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A p-adic analogue of a formula of Ramanujan

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Abstract. During his lifetime, Ramanujan provided many formulae relating binomial sums to special values of the Gamma function. Based on numerical computations, Van Hamme recently conjectured p-adic analogues to such formulae. Using a combination of ordinary and Gaussian hypergeometric series, we prove one of these conjectures.

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1. Introduction. In Ramanujan's second letter to Hardy dated February 27, 1913, the following formula appears:

$$1-5\left(\frac{1}{2}\right)^5+9\left(\frac{1\cdot 3}{2\cdot 4}\right)^5-13\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^5+\cdots=\frac{2}{\Gamma\left(\frac{3}{4}\right)^4} \tag{1.1}$$

where $\Gamma(\cdot)$ is the Gamma function. This result was proved in 1924 by Hardy [14] and a further proof was given by Watson [29] in 1931. Note that (1.1) can be expressed as

$$\sum_{k=0}^{\infty}(4k+1)\binom{-\frac{1}{2}}{k}^5=\frac{2}{\Gamma\left(\frac{3}{4}\right)^4}\ .$$

Other formulae of this type include

$$\sum_{k=0}^{\infty}(-1)^k\frac{6k+1}{4^k}\binom{-\frac{1}{2}}{k}^3=\frac{4}{\pi}=\frac{4}{\Gamma\left(\frac{1}{2}\right)^2}, \tag{1.2}$$

which is Entry 20, page 352 of [5]. It is interesting to note that a proof of (1.2) was not found until 1987 [8].

Recently, Van Hamme [27] studied a p-adic analogue of (1.1). Namely, he truncated the left-hand side and replaced the Gamma function with the p-adic Gamma function. Based on numerical computations, he posed the following.

Conjecture 1.1. Let p be an odd prime. Then

$$\sum_{k=0}^{\frac{p-1}{2}}(4k+1)\binom{-\frac{1}{2}}{k}^5 \equiv \begin{cases} -\frac{p}{\Gamma_p\left(\frac{3}{4}\right)^4} \pmod{p^3} & \text{if} \quad p \equiv 1 \pmod 4, \\ 0 \pmod{p^3} & \text{if} \quad p \equiv 3 \pmod 4 \end{cases}$$

where $\Gamma_p(\cdot)$ is the p-adic Gamma function.

The purpose of this paper is to prove the following.

Theorem 1.2. *Conjecture 1.1 is true.*

Theorem 1.2 is one example of a general phenomena called *Supercongruences.* This term first appeared in the Ph.D. thesis of Coster [9] and refers to the fact that a congruence holds modulo p^k for some $k \geq 2$. Other examples of supercongruences have been observed in the context of number theory (see [22] and the references therein), mathematical physics [17], and algebraic geometry [26].

Van Hamme states 12 other conjectures relating truncated hypergeometric series to values of the p-adic Gamma function. Motivated by Theorem 1.2, one of these conjectures has been settled in [20]. The remaining 11 include a conjectural p-adic analogue of (1.2) which states

$$\sum_{k=0}^{\frac{p-1}{2}}(-1)^k \frac{6k+1}{4^k}\binom{-\frac{1}{2}}{k}^3 \equiv -\frac{p}{\Gamma_p\left(\frac{1}{2}\right)^2} \pmod{p^4}.$$

These conjectures were motivated experimentally and as van Hamme states that "we have no real explanation for our observations", it might be worthwhile to determine whether these congruences arise from considering some appropriate algebraic surfaces (see [7] or [25]). Finally, if $p \equiv 1 \pmod 4$, then the congruence in Conjecture 1.1 appears to hold modulo p^4. This has been numerically verified for all primes less than 5000.

The paper is organized as follows. In Section 2 we recall some properties of the Gamma function, ordinary hypergeometric series, the p-adic Gamma function and Gaussian hypergeometric series. The proof of Theorem 1.2 is then given in Section 3.

