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An ultimate extremely accurate formula for approximation of the factorial function

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Abstract. We prove in this paper that for every $x \geq 0$,

$$\sqrt{2\pi e} \cdot e^{-\omega} \left(\frac{x+\omega}{e}\right)^{x+\frac{1}{2}} < \Gamma(x+1) \leq \alpha \cdot \sqrt{2\pi e} \cdot e^{-\omega} \left(\frac{x+\omega}{e}\right)^{x+\frac{1}{2}}$$

where $\omega = (3-\sqrt{3})/6$ and $\alpha = 1.072042464...$, then

$$\beta \cdot \sqrt{2\pi e} \cdot e^{-\zeta} \left(\frac{x+\zeta}{e}\right)^{x+\frac{1}{2}} \leq \Gamma(x+1) < \sqrt{2\pi e} \cdot e^{-\zeta} \left(\frac{x+\zeta}{e}\right)^{x+\frac{1}{2}},$$

where $\zeta = (3+\sqrt{3})/6$ and $\beta = 0.988503589...$ Besides the simplicity, our new formulas are very accurate, if we take into account that they are much stronger than Burnside's formula, which is considered one of the best approximation formulas ever known having a simple form.

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1. Introduction. Stirling's formula and its generalizations have a large class of applications in science as in statistical physics or probability theory. In consequence, it has been deeply studied by a large number of authors, due to its practical importance. Stirling's formula:

$$n! \approx \left(\frac{n}{\mathrm{e}}\right)^n \sqrt{2\pi n} = \sigma_n, \tag{1.1}$$

is an approximation for big factorials. In fact, the formula (1.1) was discovered by the French mathematician Abraham de Moivre (1667–1754) in the form

$$n! \approx \text{constant} \cdot \left(\frac{n}{\mathrm{e}}\right)^n \sqrt{n}$$

and the Scottish mathematician James Stirling (1692–1770) discovered the constant $\sqrt{2\pi}$ in the previous formula. For proofs and other details see [6].

Furthermore, there is a variety of approaches to Stirling's formula, ranging from elementary to advanced methods. As recent examples, we mention the estimations given by W. Schuster in [8], or the formula

$$n! \approx \sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}} = \beta_n,$$

with $n! < \beta_n$, due to W. Burnside, whose superiority over Stirling's formula was proved in [4].

2. The results. In the first part of this section we prove the following new estimation formula:

$$n! \approx \sqrt{\frac{2\pi}{e}}\left(\frac{n+1}{e}\right)^{n+\frac{1}{2}} = \alpha_n, \tag{2.1}$$

with $\sigma_n < \alpha_n < n!$, which is already stronger than the much celebrated Stirling's formula. The starting idea is the following representation of the factorial function involving a double sum:

$$n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \cdot \exp\left(\sum_{k=n}^{\infty}\sum_{j=2}^{\infty} \frac{j-1}{2j(j+1)}\left(\frac{-1}{k}\right)^j\right).$$

See [5]. In connection with this relation, we give the following

Lemma 2.1. *There exists a convergent series* $\sum_{n=1}^{\infty} a_n$ *with positive terms that satisfies, for every integer* $n \geq 1$, *the relation*

$$n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n}\exp\left(\sum_{k=n}^{\infty} a_k\right). \tag{2.2}$$

Proof. By dividing the relations

$$n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n}\exp\left(\sum_{k=n}^{\infty} a_k\right)$$

and

$$(n-1)! = \left(\frac{n-1}{e}\right)^{n-1} \sqrt{2\pi(n-1)}\exp\left(\sum_{k=n-1}^{\infty} a_k\right),$$

we get

$$a_n = \left(n+\frac{1}{2}\right)\ln\left(1+\frac{1}{n}\right) - 1,$$

for every $n \geq 2$. The obtained series

$$s = \sum_{n=2}^{\infty}\left(\left(n+\frac{1}{2}\right)\ln\left(1+\frac{1}{n}\right) - 1\right)$$

is convergent (with sum s), according to a comparison test applied in the case

$$\lim_{n\to\infty} \frac{\left(n+\frac{1}{2}\right)\ln\left(1+\frac{1}{n}\right)-1}{\frac{1}{n^2}} = \frac{1}{12}.$$

Then, for $n = 1$ in (2.2), we impose the condition

$$1 = \frac{\sqrt{2\pi}}{e}\exp\left(a_1 + s\right),$$

thus $a_1 = 1 - s - \ln\sqrt{2\pi}$.

