in Corollary 1.4.

2. Proof of the theorem. Let u be a positive solution to (1.2). Set $w = \log u$, then w satisfies the equation

$$w_t = \Delta_f w + |\nabla w|^2 + aw.$$

Lemma 2.1. *Let (M, g) be a complete Riemannian manifold with Ricci curvature bounded below by the constant $-K := -K(2R)$, where $R > 0$ and $K(2R) > 0$ in the metric ball $B_{2R}(p)$. For a smooth function w defined on $M \times [0, +\infty)$ satisfies the equation*

$$w_t = \Delta_f w + |\nabla w|^2 + aw,$$

we have

$$\left(\Delta_f - \frac{\partial}{\partial t}\right) F \geq -2\nabla w \nabla F + t\Bigg\{ \frac{2\beta}{N+n}\left((\beta-1)|\nabla w|^2 - \frac{F}{t}\right)^2 \\ -a(\beta-1)|\nabla w|^2 \Bigg\} - aF - \frac{F}{t}, \tag{2.1}$$

where $F = t(\beta|\nabla w|^2 + aw - w_t)$ *and* $\beta = e^{-2Kt}$.

Proof. Define

$$F = t(\beta|\nabla w|^2 + aw - w_t),$$

where $\beta = e^{-2Kt}$. It is well known that for the N-Bakry-Emery Ricci tensor, we have the Bochner formula [10,11]:

$$\Delta_f|\nabla w|^2 \geq \frac{2}{N+n}|\Delta_f w|^2 + 2\nabla w\nabla(\Delta_f w) - 2K|\nabla w|^2.$$

Noticing

$$\Delta_f w_t = (\Delta_f w)_t = -2\nabla w\nabla w_t - aw_t + w_{tt}$$

and

$$\Delta_f w = -|\nabla w|^2 - aw + w_t = (\beta-1)|\nabla w|^2 - \frac{F}{t} \\ = \left(1 - \frac{1}{\beta}\right)(-aw + w_t) - \frac{F}{\beta t},$$

one gets

$$\begin{aligned} \Delta_f F &= t(\beta\Delta_f|\nabla w|^2 + a\Delta_f w - \Delta_f w_t) \\ &\geq t\Bigg\{\frac{2\beta}{N+n}|\Delta_f w|^2 + 2\beta\nabla w\nabla(\Delta_f w) - 2K\beta|\nabla w|^2 \\ &\quad + a\left(\left(1-\frac{1}{\beta}\right)(-aw+w_t) - \frac{F}{\beta t}\right) - (-2\nabla w\nabla w_t - aw_t + w_{tt})\Bigg\} \\ &= t\Bigg\{\frac{2\beta}{N+n}\left((\beta-1)|\nabla w|^2 - \frac{F}{t}\right)^2 - \frac{2}{t}\nabla w\nabla F - 2(a(\beta-1)+K\beta)|\nabla w|^2 \\ &\quad + 2\beta\nabla w\nabla w_t - a^2\left(1-\frac{1}{\beta}\right)w + a\left(2-\frac{1}{\beta}\right)w_t - w_{tt} - \frac{aF}{\beta t}\Bigg\} \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} F_t &= (\beta|\nabla w|^2 + aw - w_t) + t(2\beta\nabla w\nabla w_t + aw_t - w_{tt} - 2K\beta|\nabla w|^2) \\ &= \frac{F}{t} + t(2\beta\nabla w\nabla w_t + aw_t - w_{tt} - 2K\beta|\nabla w|^2). \end{aligned} \tag{2.3}$$

It follows from (2.2) and (2.3) that

$$\begin{aligned}\left(\Delta_f-\frac{\partial}{\partial t}\right)F\geq&-2\nabla w\nabla F+t\left\{\frac{2\beta}{N+n}\left((\beta-1)|\nabla w|^2-\frac{F}{t}\right)^2\right.\\&\left.-2a(\beta-1)|\nabla w|^2-a^2\left(1-\frac{1}{\beta}\right)w+a\left(1-\frac{1}{\beta}\right)w_t\right\}-\frac{aF}{\beta}-\frac{F}{t}\\=&-2\nabla w\nabla F+t\left\{\frac{2\beta}{N+n}\left((\beta-1)|\nabla w|^2-\frac{F}{t}\right)^2\right.\\&\left.-a(\beta-1)|\nabla w|^2\right\}-aF-\frac{F}{t}.\end{aligned}$$