2. Preliminaries. We briefly discuss some preliminaries which we will need in Section 3. For further details see [3], [6], or [18]. Recall that for all complex numbers $x \neq 0, -1, -2, \ldots$, the Gamma function $\Gamma(x)$ is defined by

$$\Gamma(x) := \lim_{k\to\infty} \frac{k!\, k^{x-1}}{(x)_k}$$

where $(a)_0 := 1$ and $(a)_n := a(a+1)(a+2)\cdots(a+n-1)$ for positive integers n. The Gamma function satisfies the reflection formula

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}. \tag{2.1}$$

We also recall that the hypergeometric series ${}_pF_q$ is defined by

$$ {}_pF_q\left[\begin{matrix} a_1, & a_2, & a_3, & \ldots, & a_p \\ & b_1, & b_2, & \ldots, & b_q \end{matrix} \;\Big|\; z\right] := \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n (a_3)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_q)_n} \, \frac{z^n}{n!} \tag{2.2}$$

where a_i, b_i and z are complex numbers, with none of the b_i being negative integers or zero, and, p and q are positive integers. Note that the series terminates if some a_j is a negative integer. In [30], Whipple studied properties of *well-poised* series where $p = q+1$, $z = \pm 1$, and $a_1 + 1 = a_2 + b_1 = a_3 + b_2 = \cdots = a_p + b_q$. One such transformation property of the well-poised series (see (6.3), page 252 in [30]) is

$$ {}_6F_5\left[\begin{matrix} a, & 1+\frac{1}{2}a, & c, & d, & e, & f \\ & \frac{1}{2}a, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a-f \end{matrix} \;\Big|\; -1\right] \tag{2.3}$$
$$= \frac{\Gamma(1+a-e)\Gamma(1+a-f)}{\Gamma(1+a)\Gamma(1+a-e-f)} \, {}_3F_2\left[\begin{matrix} 1+a-c-d, & e, & f \\ & 1+a-c, & 1+a-d \end{matrix} \;\Big|\; 1\right].$$

This is Entry 31, Chapter 10 in Ramanujan's second notebook (see page 41 of [4]). Watson's proof of (1.1) is a specialization of (2.3) combined with Dixon's theorem [10].

Let p be an odd prime. For $n \in \mathbb{N}$, we define the p-adic Gamma function as

$$\Gamma_p(n) := (-1)^n \prod_{\substack{j<n \\ p\nmid j}} j$$

and extend to all $x \in \mathbb{Z}_p$ by setting

$$\Gamma_p(x) := \lim_{n \to x} \Gamma_p(n)$$

where n runs through any sequence of positive integers p-adically approaching x and $\Gamma_p(0) := 1$. This limit exists, is independent of how n approaches x and determines a continuous function on $\mathbb{Z}_p$.

In [13], Greene introduced the notion of general hypergeometric series over finite fields or *Gaussian hypergeometric series*. These series are analogous to classical hypergeometric series and have played an important role in relation to the number of points over $\mathbb{F}_p$ of Calabi-Yau threefolds [2], traces of Hecke operators [11], formulae for Ramanujan's τ-function [24], and the number of points on a family of elliptic curves [12].

We now introduce two definitions. Let $\mathbb{F}_p$ denote the finite field with p elements. We extend the domain of all characters χ of $\mathbb{F}_p^*$ to $\mathbb{F}_p$ by defining $\chi(0) := 0$. The first definition is the finite field analogue of the binomial coefficient. For characters A and B of $\mathbb{F}_p^*$, define $\binom{A}{B}$ by

$$\binom{A}{B} := \frac{B(-1)}{p} J(A, \overline{B})$$

where $J(\chi, \lambda)$ denotes the Jacobi sum for χ and λ characters of $\mathbb{F}_p^*$. The second definition is the finite field analogue of ordinary hypergeometric series. For characters $A_0, A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ of $\mathbb{F}_p^*$ and $x \in \mathbb{F}_p$, define the *Gaussian hypergeometric series* by

$${}_{n+1}F_n\left(\begin{matrix} A_0, & A_1, & \ldots, & A_n \\ & B_1, & \ldots, & B_n \end{matrix} \middle| \, x \right)_p := \frac{p}{p-1} \sum_{\chi} \binom{A_0\chi}{\chi} \binom{A_1\chi}{B_1\chi} \cdots \binom{A_n\chi}{B_n\chi} \chi(x)$$

where the summation is over all characters χ on $\mathbb{F}_p^*$.

