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Vector variational principle

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Abstract. We prove an Ekeland's type vector variational principle for monotonically semicontinuous mappings with perturbations given by a convex bounded subset of directions multiplied by the distance function. This generalizes the existing results where directions of perturbations are singletons.

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1. Introduction. Let $K \subset Y$ be a closed convex cone in a locally convex space Y (i.e., $K + K \subset K$ and $\alpha K \subset K$ for all $\alpha \in [0, \infty)$) and let $f : X \longrightarrow Y$ be a mapping defined on a complete metric space X. Let $D \subset K$ be a closed convex bounded subset of K such that $0 \notin \mathrm{cl}\,(D + K)$. We show that under mild assumptions on f and D there is $\bar{x} \in X$ such that

$$(f(\bar{x}) - K) \cap (f(z) + d(z, \bar{x})D) = \emptyset \quad \text{for } z \in X \setminus \{\bar{x}\}. \tag{1.1}$$

If $Y = \mathbb{R}$, $K = [0, \infty)$, $D = \{\epsilon\}$, $\epsilon > 0$, then (1.1) takes the form

$$f(z) + \epsilon d(z, \bar{x}) > f(\bar{x}) \quad \text{whenever } z \neq \bar{x}.$$

This is the variational inequality from Ekeland's Variational Principle (EVP) [4,5,21,22]. Generalizations of EVP to metric spaces are given, e.g., in [3,12,13,20], to locally convex spaces and to general topological spaces see, e.g., [6]. Thus (1.1) can be regarded as an extension of EVP to vector-valued mappings.

EVP is a powerful tool with many applications in optimization, control theory, subdifferential calculus, nonlinear analysis, global analysis and mathematical economy. Therefore, several formulations of EVP for vector-valued and set-valued mappings are proved, e.g., in [1,2,7,8,14,15,17–19,24].

However, the common feature of those formulations is their 'directional' character, more precisely, instead of (1.1) it is proved that

$$(f(\bar{x}) - K) \cap (f(z) + d(z, \bar{x})k_0) = \emptyset \quad \text{for } z \in X \setminus \{\bar{x}\}, \tag{1.2}$$

with $0 \neq k_0$ chosen from the ordering cone K. Under additional assumptions on the cone K, Németh (cf. Theorem 6.1 of [19]) proved (1.2) with $d(z, \bar{x})k_0$ replaced by $r(z, \bar{x})$, where $r : X \times X \to K$ is a mapping such that: (i) $r(u, v) = 0 \Leftrightarrow u = v$, $(ii) r(u, v) = r(v, u)$, (iii) $r(u, z) \in r(u, v) + r(v, z) - K$ for any $u, v, z \in X$.

As in the case of scalar-valued mappings, the validity of EVP for vector- or set-valued mappings is usually verified on the basis of topological arguments and the core of the proofs is Cantor's intersection theorem. In contrast to that, in the present paper we prove EVP for vector-valued mappings by combining topological and set-theoretic methods. The main set-theoretic tool is Theorem 3.7 of [11] providing sufficient conditions for the existence of maximal elements of countably orderable sets [11, Definition 2.1]. The application of set-theoretic methods allows us to prove (1.1) which reduces to (1.2) when $D = \{k_0\}$, $0 \neq k_0 \in K$.

The organization of the paper is as follows. In Section 2 we present basic set-theoretical tools which are used in the sequel. In Section 3 we recall semicontinuity concepts for vector-valued functions. Section 4 contains the main result of the paper, namely Theorem 4.1 together with some examples. In Section 5 the distance $d(x, \bar{x})$ is estimated in the case when x is an approximate minimal point and $\bar{x}$ satisfies (1.1).

2. Preliminaries. The following set-theoretic concepts and facts are used in the sequel. For any nonempty set X and any relation $s \subset X \times X$ by $x\ s\ y$ we mean that $(x, y) \in s$ and we write $x\ s^*\ y$ if and only if there is a finite number of elements $x_1, \dots, x_n \in X$ such that

$$x = x_1,\ x_1\ s\ x_2, \dots, x_{n-1}\ s\ x_n,\ x_n = y$$

Relation s^* is the transitive closure of s. If s is transitive, then $s = s^*$.

An element $x \in X$ is *maximal with respect to a relation* $s \subset X \times X$, we say x is s-maximal, if for every $y \in X$

$$x\ s\ y \Rightarrow y\ s^*\ x.$$

When s is a partial order, the above definition coincides with the usual definition of maximality, i.e., x is s-maximal if for every $y \in X$

$$x\ s\ y \Rightarrow\ x = y,$$

we refer to [9–11] for more information on maximality with respect to non-transitive relations.