This completes the proof of the Lemma 2.1. □

Let ξ be a cut-off function such that $\xi(r)=1$ for $r\leq 1$, $\xi(r)=0$ for $r\geq 2$, $0\leq\xi(r)\leq 1$, and

$$\begin{aligned}&0\geq\xi^{-\frac{1}{2}}(r)\xi'(r)\geq -c_1,\\&\xi''(r)\geq -c_2,\end{aligned}$$

for positive constants c_1 and c_2. Denote by $\rho(x)=d(x,p)$ the distance between x and p in M. Let

$$\varphi(x)=\xi\left(\frac{\rho(x)}{R}\right).$$

Making use of an argument of Calabi [2] (see also Cheng and Yau [5]), we can assume without loss of generality that the function φ is smooth in $B_{2R}(p)$. Then, we have

$$\frac{|\nabla\varphi|^2}{\varphi}\leq\frac{c_1^2}{R^2}. \tag{2.4}$$

It has been shown by Qian [17] that

$$\Delta_f(\rho^2)\leq n\left\{1+\sqrt{1+\frac{4K\rho^2}{n}}\right\}.$$

Hence, we have

$$\begin{aligned}\Delta_f\rho=&\frac{1}{2\rho}(\Delta_f(\rho^2)-2|\nabla\rho|^2)\\\leq&\frac{n-2}{2\rho}+\frac{n}{2\rho}\left(1+\sqrt{\frac{4K\rho^2}{n}}\right)\\=&\frac{n-1}{\rho}+\sqrt{nK}.\end{aligned}$$

It follows that

$$\begin{aligned}\Delta_f\varphi &= \frac{\xi''(r)|\nabla\rho|^2}{R^2} + \frac{\xi'(r)\Delta_f\rho}{R}\\ &\geq -\frac{(n-1+\sqrt{nK}R)c_1 + c_2}{R^2}.\end{aligned} \tag{2.5}$$

For the Laplacian comparison theorem, we refer to [1].

For $T \geq 0$, let (x,s) be a point in $B_{2R}(p)\times[0,T]$ at which φF attains its maximum value P, and we assume that P is positive (otherwise the proof is trivial). At the point (x,s), we have

$$\nabla(\varphi F) = 0, \quad \Delta_f(\varphi F) \leq 0, \quad F_t \geq 0.$$

It follows that

$$\varphi\Delta_f F + F\Delta_f\varphi - 2F\varphi^{-1}|\nabla\varphi|^2 \leq 0.$$

This inequality together with the inequalities (2.4) and (2.5) yields

$$\varphi\Delta_f F \leq AF, \tag{2.6}$$

where

$$A = \frac{(n-1+\sqrt{nK}R)c_1 + c_2 + 2c_1^2}{R^2}.$$

At (x,s), multiplying both sides of (2.1) by φ, we have

$$\begin{aligned}\varphi\Delta_f F \geq &-2\varphi\nabla w\nabla F + s\varphi\left\{\frac{2\beta}{N+n}\left((\beta-1)|\nabla w|^2 - \frac{F}{s}\right)^2\right.\\ &\left.- a(\beta-1)|\nabla w|^2\right\} - a\varphi F - \frac{\varphi F}{s}\\ \geq &-\frac{2c_1}{R}\varphi^{\frac{1}{2}}F|\nabla w| + s\varphi\left\{\frac{2\beta}{N+n}\left((\beta-1)|\nabla w|^2 - \frac{F}{s}\right)^2\right.\\ &\left.- a(\beta-1)|\nabla w|^2\right\} - a\varphi F - \frac{\varphi F}{s},\end{aligned} \tag{2.7}$$

where the last inequality in (2.7) used

$$-2\varphi\nabla w\nabla F = 2F\nabla w\nabla\varphi \geq -2F|\nabla w||\nabla\varphi| \geq -\frac{2c_1}{R}\varphi^{\frac{1}{2}}F|\nabla w|.$$