In [23], the case where $A_i = \phi_p$, the quadratic character, for all i and $B_j = \epsilon_p$, the trivial character mod p, for all j is examined and is denoted ${}_{n+1}F_n(x)$ for brevity. By [13], $p^n {}_{n+1}F_n(x) \in \mathbb{Z}$. Before stating the main result of [23], we recall that for i, $n \in \mathbb{N}$, generalized harmonic sums, $H_n^{(i)}$, are defined by

$$H_n^{(i)} := \sum_{j=1}^{n} \frac{1}{j^i}$$

and $H_0^{(i)} := 0$. For p an odd prime, $\lambda \in \mathbb{F}_p$, $n \in \mathbb{Z}^+$, we now define the quantities

(2.4)
$$X(p, \lambda, n) := \phi_p(\lambda) \sum_{j=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}+j}{j}^l \binom{\frac{p-1}{2}}{j}^l (-1)^{jl} \lambda^{-j} \Big[1 + 2(n+1)j \left(H^{(1)}_{\frac{p-1}{2}+j} - H_j^{(1)} \right) + \frac{(n+1)^2}{2} j^2 \left(H^{(1)}_{\frac{p-1}{2}+j} - H_j^{(1)} \right)^2 - \frac{(n+1)}{2} j^2 \left(H^{(2)}_{\frac{p-1}{2}+j} - H_j^{(2)} \right) \Big],$$

(2.5)
$$Y(p, \lambda, n) := \phi_p(\lambda) \sum_{j=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}+j}{j}^l \binom{\frac{p-1}{2}}{j}^l (-1)^{jl} \lambda^{-jp} \Big[1 + (n+1)j \left(H^{(1)}_{\frac{p-1}{2}+j} - H_j^{(1)} \right) - \frac{n+1}{2} j \left(H^{(1)}_{\frac{p-1}{2}+j} - H^{(1)}_{\frac{p-1}{2}-j} \right) \Big],$$

and

(2.6)
$$Z(p, \lambda, n) := \phi_p(\lambda) \sum_{j=0}^{\frac{p-1}{2}} \binom{2j}{j}^{2l} 16^{-jl} \lambda^{-jp^2},$$

where $l = \frac{n+1}{2}$. The main result in [23] provides an expression for ${}_{n+1}F_n$ modulo p^3. Precisely, we have

Theorem 2.1. *Let p be an odd prime, $\lambda \in \mathbb{F}_p$, and $n \geq 2$ be an integer. Then*

$$-p^n {}_{n+1}F_n(\lambda) \equiv (-\phi_p(-1))^{n+1} \left[p^2 X(p,\lambda,n) + pY(p,\lambda,n) + Z(p,\lambda,n)\right] \pmod{p^3}.$$

3. Proof of Theorem 1.2. By Theorem 4 in [21] (or Proposition 4.2 in [19]) and Corollary 5 in [27], we have that

$$p^3 {}_3F_2(1) = \begin{cases} -\frac{p}{\Gamma_p\left(\frac{3}{4}\right)^4} \pmod{p^3} & \text{if} \quad p \equiv 1 \pmod 4, \\ 0 \pmod{p^3} & \text{if} \quad p \equiv 3 \pmod 4 . \end{cases}$$

Thus, by Theorem 2.1 it suffices to prove

$$\sum_{k=0}^{\frac{p-1}{2}} (4k+1) \binom{-\frac{1}{2}}{k}^5 \equiv \phi_p(-1) \left[p^3 X(p,1,2) + p^2 Y(p,1,2) + pZ(p,1,2)\right] \pmod{p^3} \tag{3.1}$$

where the quantities $X(p,\lambda,n), Y(p,\lambda,n)$ and $Z(p,\lambda,n)$ are defined by (2.4), (2.5), and (2.6) respectively. We first show, via the following lemmas, that the terms involving $Y(p,1,2)$ and $X(p,1,2)$ in (3.1) vanish modulo p^3.