The following definition is essential for the results of Section 3.

Definition 2.1. (Definition 2.1 in [11]) A set X with a relation $s \subset X \times X$ is *countably orderable* with respect to s if for every nonempty subset $W \subseteq X$ the existence of a well ordering relation μ on W such that

$$v \ \mu \ w \ \Rightarrow \ v \ s^* \ w \quad \text{for every } v, w \in W, \ v \neq w \tag{2.1}$$

implies that W is at most countable.

Let us recall the existence theorem for countably orderable sets. Its proof has been originally given in [11] (see also Proposition 3.2.8, p. 90 of [14]).

Theorem 2.2. *Let X be countably orderable set by a relation $s \subseteq X \times X$. Assume that for any sequence $(x_i) \subset X$ satisfying*

$$x_i \ s \ x_{i+1} \quad \textit{for } i \in \mathbb{N}$$

there are a subsequence $(x_{i_k}) \subset (x_i)$ and an element $x \in X$ such that

$$x_{i_k} \ s \ x \quad \textit{for all } k \in \mathbb{N}. \tag{2.2}$$

Then there exists an s^*-maximal element of X.

Corollary 2.3. *Let X be countably orderable set by a transitive relation $s \subseteq X \times X$. Assume that for any sequence $(x_i) \subset X$ satisfying*

$$x_i \ s \ x_{i+1} \quad \textit{for } i \in \mathbb{N}$$

there are a subsequence $(x_{i_k}) \subset (x_i)$ and an element $x \in X$ such that

$$x_{i_k} \ s \ x \quad \textit{for all } k \in \mathbb{N}. \tag{2.3}$$

Then there exists an s-maximal element of X.

3. Semicontinuous vector-valued functions. Let X be a topological space and let Y be a locally convex space, i.e., Y is a linear topological space with a local base consisting of convex neighborhoods of the origin, see [16]. Let $K \subset Y$ be a convex cone in Y. For any $x, y \in Y$ define

$$x \leq_K y \Leftrightarrow y - x \in K.$$

We say that a subset $D \subset Y$ is *semi-complete* if every Cauchy sequence contained in D has a limit in D, see [16]. According to [9] we say that a function $f : X \to Y$ is *monotonically semicontinuous (msc) with respect to* K at $x \in X$ if for every sequence $(x_i) \subset X$, $x_i \to x$, satisfying

$$f(x_{i+1}) \leq_K f(x_i) \quad i \in \mathbb{N}$$

we have $f(x) \leq_K f(x_i)$ for $i \in \mathbb{N}$.

We say that f is msc on X if f is msc at each $x \in X$. To the best of our knowledge monotonically semicontinuous functions were first introduced in [19, p. 674] via nets. They were used in [9] to prove a general form of Weierstrass's theorem for vector-valued functions and in [19] to formulate vector variational principle (see also Corollary 3.10.19 of [14]). The sequential definition given above is more adequate for our purposes.

It is not our aim to provide a detailed discussion of relationships between msc functions and other important classes of functions. However, it is worth mentioning that the class of msc functions is sufficiently large to encompass level-closed functions and order lower semicontinuous functions [8].

Let us recall that a function f is *level-closed* if for each b in Y the set $\{x \in X : f(x) \leq_K b\}$ is closed in X. A function f is *order lower semi-continuous* (o-lsc, for short) [8] at $x_0 \in X$ if for each sequence $(x_i) \subset X$ converging to x_0 for which there exists a sequence $(\varepsilon_i) \subset Y$ converging to 0 such that the sequence $(f(x_i)+\varepsilon_i)$ is non-increasing, i.e., $f(x_i)+\varepsilon_i \geq_K f(x_{i+1})+\varepsilon_{i+1}$, there exists a sequence $(g_i) \subset Y$ converging to 0 such that $f(x_0) \leq_K f(x_i) + g_i$ for every $i \in \mathbb{N}$.