Now let us separate the term a_n from the series to obtain:

$$\begin{aligned} n! &= \left(\frac{n}{e}\right)^n \sqrt{2\pi n}\exp a_n \cdot \exp\left(\sum_{k=n+1}^{\infty} a_k\right) \\ &= \sqrt{\frac{2\pi}{e}}\left(\frac{n+1}{e}\right)^{n+\frac{1}{2}} \exp\left(\sum_{k=n+1}^{\infty} a_k\right). \end{aligned} \tag{2.3}$$

The remainder of the (convergent) series from the right-hand side of (2.3) tends to zero as n tends to infinity, so we have the following estimation:

$$n! \approx \sqrt{\frac{2\pi}{e}}\left(\frac{n+1}{e}\right)^{n+\frac{1}{2}}.$$

If we compare the remainders of the series (2.2)–(2.3) (with positive terms), we deduce that

$$\left(\frac{n}{e}\right)^n \sqrt{2\pi n} < \sqrt{\frac{2\pi}{e}}\left(\frac{n+1}{e}\right)^{n+\frac{1}{2}} < n!,$$

which proves that our new formula (2.1) is substantially stronger than Stirling's formula. Further, we mention the following formula

$$n! \approx \sqrt{2\pi}\left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}} = \beta_n, \tag{2.4}$$

established by W. Burnside in [4], then rediscovered by Y. Weissman in [10].

If we look carefully at the estimations (2.1) and (2.4), then we remark that both are estimations for $n!$ of the form

$$n! \approx \lambda\left(\frac{n+p}{e}\right)^{n+q} = \tau_n, \tag{2.5}$$

where λ, p, q are constants (α_n is obtained for $\lambda = \sqrt{2\pi/e}$, $p = 1$, $q = 1/2$, while β_n is obtained for $\lambda = \sqrt{2\pi}$, $p = q = 1/2$). Surprisingly, also Stirling's formula can be deduced from (2.5) in case $\lambda = \sqrt{2\pi e}$, $p = 0$ and $q = 1/2$.

Then a natural question appears, namely which are the constants λ, p, q such that better approximations (2.5) are obtained. First we impose the condition that the sequence $\tau_n/n!$ tends to 1. It is difficult to find this limit directly, because, at least theoretically, the computation of such a limit must repeat in some way the proof of Stirling's formula. Under these circumstances, we have the idea to interprese the sequence $\beta_n/n!$ which already tends to 1. Hence the previous condition can be written as

$$\lim_{n\to\infty} \frac{\tau_n}{\beta_n} = 1,$$

or

$$1 = \lim_{n\to\infty} \frac{\tau_n}{\beta_n} = \frac{\lambda}{\sqrt{2\pi}} \cdot e^{q-\frac{1}{2}} \lim_{n\to\infty} \frac{\left(1+\frac{p}{n}\right)^{n+q}}{\left(1+\frac{1}{2n}\right)^{n+\frac{1}{2}}} \cdot n^{q-\frac{1}{2}}$$

$$= \frac{\lambda}{\sqrt{2\pi}} \cdot e^{p+q-1} \lim_{n\to\infty} n^{q-\frac{1}{2}}.$$

First, it results $q = 1/2$, then $\lambda = e^{-p}\sqrt{2\pi e}$. Thus for every positive real p, the following sharp estimations hold:

$$n! \approx \sqrt{2\pi e} \cdot e^{-p} \left(\frac{n+p}{e}\right)^{n+\frac{1}{2}} \tag{2.6}$$

Now let us define the function $f : [0, 1] \to \mathbb{R}$ by the formula

$$f(x) = \sqrt{2\pi e} \cdot e^{-x} \left(\frac{n+x}{e}\right)^{n+\frac{1}{2}},$$

where $n \geq 1$ is any fixed integer. As we have already noted, we have

$$f(0) = \sqrt{2\pi n} \left(\frac{n}{e}\right)^{n}, \quad f\left(\frac{1}{2}\right) = \sqrt{2\pi} \left(\frac{n+\frac{1}{2}}{e}\right)^{n+\frac{1}{2}}, \quad f(1) = \sqrt{\frac{2\pi}{e}} \left(\frac{n+1}{e}\right)^{n+\frac{1}{2}}.$$