Combining (2.6) and (2.7), we obtain

$$\begin{aligned}&\frac{2s\varphi\beta}{N+n}\left((\beta-1)|\nabla w|^2 - \frac{F}{s}\right)^2\\ &\quad\leq \frac{2c_1}{R}\varphi^{\frac{1}{2}}F|\nabla w| + AF + as\varphi(\beta-1)|\nabla w|^2 + a\varphi F + \frac{\varphi F}{s}.\end{aligned}$$

Following Davies [6] (see also Negrin [15]), we set

$$\mu = \frac{|\nabla w|^2}{F}.$$

Then, we have

$$\begin{aligned}&\frac{2\varphi\beta((\beta-1)s\mu-1)^2}{(N+n)s}F^2\\ &\quad\le\frac{2c_1}{R}\varphi^{\frac12}\mu^{\frac12}F^{\frac32}+AF+as\varphi(\beta-1)\mu F+a\varphi F+\frac{\varphi F}{s}.\end{aligned}\tag{2.8}$$

Next, we consider two cases: (1) $a\le 0$; (2) $a\ge 0$.

(1) When $a\le 0$, multiplying both sides of the inequality (2.8) by $s\varphi$, we have

$$\begin{aligned}&\frac{2\beta((\beta-1)s\mu-1)^2}{N+n}(\varphi F)^2\\ &\le\frac{2c_1}{R}s\mu^{\frac12}(\varphi F)^{\frac32}+As\varphi F+as^2(\beta-1)\mu\varphi F+as\varphi^2F+\varphi F\\ &\le\frac{2c_1}{R}s\mu^{\frac12}(\varphi F)^{\frac32}+As\varphi F+as^2(\beta-1)\mu\varphi F+\varphi F\\ &\le\frac{2\delta\beta((\beta-1)s\mu-1)^2}{N+n}(\varphi F)^2+\frac{(N+n)c_1^2s^2\mu}{2\delta\beta((\beta-1)s\mu-1)^2R^2}\varphi F\\ &\quad+As\varphi F+as^2(\beta-1)\mu\varphi F+\varphi F,\end{aligned}$$

where $0<\delta<1$. That is,

$$\begin{aligned}P\le&\frac{N+n}{2(1-\delta)\beta((\beta-1)s\mu-1)^2}\\ &\times\left(\frac{(N+n)c_1^2s^2\mu}{2\delta\beta((\beta-1)s\mu-1)^2R^2}+As+as^2(\beta-1)\mu+1\right).\end{aligned}\tag{2.9}$$

Since

$$((\beta-1)s\mu-1)^2\ge 2(1-\beta)s\mu+1,$$

one gets

$$\begin{aligned}&\frac{1}{((\beta-1)s\mu-1)^2}\left(\frac{(N+n)c_1^2s^2\mu}{2\delta\beta((\beta-1)s\mu-1)^2R^2}+as^2(\beta-1)\mu\right)\\ &\quad\le\frac{(N+n)c_1^2s}{4\delta\beta(1-\beta)R^2}-\frac{as}{2}.\end{aligned}$$

Hence, the inequality (2.9) yields

$$P\le\frac{N+n}{2(1-\delta)\beta}\left(\frac{(N+n)c_1^2s}{4\delta\beta(1-\beta)R^2}-\frac{as}{2}+As+1\right).$$

Now, (1) of Theorem 1.2 follows easily from the inequality above.