Level-closed functions are contained in the class of msc functions. Indeed, suppose on the contrary that a function f defined on a metric space X is not msc at some x_0. Then, there exists a sequence $(x_j) \subset X$, $x_j \to x_0$, such that $f(x_{j+1}) \leq_K f(x_j)$ for all $j \in \mathbb{N}$ and $f(x_0) \notin f(x_i) - K$ for some $i \in \mathbb{N}$. Then $f(x_0) \notin f(x_{i+l}) - K$ for all $l \in \mathbb{N}$. This shows that the level set corresponding to the level $b = f(x_i)$ is not closed. In general, msc functions are not level-closed. Below we prove that o-lsc functions are also included in the class of msc functions.

Proposition 3.1. *Let X be a metric space and Y be a real Banach space. Let $K \subset Y$ be a closed convex cone and let $f : X \longrightarrow Y$ be an o-lsc function at x_0. Then f is msc at x_0.*

Proof. Let $x_i \longrightarrow x_0$ and $f(x_i) \geq_K f(x_{i+1})$ for every $i \in \mathbb{N}$. By o-lsc, there is a sequence $(g_i) \subset Y$ such that $g_i \longrightarrow 0$ and $f(x_0) \leq_K f(x_i) + g_i$. Hence, for every $i, l \in \mathbb{N}$ we get

$$f(x_i) \in f(x_{i+l}) + K \subset f(x_0) + K - g_{i+l},$$

thus $f(x_i) \geq_K f(x_0)$ for every $i \in \mathbb{N}$. □

We say that f is *K-bounded* if there exists a bounded subset M of Y such that $f(X) \subset K + M$. The topological closure is denoted by cl.

4. Vector variational principle. The following theorem provides vector Ekeland's principle.

Theorem 4.1. *Let X be a complete metric space and let Y be a locally convex space. Let $K \subset Y$ be a closed and convex cone in Y and let $D \subset K$ be a closed semi-complete convex and bounded subset of K such that $0 \notin \mathrm{cl}\,(D + K)$.*

Let $f : X \to Y$ be msc with respect to K and K-bounded. Then for every $x \in X$ there exists $\bar{x} \in X$ such that

(i) $(f(x) - K) \cap (f(\bar{x}) + d(x, \bar{x})D)) \neq \emptyset$,
(ii) $(f(\bar{x}) - K) \cap (f(z) + d(z, \bar{x})D) = \emptyset$ *for every $z \neq \bar{x}$.*

Proof. Let $x \in X$ and

$$A := \{v \in X \ : \ (f(x) - K) \cap (f(v) + d(x, v)D) \neq \emptyset\}.$$

Let $r \subset X \times X$ be a relation defined as follows: for any $u, v \in X$

$$u \ r \ v \ \Leftrightarrow (f(u) - K) \cap (f(v) + d(u, v)D) \neq \emptyset.$$

By the convexity of D, the relation r is transitive. In fact, let us observe that $u \ r \ v$ and $v \ r \ w$ entails $f(u) = f(v) + k_v + d(u, v)d_v$ and $f(v) = f(w) + k_w +$

$d(v,w)d_w$ for some $k_v, k_w \in K$ and $d_v, d_w \in D$. If $d(u,v) + d(v,w) = 0$ then $f(u) = f(w)$ and we are done. If $d(u,v) + d(v,w) > 0$, then

$$\frac{d(u,v)}{d(u,v) + d(v,w)} d_v + \frac{d(v,w)}{d(u,v) + d(v,w)} d_w \in D,$$

so $d(u,v)d_v + d(v,w)d_w \in (d(u,v) + d(v,w))D$, hence $d(u,v)d_v + d(v,w)d_w \in d(u,v)D + K$. Thus $f(u) \in f(w) + d(u,w)D + K$, which implies $u\ r\ w$.

The main step of the proof consists in applying Theorem 2.2 to show that A has an r-maximal element. Observe that $A = \{v \in X : x\ r\ v\}$, and hence any r-maximal element of A is an r-maximal element of X.

Since $0 \notin \mathrm{cl}\,(D + K)$, by separation arguments, there exists $y^* \in Y^*$ such that

$$\langle y^*, d\rangle + \langle y^*, k\rangle > \varepsilon > 0$$

for some $\varepsilon > 0$ and any $d \in D$, $k \in K$. Hence, $\inf_{d\in D}\langle y^*, d\rangle > 0$ and $\langle y^*, k\rangle \geq 0$ for any $k \in K$. Moreover,

$$u\ r\ v,\ u \neq v \Rightarrow \langle y^*, f(u)\rangle > \langle y^*, f(v)\rangle. \tag{4.1}$$

Indeed, if $u\ r\ v$, then $f(u) = f(v) + d(u,v)d + k$, where $d \in D$ and $k \in K$. Consequently, $\langle y^*, f(u)\rangle = \langle y^*, f(v) + d(u,v)d + k\rangle > \langle y^*, f(v)\rangle$.