Lemma 3.1. *Let p be an odd prime. Then*

$$Y(p,1,2) \equiv 0 \pmod{p} .$$

Proof. Substituting $\lambda = 1$ and $n = 2$ in equation (2.5), we get

$$\begin{aligned} Y(p,1,2) = \sum_{j=0}^{\frac{p-1}{2}} \binom{\frac{p-1}{2}+j}{j}^{\frac{3}{2}} \binom{\frac{p-1}{2}}{j}^{\frac{3}{2}} (-1)^{\frac{3}{2}j} \Big[1 + 3j \left(H^{(1)}_{\frac{p-1}{2}+j} - H^{(1)}_j \right) \\ - \frac{3}{2} j \left(H^{(1)}_{\frac{p-1}{2}+j} - H^{(1)}_{\frac{p-1}{2}-j} \right) \Big] . \end{aligned}$$

Noting that $\binom{u+k}{k} = (-1)^k \binom{-1-u}{k}$, we get

$$\begin{aligned} \binom{\frac{p-1}{2}+j}{j} \binom{\frac{p-1}{2}}{j} &= (-1)^j \binom{-\frac{1}{2}-\frac{p}{2}}{j} \binom{-\frac{1}{2}+\frac{p}{2}}{j} \\ &\equiv (-1)^j \binom{-\frac{1}{2}}{j}^2 \pmod{p^2} . \end{aligned} \tag{3.2}$$

Also,

$$\begin{aligned} H^{(1)}_{\frac{p-1}{2}+j} - H^{(1)}_{\frac{p-1}{2}-j} &= \frac{1}{\frac{p-1}{2}-j+1} + \frac{1}{\frac{p-1}{2}-j+2} + \cdots + \frac{1}{\frac{p-1}{2}} \\ &\quad + \frac{1}{\frac{p+1}{2}} + \cdots + \frac{1}{\frac{p-1}{2}+j} \\ &= \sum_{r=0}^{j-1} \frac{1}{\frac{p-1}{2}-r} + \frac{1}{\frac{p+1}{2}+r} \\ &= \sum_{r=0}^{j-1} \frac{4p}{p^2-(2r+1)^2} \\ &\equiv 0 \pmod p . \end{aligned}$$

So we need only show

$$\sum_{j=0}^{\frac{p-1}{2}} \binom{-\frac{1}{2}}{j}^3 (-1)^{3j} \left[1 + 3j\left(H^{(1)}_{\frac{p-1}{2}+j} - H^{(1)}_j\right)\right] \equiv 0 \pmod p . \tag{3.3}$$

For $j \geq 1$, note that

$$\binom{-\frac{1}{2}}{j}(-1)^j \equiv \frac{(j+1)_{\frac{p-1}{2}}}{\left(\frac{p-1}{2}\right)!} \pmod p . \tag{3.4}$$

As $\gcd\left(\left(\frac{p-1}{2}\right)!^3, p\right) = 1$, it now suffices to show

$$\left(\frac{p-1}{2}\right)!^3 + \sum_{j=1}^{\frac{p-1}{2}} (j+1)^3_{\frac{p-1}{2}} \left[1 + 3j\left(H^{(1)}_{\frac{p-1}{2}+j} - H^{(1)}_j\right)\right] \equiv 0 \pmod p . \tag{3.5}$$

