Arch. Math. 93 (2009), 491–496

0003-889X/09/050491-6
published online November 3, 2009
DOI 10.1007/s00013-009-0062-z

Archiv der Mathematik

A note on nonnegative Bakry–Émery Ricci curvature

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Abstract. In this note, we show that if M^n is a nonnegatively Bakry–Émery-Ricci curved manifold with bounded potential function, any finitely generated subgroup of the fundamental group of M has polynomial growth of degree less than or equal to n.

Mathematics Subject Classification (2000). Primary 53C20, 57M05.

Keywords. Bakry–Émery Ricci curvature, Fundamental group, Polynomial growth.

1. Introduction. Let (M, g) be a complete Riemannian n-dimensional manifold with a vector field X such that, for some real constant λ,

$$\mathrm{Ric} + \mathcal{L}_X g \geq \lambda g, \tag{1.1}$$

where Ric and $\mathcal{L}_X g$ are the Ricci tensor and the Lie derivative of g with respect to the vector field X, respectively.

This class of manifolds includes two well known subclasses. The first is the class of complete manifolds satisfying $\mathrm{Ric} + \mathcal{L}_X g = \lambda g$, i.e., complete Ricci solitons. Ricci solitons are of great importance in the study of Ricci flow [2]. The second is the class of smooth metric measure spaces with Bakry–Émery tensor [1,6,8] bounded below by λ. This class consists of complete manifolds satisfying (1.1) with X being the gradient vector field of some $f \in C^\infty(M)$, where f is called the potential function. In addition, if the equality holds, M is a gradient Ricci soliton.

For positive λ, Fernández-López and García-Rió [3] and Zhang [11] showed that compact manifolds satisfying (1.1) have a finite fundamental group, which was first proved by Li [5]. Later, Wylie [10] proved the same result in the noncompact case.

In this note, we are interested in studying the second class of manifolds stated above. Precisely, we study complete Riemannian manifold (M, g) satisfying

$$\mathrm{Ric} + \mathrm{Hess}(f) \geq 0, \tag{1.2}$$

with potential function $f \in C^\infty(M)$, i.e., manifolds with nonnegative Bakry–Émery Ricci curvature.

Before stating our main result, we need the following definitions (see [4] or [7]). Let H be a finitely generated subgroup of a group G, and $\{g_1, g_2, \ldots, g_l\}$ be a specified set of generators of H. Define the *growth function* ϕ of H with respect to the specified generators for G as follows: for each positive integer s, let $\phi(s)$ be the number of distinct group elements in G which can be expressed as words of length $\leq s$ in the specified generators and their inverses. If for some constant m, $\phi(s) \leq \mathit{constant} \cdot s^m$ for all $s \in \mathbb{Z}^+$, we say that H has *polynomial growth*. The number $\inf\{m : \phi(s) \leq \mathit{constant} \cdot s^m\}$ is called the degree of the polynomial growth.

In [4], Ho proved that for a complete manifold M satisfying (1.1) with some vector field X and $\lambda \geq 0$, any finitely generated subgroup of the fundamental group of M has polynomial growth, under some technical assumption on the universal covering of M (a little different from the original assumption in [4]).

Remark 1.1. In [4], the definition of $M(r)$ might be replaced by $M(r) = \max\{|\tilde{X}(x)| : \mathrm{dist}(x, p) = r\}$ and hence the conclusion of the growth function turns out to be $\phi(s) \leq \mathrm{constant} \cdot s^{n+6C}$.

In [7], Milnor showed that if M is a nonnegatively Ricci curved manifold, any finitely generated subgroup of the fundamental group of M has polynomial growth of degree $\leq n$. In [9], Wei and Wylie proved that if M is a n-dimensional nonnegatively Bakry–Émery Ricci curved manifold with $|f| \leq k$ for some constant k, any finitely generated subgroup of the fundamental group of M has polynomial growth of degree $\leq n + 4k$. In this note, we prove that the degree of polynomial growth is less than or equal to n.

Theorem 1.2. *Let M^n be a complete manifold with nonnegative Bakry–Émery Ricci curvature, i.e.,* $\mathrm{Ric} + \mathrm{Hess}(f) \geq 0$ *for some $f \in C^\infty(M)$. If $|f| \leq k$ for some constant k, any finitely generated subgroup H of the fundamental group of M has polynomial growth of degree $\leq n$. More precisely, the growth function of H satisfies $\phi(s) \leq \mathit{constant} \cdot s^n$.*

The techniques used in the proof of Theorem 1.2 are based on [4,7].