The function f is strictly increasing on $[0, 1/2]$ and strictly decreasing on $[1/2, 1]$, if we take into account that

$$\frac{d}{dx} (\log f(x)) = \frac{\frac{1}{2} - x}{n+x}.$$

The upper and the lower bound of the function f are estimations for $n!$ and

$$f(0) < n! < f\left(\frac{1}{2}\right) > n! > f(1),$$

so performant approximations (2.6) can be obtained for $p \in [0, 1]$.

As usually, we associate to the approximation (2.6) the following function

$$F_a : [0, \infty) \to \mathbb{R}, \quad F_a(x) = \log \frac{\Gamma(x+1)}{\sqrt{2\pi e} \cdot e^{-a} \left(\frac{x+a}{e}\right)^{x+\frac{1}{2}}},$$

where $a \in [0, 1]$ is a parameter. We have

$$F_a(x) = -\log\sqrt{2\pi e} + a - \left(x + \frac{1}{2}\right) \log(x+a) + x + \frac{1}{2} + \log\Gamma(x+1),$$

so

$$F_a'(x) = -\log(x+a) - \frac{x+\frac{1}{2}}{x+a} + 1 + \psi(x) + \frac{1}{x},$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the Euler's digamma function, then

$$F_a''(x) = -\frac{1}{x+a} + \frac{1-2a}{2(a+x)^2} + \psi'(x) - \frac{1}{x^2}.$$

Using the integral representations (*e.g.*, [1,9]):

$$\psi'(x) = \int_0^\infty \frac{te^{-xt}}{1-e^{-t}}dt, \quad \frac{1}{x} = \int_0^\infty e^{-xt}dt, \quad \frac{1}{x^2} = \int_0^\infty te^{-xt}dt,$$

we obtain

$$F_a''(x) = \int_0^\infty \frac{e^{-(1+x+a)t}}{1-e^{-t}} \left(\left(1 + \left(a - \frac{1}{2} \right) t + te^{at} \right) \left(e^t - 1 \right) - te^{(1+a)t} \right) dt,$$

or

$$F_a''(x) = \int_0^\infty \frac{e^{-(1+x+a)t}}{1-e^{-t}} \varphi(t) dt, \tag{2.7}$$

where

$$\varphi(t) = te^{at} + \left(e^t - 1 \right) \left(\left(\frac{1}{2} - a \right) t - 1 \right).$$

The function φ can be easily expanded in powers series as

$$\varphi(t) = \frac{1}{12} \left(6a^2 - 6a + 1 \right) t^3 + \sum_{n=4}^\infty \left(n \left(\frac{1}{2} - a \right) + na^{n-1} - 1 \right) \frac{t^n}{n!}. \tag{2.8}$$

Lemma 2.2. *Let*

$$x_n = n \left(\frac{1}{2} - a \right) + na^{n-1} - 1\,, \quad a > 0.$$

Then x_n is positive for all $a \in \left[0, \left(3 - \sqrt{3} \right) /6 \right]$, and negative for all $a \in \left[1/2, \left(3 + \sqrt{3} \right) /6 \right]$ and $n \geq 4$.

Proof. First, for $a \in \left[0, \left(3 - \sqrt{3} \right) /6 \right]$ and $n \geq 4$, we have

$$n \left(\frac{1}{2} - a \right) + na^{n-1} - 1 \geq \left(4 \left(\frac{1}{2} - \frac{3-\sqrt{3}}{6} \right) - 1 \right) + na^{n-1} > 0.$$

We show that $x_n < 0$, for every $n \geq 4$ and $a \in [1/2, 4/5]$.

In this sense, let us define the function $t_n : [1/2, 4/5] \to \mathbb{R}$, by

$$t_n(a) = n \left(\frac{1}{2} - a \right) + na^{n-1} - 1.$$

Then t_n is negative, since it is convex, with $t_n(1/2) < 0$ and $t_n(4/5) < 0$.