We now use an argument similar to that in Section 4 of [16] (see also [19]). Let

$$P(z) := \frac{d}{dz}\left[z(z+1)^3_{\frac{p-1}{2}}\right] = \sum_{k=0}^{\frac{3p-3}{2}} a_k z^k \tag{3.6}$$

for some integers a_k. By a computation, we have

$$P(z) = (z+1)^3_{\frac{p-1}{2}} \left[1 + 3z\left(H^{(1)}_{\frac{p-1}{2}+z} - H^{(1)}_z\right)\right] .$$

Combining this with (3.5), it is enough to show that

$$\left(\frac{p-1}{2}\right)!^3 + \sum_{j=1}^{\frac{p-1}{2}} P(j) \equiv 0 \pmod p . \tag{3.7}$$

Note that, for $\frac{p-1}{2} < j < p$, $(j+1)_{\frac{p-1}{2}}$ is divisible by p and $H^{(i)}_{\frac{p-1}{2}+j} - H^{(i)}_j \in \frac{1}{p^i}\mathbb{Z}_p$, so that $P(j) \equiv 0 \pmod p$ for such j. Hence (3.7) will hold if we can show

$$\left(\frac{p-1}{2}\right)!^3 + \sum_{j=1}^{p-1} P(j) \equiv 0 \pmod p\ . \tag{3.8}$$

We now recall the following elementary fact about exponential sums. For a positive integer k, we have

$$\sum_{j=1}^{p-1} j^k \equiv \begin{cases} -1 \pmod p & \text{if } (p-1)|k\ , \\ \ \ 0 \pmod p & \text{otherwise}\ . \end{cases} \tag{3.9}$$

By (3.6), (3.9) and the fact that $\frac{3p-3}{2} < 2p-2$, we see that

$$\begin{aligned} \sum_{j=1}^{p-1} P(j) &= \sum_{j=1}^{p-1} \sum_{k=0}^{\frac{3p-3}{2}} a_k j^k \\ &= \sum_{k=0}^{\frac{3p-3}{2}} a_k \sum_{j=1}^{p-1} j^k \\ &\equiv -a_0 - a_{p-1} \pmod p\ . \end{aligned}$$

Additionally, by (3.6)

$$(z+1)^3_{\frac{p-1}{2}} = \cdots + \frac{a_{p-1}}{p} z^{p-1} + \cdots \qquad .$$

As $(z+1)^3_{\frac{p-1}{2}}$ has integer coefficients, p divides a_{p-1}. Hence $a_{p-1} \equiv 0 \pmod p$. One can also check that

$$a_0 = \left(\frac{p-1}{2}\right)!^3\ .$$

Thus

$$\sum_{j=1}^{p-1} P(j) \equiv -\left(\frac{p-1}{2}\right)!^3 \pmod p$$

and (3.8) holds. This proves the result. □

Now we would like to show that $\mathrm{ord}_p(X(p,1,2)) \geq 0$ which ensures that the term involving $X(p,1,2)$ in equation (3.1) vanishes modulo p^3. In fact, in the following lemma, we show that $\mathrm{ord}_p(X(p,1,2)) \geq 1$.

Lemma 3.2. *Let p be an odd prime. Then*

$$X(p,1,2) \equiv 0 \pmod p\ .$$

Proof. Substituting $\lambda = 1$ and $n = 2$ in equation (2.4) and applying (3.2) and (3.3) yields

$$X(p,1,2) \equiv \sum_{j=0}^{\frac{p-1}{2}} \binom{-\frac{1}{2}}{j}^3 (-1)^{3j} \left[3j \left(H^{(1)}_{\frac{p-1}{2}+j} - H^{(1)}_j \right) \right.$$
$$\left. + \frac{9}{2} j^2 \left(H^{(1)}_{\frac{p-1}{2}+j} - H^{(1)}_j \right)^2 - \frac{3}{2} j^2 \left(H^{(2)}_{\frac{p-1}{2}+j} - H^{(2)}_j \right) \right] \pmod{p} .$$