We start by showing that A is countably orderable with respect to r. Let $\emptyset \neq W \subseteq A$ be any subset of A well ordered by a relation μ satisfying (2.1). Then for any $u, v \in W$, $u \neq v$

$$u\ \mu\ v \quad\Rightarrow\quad u\ r\ v \quad\Rightarrow\quad \langle y^*, f(u)\rangle > \langle y^*, f(v)\rangle.$$

Thus, $y^* \circ f(W) \subset \mathbb{R}$ is well ordered by the relation '$>$' and therefore $y^* \circ f(W)$ is at most countable. This entails that W is at most countable since $y^* \circ f$ is a one-to-one mapping on W.

Now we show that (2.3) holds for A, i.e., for any sequence $(x_n) \subset A$

$$\forall n \in \mathbb{N},\ x_n\ r\ x_{n+1} \quad\Rightarrow\quad \exists x_0 \in A:\ \forall n \in \mathbb{N},\ x_n\ r\ x_0.$$

Let us observe that if $x_m = x_{m+1} = \cdots$ for some $m \in \mathbb{N}$, then by putting $x_0 := x_m$ we get (2.3) immediately. So, it is enough to consider the case when $\sum_{i=n}^{\infty} d(x_i, x_{i+1}) > 0$ for every $n \in \mathbb{N}$. By the definition of r, for each $n \in \mathbb{N}$

$$x_n\ r\ x_{n+1} \Leftrightarrow f(x_n) - f(x_{n+1}) = k_n + d(x_n, x_{n+1})d_n \in K,$$

where $d_n \in D$ and $k_n \in K$. Moreover, in view of the K-boundedness of f, for any $m \in \mathbb{N}$

$$\begin{aligned} f(x_1) &= f(x_1) - f(x_2) + f(x_2) - \cdots - f(x_{m+1}) + f(x_{m+1}) \\ &= f(x_{m+1}) + \sum_{i=1}^{m} k_i + \sum_{i=1}^{m} d(x_i, x_{i+1})d_i \\ &\in M + K + \sum_{i=1}^{m} d(x_i, x_{i+1})d_i. \end{aligned}$$

Hence, for each $m \in \mathbb{N}$ we have

$$\langle y^*, f(x_1)\rangle \geq \inf_{z\in M}\langle y^*, z\rangle + \sum_{i=1}^{m} d(x_i, x_{i+1}) \inf_{d\in D}\langle y^*, d\rangle.$$

By the boundedness of M and the fact that $\inf_{d\in D}\langle y^*, d\rangle > 0$, the sequence $(\sum_{i=1}^{m} d(x_i, x_{i+1}))$ is bounded from above, hence the series

$$\sum_{i=1}^{\infty} d(x_i, x_{i+1})$$

converges. Let us fix $n \in \mathbb{N}$. By the boundedness of D, the sequence

$$\left(\frac{\sum_{i=n}^{m} d(x_i, x_{i+1})d_i}{\sum_{i=n}^{m} d(x_i, x_{i+1})}\right)_{m=n}^{\infty} \subset D$$

is a Cauchy sequence (the denominators are positive for m large enough). By the semi-completness of D, it converges to a point from D when $m \longrightarrow \infty$. Moreover,

$$\bar{d_n} := \sum_{i=n}^{\infty} \frac{d(x_i, x_{i+1})}{\sum_{i=n}^{\infty} d(x_i, x_{i+1})} d_i = \lim_{m\longrightarrow\infty} \sum_{i=n}^{m} \frac{d(x_i, x_{i+1})}{\sum_{i=n}^{m} d(x_i, x_{i+1})} d_i$$

for any $n \in \mathbb{N}$.