(2) When $a \geq 0$, multiplying both sides of the inequality (2.8) by $s\varphi$, we have

$$\begin{aligned}
&\frac{2\beta((\beta-1)s\mu-1)^2}{N+n}(\varphi F)^2 \\
&\quad\leq \frac{2c_1}{R}s\mu^{\frac{1}{2}}(\varphi F)^{\frac{3}{2}} + As\varphi F + as^2(\beta-1)\mu\varphi F + as\varphi^2 F + \varphi F \\
&\quad\leq \frac{2c_1}{R}s\mu^{\frac{1}{2}}(\varphi F)^{\frac{3}{2}} + As\varphi F + as\varphi F + \varphi F \\
&\quad\leq \frac{2\delta\beta((\beta-1)s\mu-1)^2}{N+n}(\varphi F)^2 + \frac{(N+n)c_1^2 s^2\mu}{2\delta\beta((\beta-1)s\mu-1)^2R^2}\varphi F \\
&\qquad + As\varphi F + as\varphi F + \varphi F.
\end{aligned}$$

That is,

$$P \leq \frac{N+n}{2(1-\delta)\beta((\beta-1)s\mu-1)^2}\left(\frac{(N+n)c_1^2 s^2\mu}{2\delta\beta((\beta-1)s\mu-1)^2R^2} + As + as + 1\right).$$

Similarly, we obtain (2) of Theorem 1.2.

Proof of Corollary 1.4. (1) Letting $a = 0$ in the inequality (1.6), one gets

$$\beta\frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} \leq \frac{N+n}{2(1-\delta)\beta t}.$$

Then, the inequality (1) of Corollary 1.4 follows from letting $\delta \to 0$ in the inequality above.

(2) For any points (x_1, t_1) and (x_2, t_2) in $M \times (0, +\infty)$, we can take a curve $\gamma(t)$ parameterized with $\gamma(t_1) = x_1$ and $\gamma(t_2) = x_2$. One gets from (1.8) that

$$\begin{aligned}
&\log u(x_2,t_2) - \log u(x_1,t_1) \\
&\quad= \int_{t_1}^{t_2}((\log u)_t + \langle\nabla\log u, \dot\gamma\rangle)\,\mathrm{d}t \\
&\quad\geq \int_{t_1}^{t_2}\left(e^{-2Kt}|\nabla\log u|^2 - e^{2Kt}\frac{N+n}{2t} - |\nabla\log u||\dot\gamma|\right)\mathrm{d}t \\
&\quad\geq -\int_{t_1}^{t_2}\left(4e^{2Kt}|\dot\gamma|^2 + e^{2Kt}\frac{N+n}{2t}\right)\mathrm{d}t \\
&\quad\geq -\int_{t_1}^{t_2}\left(4e^{2Kt}|\dot\gamma|^2 + \frac{N+n}{2}\left(2Ke^{2Kt} + \frac{1}{t}\right)\right)\mathrm{d}t \\
&\quad= -\left(\int_{t_1}^{t_2} 4e^{2Kt}|\dot\gamma|^2\,\mathrm{d}t + \log\left(\frac{t_2}{t_1}\right)^{\frac{N+n}{2}} + \frac{N+n}{2}(e^{2Kt_2} - e^{2Kt_1})\right),
\end{aligned}$$

which means that

$$\log \frac{u(x_1,t_1)}{u(x_2,t_2)} \le \int_{t_1}^{t_2} 4e^{2Kt}|\dot{\gamma}|^2 \,\mathrm{d}t + \log\left(\frac{t_2}{t_1}\right)^{\frac{N+n}{2}} + \frac{N+n}{2}(e^{2Kt_2} - e^{2Kt_1}).$$

Therefore,

$$u(x_1,t_1) \le u(x_2,t_2)\left(\frac{t_2}{t_1}\right)^{\frac{N+n}{2}} e^{\phi(x_1,x_2,t_1,t_2)+B},$$

where

$$\phi(x_1,x_2,t_1,t_2) = \inf_{\gamma}\int_{t_1}^{t_2} 4e^{2Kt}|\dot{\gamma}|^2\,\mathrm{d}t, \quad B = \frac{N+n}{2}(e^{2Kt_2} - e^{2Kt_1}).$$

(3) Under the assumption of $\lim_{t\to 0}(4\pi t)^{\frac{N+n}{2}} H(x,x,t) \ge C$, we obtain

$$H(x,y,t) \ge C(4\pi t)^{-\frac{N+n}{2}} e^{-\phi(x,y,t)-B'}$$

from letting $t_1 \to 0$ in (1.9). This completes the proof of Corollary 1.4. □

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