Indeed, $t_n''(a) = n(n-1)(n-2)a^{n-3} > 0$,

$$t_n \left(\frac{1}{2} \right) = \frac{n}{2^{n-1}} - 1 < 0$$

and $t_4(4/5) = -0.152$, $t_5(4/5) = -0.452$, $t_6(4/5) = -0.83392$, and for every $n \geq 7$,

$$t_n\left(\frac{4}{5}\right) = -\frac{3n}{10} + n\left(\frac{4}{5}\right)^{n-1} - 1 \leq -\left(\frac{3}{10} - \left(\frac{4}{5}\right)^6\right)n - 1 < 0.$$

Now we are in position to give the following

Theorem 2.1. a) *For every $a \in \left[0, \frac{3-\sqrt{3}}{6}\right]$, the function F_a is decreasing and for every $x \geq 0$, the following inequalities hold:*

$$\sqrt{2\pi e} \cdot e^{-a}\left(\frac{x+a}{e}\right)^{x+\frac{1}{2}} < \Gamma(x+1) \leq \sqrt{\frac{e}{a}}\left(\frac{x+a}{e}\right)^{x+\frac{1}{2}}. \tag{2.9}$$

b) *For every $b \in \left[\frac{1}{2}, \frac{3+\sqrt{3}}{6}\right]$, the function F_b is increasing and for every $x \geq 0$, the following inequalities hold:*

$$\sqrt{\frac{e}{b}}\left(\frac{x+b}{e}\right)^{x+\frac{1}{2}} \leq \Gamma(x+1) < \sqrt{2\pi e} \cdot e^{-b}\left(\frac{x+b}{e}\right)^{x+\frac{1}{2}}. \tag{2.10}$$

Proof. a) We can see from the relation (2.8) that $6a^2 - 6a + 1 \geq 0$, for every $a \in \left[0, \frac{3-\sqrt{3}}{6}\right]$ and each coefficient $n\left(\frac{1}{2} - a\right) + na^{n-1} - 1$ is positive, for $n \geq 4$, thus $\varphi > 0$. From (2.7) it results that $F_a'' > 0$, so F_a is convex. As we proved,

$$\lim_{x\to\infty} F_a(x) = \lim_{x\to\infty} F_a'(x) = 0,$$

so $F_a(x) > 0$ and $F_a'(x) < 0$, for $x \in [0, \infty)$. This implies that F_a is strictly decreasing,

$$0 = \lim_{x\to\infty} F_a(x) < F_a(x) \leq F_a(0),$$

which is equivalent with (2.9).

b) Now, for $b \in \left[\frac{1}{2}, \frac{3+\sqrt{3}}{6}\right]$, we have $6b^2 - 6b + 1 \leq 0$, and each coefficient $\left(nb^{n-1} - 1\right) - n\left(b - \frac{1}{2}\right)$ is negative, for $n \geq 4$, thus $\varphi < 0$. From (2.7) it results that $F_b'' < 0$, so F_b is concave. As we proved,

$$\lim_{x\to\infty} F_b(x) = \lim_{x\to\infty} F_b'(x) = 0,$$

so $F_b(x) < 0$ and $F_b'(x) > 0$, for $x \in [0, \infty)$. This implies that F_b is strictly increasing,

$$F_b(0) \leq F(x) < \lim_{x\to\infty} F_b(x) = 0,$$

which is equivalent with (2.10). □

If we take $a = \omega = \left(3 - \sqrt{3}\right)/6$ in (2.9), we obtain the estimations

$$\sqrt{2\pi e} \cdot e^{-\omega}\left(\frac{x+\omega}{e}\right)^{x+\frac{1}{2}} < \Gamma(x+1) \leq \alpha \cdot \sqrt{2\pi e} \cdot e^{-\omega}\left(\frac{x+\omega}{e}\right)^{x+\frac{1}{2}},$$

where

$$\alpha = \frac{e^{\omega}}{\sqrt{2\pi e}\left(\frac{\omega}{e}\right)^{\frac{1}{2}}} = 1.072042464\ldots.$$