By (3.4) and as $\gcd\left(\left(\frac{p-1}{2}\right)!^3, p \right) = 1$, it suffices to prove that

$$\sum_{j=1}^{\frac{p-1}{2}} (j+1)^3_{\frac{p-1}{2}} \left[3j \left(H^{(1)}_{\frac{p-1}{2}+j} - H^{(1)}_j \right) + \frac{9}{2} j^2 \left(H^{(1)}_{\frac{p-1}{2}+j} - H^{(1)}_j \right)^2 \right.$$
$$\left. - \frac{3}{2} j^2 \left(H^{(2)}_{\frac{p-1}{2}+j} - H^{(2)}_j \right) \right] \equiv 0 \pmod{p} . \tag{3.10}$$

Similar to the proof of Lemma 3.1, we now let

$$Q(z) := \frac{z}{2} \frac{d^2}{dz^2} \left[z(z+1)^3_{\frac{p-1}{2}} \right] = \sum_{k=0}^{\frac{3p-3}{2}} a_k z^k \tag{3.11}$$

for some integers a_k. One can check that it now suffices to show

$$\sum_{j=1}^{p-1} Q(j) \equiv 0 \pmod{p} . \tag{3.12}$$

By (3.9), (3.11) and the fact that $\frac{3p-3}{2} < 2p-2$, we have

$$\begin{aligned} \sum_{j=1}^{p-1} Q(j) &= \sum_{j=1}^{p-1} \sum_{k=0}^{\frac{3p-3}{2}} a_k j^k \\ &= \sum_{k=0}^{\frac{3p-3}{2}} a_k \sum_{j=1}^{p-1} j^k \\ &\equiv -a_{p-1} \pmod{p} . \end{aligned}$$

Here we have used that $a_0 = 0$ as $z | Q(z)$. One can check that

$$(z+1)^3_{\frac{p-1}{2}} = \cdots + \frac{2a_{p-1}}{p(p-1)} z^{p-1} + \cdots \quad .$$

As $(z+1)^3_{\frac{p-1}{2}}$ has integer coefficients, p divides a_{p-1}. Hence $a_{p-1} \equiv 0 \pmod{p}$. Thus (3.12) holds and the result is proven □

Via (3.1), Lemmas 3.1 and 3.2, the proof of Theorem 1.2 is complete upon proving the following Proposition.

Proposition 3.3. *Let p be an odd prime. Then*

$$\sum_{k=0}^{\frac{p-1}{2}}(4k+1)\binom{-\frac{1}{2}}{k}^5 \equiv \phi_p(-1)\, p\, Z(p,1,2) \pmod{p^3}.$$

Proof. Substituting $\lambda = 1$ and $n = 2$ in equation (2.6), we get

$$Z(p,1,2) = \sum_{j=0}^{\frac{p-1}{2}}\binom{2j}{j}^3 16^{-\frac{3}{2}j}. \tag{3.13}$$

Noting that

$$\binom{2j}{j} = 2^{2j}(-1)^j\binom{-\frac{1}{2}}{j},$$

it suffices to prove

$$\sum_{k=0}^{\frac{p-1}{2}}(4k+1)\binom{-\frac{1}{2}}{k}^5 \equiv \phi_p(-1)\, p\left[\sum_{j=0}^{\frac{p-1}{2}}(-1)^j\binom{-\frac{1}{2}}{j}^3\right] \pmod{p^3}. \tag{3.14}$$