In view of the completeness of X, (x_n) converges to a certain $x_0 \in X$. By msc of f, $f(x_n) - f(x_0) \in K$. Now we show that for all $n \in \mathbb{N}$

$$f(x_n) - f(x_0) - d(x_n, x_0)\bar{d_n} \in K. \tag{4.2}$$

By the definition of r, for any $n \in \mathbb{N}$ and $m \geq n$ we have

$$\begin{aligned} f(x_n) &= f(x_n) - f(x_{n+1}) + f(x_{n+1}) - \cdots - f(x_{m+1}) + f(x_{m+1}) \\ &= f(x_{m+1}) + \sum_{i=n}^{m} k_i + \sum_{i=n}^{m} d(x_i, x_{i+1})d_i. \end{aligned}$$

Since $f(x_{m+1}) - f(x_0) \in K$ this gives

$$f(x_n) - f(x_0) - d(x_n, x_{m+1}) \left(\frac{\sum_{i=n}^{m} d(x_i, x_{i+1})d_i}{\sum_{i=n}^{m} d(x_i, x_{i+1})}\right) \in K.$$

Passing to the limit with $m \to +\infty$ and taking into account the closedness of K we get (4.2). By Theorem 2.2, there exists an r-maximal element $\bar{x} \in A$. Thus, (i) holds for $\bar{x}$. We show that $\bar{x}$ satisfies (ii). Since r is transitive, $\bar{x}$ is also an r-maximal element of X. If (ii) were not satisfied, we would have $\bar{x}rz$ for some $z \neq \bar{x}$ and, by the r-maximality, $zr\bar{x}$. Consequently, by (4.1), $\langle y^*, f(\bar{x})\rangle > \langle y^*, f(z)\rangle$ and $\langle y^*, f(z)\rangle > \langle y^*, f(\bar{x})\rangle$, a contradiction. □

Let us note that conclusions (i) and (ii) are equivalent respectively to

(i′) $(f(x) - K) \cap (f(\bar{x}) + d(x, \bar{x})(D + K)) \neq \emptyset$,
(ii′) $(f(\bar{x}) - K) \cap (f(z) + d(z, \bar{x})(D + K)) = \emptyset$ for every $z \neq \bar{x}$,

where D is as in Theorem 4.1.

Let us also observe that if Y is a Banach space, the closedness of D implies its semi-completness, and $0 \notin \mathrm{cl}\,(D+K) \Leftrightarrow d(D+K,0)>0$, where $d(D+K,0)$ stands for the distance of 0 from the set $D+K$. In the two examples below it is shown that it is not difficult to verify the inequality $d(D+K,0)>0$.

Examples 4.2. 1. Let Y be a real Banach space, φ be from the dual space to Y and $\alpha>0$ be given. If K is contained in a Bishop-Phelps cone generated by φ and $\alpha>0$, i.e.,

$$K \subset \mathcal{K}_\alpha := \{z \in Y \mid \varphi(z) \geq \alpha\|z\|\}$$

and $d(D,0)>0$, $D \subset K$, then $d(D+K,0)>0$. Indeed, let us observe that for every $d \in D$, $k \in K$ we have

$$\|\varphi\|\|d+k\| \geq \varphi(d+k) \geq \varphi(d) \geq \alpha\|d\| \geq \alpha d(D,0) > 0.$$

2. If K is scalarized by a norm according to the idea of Rolewicz [23] (i.e., $u-v \in K$ implies $\|v\| \leq \|u\|$) and $d(D,0)>0$, then $d(D+K,0)>0$.

In the above examples the implication

$$0 \notin \mathrm{cl}\ D \Longrightarrow 0 \notin \mathrm{cl}\,(D+K) \tag{4.3}$$

holds.

The following fact holds true. If Y is a reflexive Banach space, K is a closed convex cone such that $K \cap -K = \{0\}$ and $D \subset K$ is closed convex and bounded, then (4.3) holds. Indeed, if $d_i + k_i \longrightarrow 0$ with $i \longrightarrow \infty$, where $(d_i) \subset D$, $(k_i) \subset K$, then by the reflexivity of Y, there exist weakly converging subsequences (d_{i_j}) and (k_{i_j}) with limit points $d_0 \in D$ and $k_0 \in K$, respectively. Hence, $0 = d_0 + k_0$ which implies $d_0 = k_0 = 0$.

The above fact allows to construct easily examples of sets D and cones K for which (4.3) fails to hold. For instance, (4.3) does not hold if

- $K = Y$, $D \subset Y$, $0 \notin clD$;
- $Y := l_\infty$, $K := \{x = (x_1, x_2, \ldots) \in l_\infty \ : \ \forall i \in \mathbb{N},\ x_{2i-1} + 2^{-i} \mid x_{2i} \mid \leq 0\}$, $D := \{x \in K : \|x\| \leq 1, \lim_{i \longrightarrow \infty} x_{2i} = 1\}$;
- $Y := C([0,1])$, $K := \{f \in Y : \int_{[0,1]} f d\mu \geq 0\}$, where μ stands for the Lebesgue measure, $D := \{f \in K : \|f\| \leq 10,\ f(1) \in [2,4]\}$.