2. Comparison theorem. Let M be a complete manifold satisfying (1.2) for some $f \in C^\infty(M)$. Fix $p \in M$ as a base point and let $r(*) = \mathrm{dist}(p, *)$, the distance function. For any $x \in M$, let $\gamma : [0, r] \to M$ be a minimal normal geodesic from p to x. Let $\{E_i(t)\}_{i=1}^{n-1}$ be parallel orthonormal vector fields along γ which are orthogonal to $\dot{\gamma}$. Constructing vector fields $\{X_i(t) = \frac{t}{r} E_i(t)\}_{i=1}^{n-1}$ along γ, then by the second variation formula, we have

$$\begin{aligned}
\Delta r(x) &\leq \int_0^r \sum_{i=1}^{n-1} (|\nabla_{\dot\gamma} X_i|^2 - \langle X_i, \mathrm{R}_{X_i,\dot\gamma}\dot\gamma\rangle)dt \\
&= \int_0^r \left(\frac{n-1}{r^2} - \frac{t^2}{r^2}\mathrm{Ric}(\dot\gamma,\dot\gamma)\right) dt \\
&\leq \frac{n-1}{r} + \int_0^r \frac{t^2}{r^2}\mathrm{Hess}(f)(\dot\gamma,\dot\gamma)dt \\
&= \frac{n-1}{r} + \frac{1}{r^2}\int_0^r t^2 \frac{d^2}{dt^2}(f\circ\gamma)dt \\
&= \frac{n-1}{r} + \langle \nabla f, \dot\gamma\rangle(x) - \frac{2}{r^2}\int_0^r t\frac{d}{dt}(f\circ\gamma)dt \\
&= \frac{n-1}{r} + \langle \nabla f, \dot\gamma\rangle(x) - \frac{2}{r}f(x) + \frac{2}{r^2}\int_0^r (f\circ\gamma)dt.
\end{aligned} \tag{2.1}$$

Now, we start to derive the volume comparison theorem. Consider the polar coordinate system (r,θ), by Gauss Lemma we have

$$ds_M^2 = dr^2 + g_{\alpha\beta}d\theta^\alpha d\theta^\beta, \quad \alpha,\beta = 2,\ldots,n.$$

Denote $J(r,\theta) = \sqrt{\det(g_{\alpha\beta})}$ the area element of the geodesic sphere $\partial B(p,r)$ with radius r centered at p, then we have

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{\partial}{\partial r}(\ln J(r,\theta))\frac{\partial}{\partial r} + \Delta_{\partial B_p(r)}.$$

Therefore,

$$\Delta r = \frac{\partial}{\partial r}(\ln J(r,\theta)) \tag{2.2}$$

on $\exp_p(\Sigma(p)) \setminus \{p\}$.

Combining (2.1) and (2.2), we obtain

$$\frac{\partial}{\partial r}(\ln J(r,\theta)) \leq \frac{n-1}{r} + \langle \nabla f, \dot\gamma\rangle - \frac{2}{r}f + \frac{2}{r^2}\int_0^r (f\circ\gamma)dt.$$

Integrating it from positive r_1 to r_2, together with $|f| \leq k$, we have

$$\begin{aligned}\ln\left(\frac{J(r_2,\theta)}{J(r_1,\theta)}\right) &\leq \ln\left(\frac{r_2}{r_1}\right)^{n-1} + f|_{r_1}^{r_2} + 2\int_{r_1}^{r_2}\left(\frac{1}{r^2}\int_0^r f(t)dt - \frac{1}{r}f(r)\right)dr \\ &= \ln\left(\frac{r_2}{r_1}\right)^{n-1} + f|_{r_1}^{r_2} - \left(\frac{2}{r}\int_0^r f dt\right)|_{r_1}^{r_2} + 2\int_{r_1}^{r_2}\frac{1}{r}f dr - 2\int_{r_1}^{r_2}\frac{1}{r}f dr \\ &= \ln\left(\frac{r_2}{r_1}\right)^{n-1} + f|_{r_1}^{r_2} - \left(\frac{2}{r}\int_0^r f dt\right)|_{r_1}^{r_2} \\ &\leq \ln\left(\frac{r_2}{r_1}\right)^{n-1} + 6k,\end{aligned}$$

Thus, we have proved the following

Lemma 2.1. *Let M be a complete manifold satisfying (1.2) for some $f \in C^\infty(M)$. Suppose $|f| \leq k$, then for positive r_1, r_2,*

$$J(r_2,\theta) \leq e^{6k}\left(\frac{r_2}{r_1}\right)^{n-1} J(r_1,\theta). \tag{2.3}$$

Let $V(B(p,r))$ be the volume of the geodesic ball $B(p,r)$. We prove the following volume comparison theorem:

Theorem 2.2. *Let M be a complete manifold satisfying (1.2) for some $f \in C^\infty(M)$. Suppose $|f| \leq k$, then for $0 < R_1 \leq R_2$,*

$$\frac{V(B(p,R_2))}{V(B(p,R_1))} \leq e^{6k}\left(\frac{R_2}{R_1}\right)^n.$$

Proof. Integrating (2.3) over $\mathbb{S}^{n-1}$ with respect to θ, we obtain

$$r_1^{n-1}\int_{\mathbb{S}^{n-1}} J(r_2,\theta)d\theta \leq e^{6k} r_2^{n-1}\int_{\mathbb{S}^{n-1}} J(r_1,\theta)d\theta.$$

Integrating it from 0 to R_1 with respect to r_1, we obtain

$$\frac{R_1^n}{n}\int_{\mathbb{S}^{n-1}} J(r_2,\theta)d\theta \leq e^{6k} r_2^{n-1} V(B(p,R_1)).$$

Integrating it from 0 to R_2 with respect to r_2, we have

$$\begin{aligned}\frac{R_1^n}{n}V(B(p,R_2)) &\leq \left(\int_0^{R_2} e^{6k} r_2^{n-1} dr_2\right) V(B(p,R_1)) \\ &= e^{6k}\frac{R_2^n}{n}V(B(p,R_1)).\end{aligned}$$

This proves the theorem. □

3. Proof of Theorem 1.2.

Proof of Theorem 1.2. Let $\tilde{M}$ be the universal covering of M with induced metric by g. Then $\tilde{M}$ also satisfies (1.2) with potential function $\tilde{f}$, the pull-back of f. Fix a point $p \in \tilde{M}$. We identify $\pi_1(M)$ with the group of all deck transformations of $\tilde{M}$ over M. Let $\tilde{g}_i : \tilde{M} \to \tilde{M}$ be the deck transformation corresponding to the specified generator g_i of H $(i = 1, \ldots, l)$. Set $\mu = \max_i \{\mathrm{dist}(p, g_i(p))\}$. Note that, for any $s \in \mathbb{Z}^+$, $B(p, \mu s)$ contains at least $\phi(s)$ distinct points of the form $g(p)$ with $g \in \pi_1(M)$.

Choose $\epsilon > 0$ which is small enough such that the geodesic ball $B(p, \epsilon)$ is disjoint from $g(B(p, \epsilon))$ for any $g \in \pi_1(M) \setminus \{1\}$. Then we have

$$\phi(s)V(B(p, \epsilon)) \leq V(B(p, \mu s + \epsilon)).$$

Applying Theorem 2.2 with $R_1 = \epsilon$ and $R_2 = \mu s + \epsilon$, we obtain

$$\phi(s) \leq e^{6k} \left(\frac{\mu s + \epsilon}{\epsilon}\right)^n,$$

which implies that $\phi(s) \leq \mathit{constant} \cdot s^n$. □

Acknowledgement. The author would like to thank Prof. Fuquan Fang for introducing him to this issue. Thanks also go to Dr. Zhenlei Zhang for helpful discussions.

References

[1] D. Bakry and M. Émery, Diffusion hypercontractivitives, in Séminaire de Probabilités XIX 177-206, 1983/84, Lect. Notes in Math. **1123**, Springer, Berlin, 1985.

[2] B. Chow, P. Lu, and L. Ni, Hamilton's Ricci flow, Graduate Studies in Mathematics **77**, American Mathematical Society, Provindence, RI, 2006

[3] M. Fernández-López and E. García-Río, A remark on compact Ricci solitons, Math. Ann. **340**, 893–896 (2008).

[4] P. T. Ho, A remark on complete non-expanding Ricci solitons, Arch. Math. **91**, 284–288 (2008).

[5] X.-M. Li, On extensions of Myers' theorem, Bull. London Math. Soc. (4) **27**, 392–396 (1995).

[6] J. Lott, Some geometric properties of the Bakry–Émery Ricci tensor, Comment. Math. Helv. **78**, 865–883 (2003).

[7] J. Milnor, A note on curvature and fundamental group, J. Diff. Geom. **2**, 1–7 (1968).

[8] Z. Qian, Estimates for weighted volumes and applications, Quart. J. Math. Oxford Ser. (2) **48**, 235–242 (1997).

[9] G. F. Wei and W. Wylie, Comparison geometry for the Bakry–Émery Ricci tensor, arXiv:math.DG/0706.1120v1.

[10] W. Wylie, Complete shrinking Ricci solitons have finite fundamental group, Proc. Amer. Math. Soc. **136**, 1803–1806 (2008).

[11] Z. L. Zhang, On the Finiteness of the fundamental group of a compact shrinking Ricci soliton, Colloq. Math. **107**, 297–299 (2007).

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Received: 12 June 2009