Letting $a = \frac{1}{2}$, $c = \frac{1}{2}+i\frac{p}{2}$, $d = \frac{1}{2}-i\frac{p}{2}$, $e = \frac{1}{2}+\frac{p}{2}$ and $f = \frac{1}{2}-\frac{p}{2}$ in (2.3), we get

$$\begin{aligned} &{}_6F_5\left[\begin{matrix}\frac{1}{2}, & \frac{5}{4}, & \frac{1}{2}+i\frac{p}{2} & \frac{1}{2}-i\frac{p}{2}, & \frac{1}{2}+\frac{p}{2}, & \frac{1}{2}-\frac{p}{2} \\ & \frac{1}{4}, & 1-i\frac{p}{2}, & 1+i\frac{p}{2}, & 1-\frac{p}{2}, & 1+\frac{p}{2}\end{matrix}\;\middle|\; -1\right] \\ &= \frac{\Gamma\left(1-\frac{p}{2}\right)\Gamma\left(1+\frac{p}{2}\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}\;{}_3F_2\left[\begin{matrix}\frac{1}{2} & \frac{1}{2}+\frac{p}{2}, & \frac{1}{2}-\frac{p}{2} \\ & 1-i\frac{p}{2}, & 1+i\frac{p}{2}\end{matrix}\;\middle|\; 1\right]. \end{aligned} \tag{3.15}$$

By (2.2) and the fact that $\frac{1}{2}-\frac{p}{2}$ is a negative integer,

$$\begin{aligned} &{}_6F_5\left[\begin{matrix}\frac{1}{2}, & \frac{5}{4}, & \frac{1}{2}+i\frac{p}{2} & \frac{1}{2}-i\frac{p}{2}, & \frac{1}{2}+\frac{p}{2}, & \frac{1}{2}-\frac{p}{2} \\ & \frac{1}{4}, & 1-i\frac{p}{2}, & 1+i\frac{p}{2}, & 1-\frac{p}{2}, & 1+\frac{p}{2}\end{matrix}\;\middle|\; -1\right] \\ &= \sum_{k=0}^{\frac{p-1}{2}} \frac{\left(\frac{1}{2}\right)_k\left(\frac{5}{4}\right)_k\left(\frac{1}{2}+i\frac{p}{2}\right)_k\left(\frac{1}{2}-i\frac{p}{2}\right)_k\left(\frac{1}{2}+\frac{p}{2}\right)_k\left(\frac{1}{2}-\frac{p}{2}\right)_k}{\left(\frac{1}{4}\right)_k\left(1-i\frac{p}{2}\right)_k\left(1+i\frac{p}{2}\right)_k\left(1-\frac{p}{2}\right)_k\left(1+\frac{p}{2}\right)_k}\,\frac{(-1)^k}{k!}. \end{aligned} \tag{3.16}$$

Now,

$$\left(\frac{1}{2}\right)_k \frac{(-1)^k}{k!} = \binom{-\frac{1}{2}}{k}, \tag{3.17}$$

$$\frac{\left(\frac{5}{4}\right)_k}{\left(\frac{1}{4}\right)_k} = 4k+1\,, \tag{3.18}$$

and

$$\frac{\left(\frac{1}{2}+i\frac{p}{2}\right)_k\left(\frac{1}{2}-i\frac{p}{2}\right)_k\left(\frac{1}{2}+\frac{p}{2}\right)_k\left(\frac{1}{2}-\frac{p}{2}\right)_k}{\left(1-i\frac{p}{2}\right)_k\left(1+i\frac{p}{2}\right)_k\left(1-\frac{p}{2}\right)_k\left(1+\frac{p}{2}\right)_k} \equiv \binom{-\frac{1}{2}}{k}^4 \pmod{p^4}\,. \tag{3.19}$$

Therefore, substituting (3.17), (3.18) and (3.19) into equation (3.16), we get

$$\begin{aligned} &{}_6F_5\left[\begin{matrix} \frac{1}{2}, & \frac{5}{4}, & \frac{1}{2}+i\frac{p}{2}, & \frac{1}{2}-i\frac{p}{2}, & \frac{1}{2}+\frac{p}{2}, & \frac{1}{2}-\frac{p}{2} \\ & \frac{1}{4}, & 1-i\frac{p}{2}, & 1+i\frac{p}{2}, & 1-\frac{p}{2}, & 1+\frac{p}{2} \end{matrix} \;\Big|\; -1\right] \\ &\equiv \sum_{k=0}^{\frac{p-1}{2}} (4k+1)\binom{-\frac{1}{2}}{k}^5 \pmod{p^4}\,. \end{aligned} \tag{3.20}$$