5. Approximate solutions. For any $d \in D$ put $\tilde{f}_d(z) := f(z) + d(z,\bar{x})d$. By conclusion (ii) of Theorems 4.1,

$$(\tilde{f}_d(\bar{x}) - K) \cap \tilde{f}_d(X) = \{\tilde{f}_d(\bar{x})\}$$

and $\bar{x}$ is unique in the sense that $(\tilde{f}_d(\bar{x}) - K) \cap \tilde{f}_d(X \setminus \{\bar{x}\}) = \emptyset$. Thus, for any $d \in D$, $\bar{x}$ is a unique (in the above sense) minimal solution to problem

$$(P)\ K\text{-}\min\left\{\tilde{f}_d(x) \ : \ x \in X\right\}.$$

Let $D \subset K$ and $0 \notin D$, $\varepsilon > 0$. We say that $x \in X$ is an *ε-approximate solution with respect to* D for (P) with the objective f if

$$(f(x) - \varepsilon D - K) \cap f(X) = \emptyset.$$

For K being a closed convex pointed cone the above definition was given by Németh [19].

Theorem 5.1. *Let X be a complete metric space and let Y be locally convex space. Let $K \subset Y$ be a closed and convex cone in Y and let $D \subset K$ be a closed semi-complete convex and bounded subset of K such that $0 \notin \mathrm{cl}\,(D + K)$.*

Let $f : X \to Y$ be msc with respect to K and K-bounded. Then for every $x \in X$, $\varepsilon > 0$ and $\lambda > 0$ there exists $\bar{x} \in X$ such that

(i) $(f(x) - K) \cap (f(\bar{x}) + \varepsilon d(x, \bar{x})D)) \neq \emptyset$,
(ii) $(f(\bar{x}) - K) \cap (f(z) + \varepsilon d(z, \bar{x})D) = \emptyset$ *for every $z \neq \bar{x}$,*
(iii) *Moreover, if x is an $\varepsilon\lambda$-approximate solution with respect to D, then*

$$d(x, \bar{x}) < \lambda.$$

Proof. To get the statements (i) and (ii) it is enough to apply Theorem 4.1 with the metric $d(\cdot, \cdot)$ replaced by the metric $\varepsilon d(\cdot, \cdot)$.

Now we prove (iii). By (i),

$$f(x) = f(\bar{x}) + \varepsilon d(x, \bar{x})\bar{d} + \bar{k}, \quad \text{where } \bar{d} \in D, \ \ \bar{k} \in K.$$

If it were $d(x, \bar{x}) \geq \lambda$, then

$$f(x) = f(\bar{x}) + \varepsilon\lambda\bar{d} + \varepsilon(d(x, \bar{x}) - \lambda)\bar{d} + \bar{k} \in f(\bar{x}) + \varepsilon\lambda\bar{d} + K$$

which would contradict the fact that x is an $\varepsilon\lambda$-approximate solution. □

6. Comments. For $D = \{k_0\}$ with $0 \neq k_0 \in K$, Theorem 5.1 reduces to the results proved, e.g., in [1,14] and the references therein. Moreover, the proofs given in those references essentially work for $D = \{k_0\} + K$ and only minor changes are required. Let us notice, however, that Theorem 5.1 can be applied, e.g., to $D = \{(x_1, x_2) \ : \ x_1 + x_2 = 1, \ x_1 \geq 0, \ x_2 \geq 0\} \subset K = \mathbb{R}^2_+ \subset \mathbb{R}^2$, and observe that D cannot be represented in the form $\{k_0\} + K$ moreover $D \setminus (\{k_0\} + K) \neq \emptyset$ for every $k_0 \in K \setminus \{0\}$.

A generalization of Theorem 5.1 in the spirit of [19], where $d(z, x)D$ is replaced by a set-valued mapping $r\colon X \times X \rightrightarrows K$, is conceivable. The resulting EVP would require, however, some stringent assumptions on K which we managed to avoid in Theorem 4.1 and Theorem 5.1. For instance, in [19] the cone is assumed to be normal and salient. Finally, it is an open question whether the vector counterpart of the Deville–Godefroy–Zizler principle proved in [8] can be generalized in the spirit of Theorem 4.1, i.e., by admitting a larger class of vector bump functions.

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