Analogously, for $b = \zeta = \left(3+\sqrt{3}\right)/6$ in (2.10), we obtain the estimations

$$\beta \cdot \sqrt{2\pi e} \cdot e^{-\zeta}\left(\frac{x+\zeta}{e}\right)^{x+\frac{1}{2}} \leq \Gamma(x+1) < \sqrt{2\pi e} \cdot e^{-\zeta}\left(\frac{x+\zeta}{e}\right)^{x+\frac{1}{2}},$$

where

$$\beta = \frac{e^{\zeta}}{\sqrt{2\pi e}\left(\frac{\zeta}{e}\right)^{\frac{1}{2}}} = 0.988503589\ldots.$$

3. Conclusions. The basic estimations (2.9) and (2.10) can be viewed as very accurate approximations for the factorial function. In order to show the practical utility of our estimations

$$\theta_n := \sqrt{2\pi e} \cdot e^{-\omega}\left(\frac{n+\omega}{e}\right)^{n+\frac{1}{2}} < n! < \sqrt{2\pi e} \cdot e^{-\zeta}\left(\frac{n+\zeta}{e}\right)^{n+\frac{1}{2}} := \xi_n \quad (3.1)$$

we give next some of their numeric values:

n	$\theta_n/n!$	$\xi_n/n!$
5	0.99971	1.0002
10	0.99992	1.0001
20	0.99998	1.0000
25	0.99999	1.0000
40	1.00000	1.0000

In this table, the values 1.0000, or 1.00000 mean that $\theta_n/n!$ and $\xi_n/n!$ differ from the unity by a quantity less than 10^{-4}, respective 10^{-5}.

If we want to compare (3.1) with other results, then it is to be noted that, more recently, the double inequality

$$\frac{n^{n+1}e^{-n}\sqrt{2\pi}}{\sqrt{n-\alpha}} < n! < \frac{n^{n+1}e^{-n}\sqrt{2\pi}}{\sqrt{n-\beta}}$$

was established in [7] in case $\alpha = 0$ and $\beta = 1$. The best possible constants $\alpha = 1 - 2\pi e^{-2}$ and $\beta = 1/6$ were discovered by N. Batir in the very recent paper [2]. The author of [2] proves the superiority of his approximation formula

$$n! \approx \frac{n^{n+1}e^{-n}\sqrt{2\pi}}{\sqrt{n-1/6}} = \gamma_n \quad (3.2)$$

over Burnside's formula (2.4), using numerical computations. As we can see from the next table, our formula $n! \approx \xi_n$ from (3.1) is more accurate than (3.2).

n	$n!$	ξ_n	γ_n
1	1	**1.0024785**	1.01015
2	2	**2.0020656**	2.0043347
3	6	**6.0033665**	6.0054101
5	120	**120.028847**	120.036736
10	3628800	**3629050.545**	3629064.897

Finally, we give the comparison table of our under-approximation θ_n from (3.1), with the corresponding under-approximation

$$n! \approx \frac{n^{n+1}e^{-n}\sqrt{2\pi}}{\sqrt{n-(1-2\pi e^{-2})}} = \nu_n \quad \text{(N. Batir [2])}$$

from which we can see that (3.1) has at least one order advantage over the formula ν_n. Our approximation θ_n is comparable with the following approximation

$$n! \approx n^n e^{-n}\sqrt{2\pi\left(n+\frac{1}{6}\right)} = \rho_n \quad \text{(N. Batir [3]).}$$

n	$n! - \theta_n$	$n! - \nu_n$	$n! - \rho_n$
5	**0.03481**	0.17385	0.02997
7	**0.76766**	5.49330	0.66252
10	**276.56**	2868.64	239.18
15	$\mathbf{4.4989 \times 10^7}$	7.0713×10^8	3.8988×10^7
20	$\mathbf{4.7425 \times 10^{13}}$	9.9891×10^{14}	4.116×10^{13}
30	$\mathbf{2.3125 \times 10^{27}}$	7.3476×10^{28}	2.0119×10^{27}
70	$\mathbf{1.9215 \times 10^{94}}$	1.4409×10^{96}	1.6853×10^{94}
120	$\mathbf{3.6277 \times 10^{192}}$	4.713×10^{194}	3.2123×10^{192}

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