Next we examine the right hand side of (3.15). By (2.2),

$$\begin{aligned} &\frac{\Gamma\left(1-\frac{p}{2}\right)\Gamma\left(1+\frac{p}{2}\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}\,{}_3F_2\left[\begin{matrix} \frac{1}{2} & \frac{1}{2}+\frac{p}{2}, & \frac{1}{2}-\frac{p}{2} \\ & 1-i\frac{p}{2}, & 1+i\frac{p}{2} \end{matrix} \;\Big|\; 1\right] \\ &= \frac{\Gamma\left(1-\frac{p}{2}\right)\Gamma\left(1+\frac{p}{2}\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)} \sum_{k=0}^{\frac{p-1}{2}} \frac{\left(\frac{1}{2}\right)_k\left(\frac{1}{2}+\frac{p}{2}\right)_k\left(\frac{1}{2}-\frac{p}{2}\right)_k}{\left(1-i\frac{p}{2}\right)_k\left(1+i\frac{p}{2}\right)_k}\,\frac{1}{k!}\,. \end{aligned} \tag{3.21}$$

Now, via (2.1) and the fact that $\Gamma(x+1) = x\Gamma(x)$ and $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, we have

$$\begin{aligned} \frac{\Gamma\left(1-\frac{p}{2}\right)\Gamma\left(1+\frac{p}{2}\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)} &= \frac{\Gamma\left(1-\frac{p}{2}\right)\left(\frac{p}{2}\right)\Gamma\left(\frac{p}{2}\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)} \\ &= \frac{p}{\sin\left(\frac{p}{2}\pi\right)} \\ &= \phi_p(-1)\,p\,. \end{aligned} \tag{3.22}$$

Also, we have

$$\frac{\left(\frac{1}{2}+\frac{p}{2}\right)_k\left(\frac{1}{2}-\frac{p}{2}\right)_k}{\left(1-i\frac{p}{2}\right)_k\left(1+i\frac{p}{2}\right)_k} \equiv \binom{-\frac{1}{2}}{k}^2 \pmod{p^2}\,. \tag{3.23}$$

Using (3.17) and substituting (3.22), (3.23) into (3.21), we get

$$\frac{\Gamma\left(1-\frac{p}{2}\right)\Gamma\left(1+\frac{p}{2}\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}\,{}_3F_2\left[\begin{matrix}\frac{1}{2} & \frac{1}{2}+\frac{p}{2}, & \frac{1}{2}-\frac{p}{2}\\ & 1-i\frac{p}{2}, & 1+i\frac{p}{2}\end{matrix}\;\Big|\;1\right]$$

$$\equiv \phi_p(-1)\,p\left[\sum_{j=0}^{\frac{p-1}{2}}(-1)^j\binom{-\frac{1}{2}}{j}^3\right] \pmod{p^3}. \tag{3.24}$$

Finally, combining (3.15), (3.20), and (3.24) yields (3.14) and hence the result follows. □

Remark 3.4. We would like to mention another approach, kindly pointed out to us by Eric Mortenson, which confirms Theorem 1.2. By [27], the right hand side in Conjecture 1.1 is equal to $p\cdot a(p)$ where $a(p)$ is the p-th Fourier coefficient of $\eta^6(4z)$. Here $\eta(z)$ is the Dedekind eta-function. Thus, in conjunction with (3.14), Conjecture 1.1 follows from

$$\phi_p(-1)\left[\sum_{j=0}^{\frac{p-1}{2}}(-1)^j\binom{-\frac{1}{2}}{j}^3\right]\equiv a(p) \pmod{p^2}.$$

This congruence has been proven in [1], [15], [19], and [28].

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