Arch. Math. 91 (2008), 166–177
© 2008 Birkhäuser Verlag Basel/Switzerland
0003/889X/020166-12, *published online* 2008-07-29
DOI 10.1007/s00013-008-2545-8

Archiv der Mathematik

Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces

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Abstract. In this paper, the class of nonspreading mappings in Banach spaces is introduced. This class contains the recently introduced class of firmly nonexpansive type mappings in Banach spaces and the class of firmly nonexpansive mappings in Hilbert spaces. Among other things, we obtain a fixed point theorem for a single nonspreading mapping in Banach spaces. Using this result, we also obtain a common fixed point theorem for a commutative family of nonspreading mappings in Banach spaces.

Mathematics Subject Classification (2000). Primary 47H10; Secondary 47H05.

Keywords. Firmly nonexpansive mapping, firmly nonexpansive type mapping, fixed point theorem, nonspreading mapping, resolvent of monotone operator.

1. Introduction. The following Browder-Göhde-Kirk fixed point theorem is well-known; see also Goebel and Kirk [14] and Takahashi [31]:

Theorem 1.1 (Browder-Göhde-Kirk [4, 15, 19]**).** *Let E be a uniformly convex Banach space, let C be a nonempty bounded closed convex subset of E and let T be a nonexpansive mapping from C into itself. Then T has a fixed point.*

The fixed point problem for nonexpansive mappings in Hilbert spaces is connected with the problem of finding zero points of maximal monotone operators in the spaces. Let H be a (real) Hilbert space and let $A \subset H \times H$ be a maximal monotone operator. Then, for each $r > 0$, the *resolvent* J_r of A is defined by $J_r x = (I + rA)^{-1}x$ for all $x \in H$. It is well-known that J_r is a single-valued *firmly nonexpansive* mapping, that is,

$$\|J_r x - J_r y\|^2 \le \langle x - y, J_r x - J_r y\rangle \tag{1}$$

for all $x, y \in H$. It also holds that $F(J_r) = A^{-1}0$, where $F(J_r)$ denotes the set of fixed points of J_r. Thus the problem of finding zero points of maximal monotone operators in Hilbert spaces is reduced to the fixed point problem for firmly nonexpansive mappings. In particular, if A is the subdifferential ∂f of a proper lower semicontinuous convex function f from H into $(-\infty, \infty]$, then J_r is given by

$$J_r x = \arg\min_{y \in H} \left\{ f(y) + \frac{1}{2r} \|y - x\|^2 \right\} \tag{2}$$

for all $x \in H$. In this case, $F(J_r) = \{z \in H : f(z) = \inf_{y \in H} f(y)\}$.

There are two generalizations of the class of maximal monotone operators in Hilbert spaces to Banach spaces. One of them is the class of m-accretive operators and the other is that of maximal monotone operators. By Rockafellar's theorem [25, 26], the subdifferential ∂f of a proper lower semicontinuous convex function f from a Banach space E into $(-\infty, \infty]$ is a maximal monotone operator.

Let E be a Banach space and let C be a nonempty closed convex subset of E. Then a mapping T from C into itself is said to be *firmly nonexpansive* (Bruck [6]) if

$$\|Tx - Ty\| \le \|r(x - y) + (1 - r)(Tx - Ty)\| \tag{3}$$

for all $r > 0$, $x, y \in C$. It is known that T is firmly nonexpansive if and only if there exists an accretive operator $A \subset E \times E$ such that $D(A) \subset C \subset R(I + A)$ and $Tx = (I + A)^{-1}x$ for all $x \in C$. In this case, $F(T) = A^{-1}0$ holds. It is also known that T is firmly nonexpansive if and only if for all $x, y \in C$, there exists $j \in J(Tx - Ty)$ such that

$$\|Tx - Ty\|^2 \le \langle x - y, j \rangle, \tag{4}$$

where J is the normalized duality mapping from E into 2^{E^*}. Bruck and Reich [7] studied the asymptotic behavior of the sequence $\{T^n x\}$ for all $x \in C$, where T is a strongly nonexpansive mapping in a Banach space. As a corollary, they deduced a weak convergence theorem for a firmly nonexpansive mapping; see also Reich and Shafrir [24] for similar results on this subject. On the other hand, Smarzewski [28] obtained a fixed point theorem for λ-firmly nonexpansive mappings defined on a nonconvex subset of a Banach space; see also Kaczor [16] for a generalization of Smarzewski's result.

Recently, the authors [20] introduced the class of firmly nonexpansive type mappings in Banach spaces. Let C be a nonempty closed convex subset of a smooth Banach space E and let T be a mapping from C into itself. Then T is said to be of *firmly nonexpansive type* if

$$\langle Tx - Ty, JTx - JTy \rangle \le \langle Tx - Ty, Jx - Jy \rangle \tag{5}$$

for all $x, y \in C$. In [20], it was shown that T is of firmly nonexpansive type if and only if

$$\phi(Tx, Ty) + \phi(Ty, Tx) + \phi(Tx, x) + \phi(Ty, y) \le \phi(Tx, y) + \phi(Ty, x) \tag{6}$$

for all $x, y \in C$, where ϕ is the mapping from $E \times E$ into $[0, \infty)$ defined by

$$\phi(u, v) = \|u\|^2 - 2\langle u, Jv\rangle + \|v\|^2 \tag{7}$$

for all $u, v \in E$. We also know that if E is smooth, strictly convex and reflexive, C is a nonempty closed convex subset of E and $A \subset E \times E^*$ is a monotone operator such that

$$D(A) \subset C \subset J^{-1}R(J + rA) \tag{8}$$

for all $r > 0$, then for each $r > 0$, the resolvent J_r of A which is defined by $J_r x = (J + rA)^{-1}Jx$ for all $x \in C$ is a firmly nonexpansive type mapping and $F(J_r) = A^{-1}0$. In particular, if $A \subset E \times E^*$ is a maximal monotone operator, then $R(J + rA) = E^*$ for all $r > 0$; see [2, 5, 27, 30]. In this case, we can define the resolvent J_r of A by $J_r x = (J + rA)^{-1}Jx$ for all $x \in E$; see, for instance, [17, 18]. We know that J_r is a firmly nonexpansive type mapping from E into itself.

In this paper, we say that a mapping T from C into itself is *nonspreading* if

$$\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(Tx, y) + \phi(Ty, x) \tag{9}$$

for all $x, y \in C$. In view of (6) and

$$\phi(u, v) \geq (\|u\| - \|v\|)^2 \geq 0 \tag{10}$$

for all $u, v \in E$, it is obvious that every firmly nonexpansive type mapping is nonspreading. In Section 3, we first show that the class of firmly nonexpansive type mappings coincides with that of resolvents of monotone operators in Banach spaces (Proposition 3.1). After that, we prove that every nonspreading mapping in a Banach space with a fixed point is *relatively nonexpansive* in the sense of Matsushita and Takahashi [21, 22] (Theorem 3.3). In Section 4, we first obtain a fixed point theorem for a single nonspreading mapping in Banach spaces (Theorem 4.1). Using this result, we also obtain a common fixed point theorem for a commutative family of nonspreading mappings in Banach spaces (Theorem 4.6).

2. Preliminaries. Throughout this paper, all linear spaces are real. Let $\mathbb{N}$ be the set of all positive integers. Let E be a Banach space and let E^* be the dual space of E. For a sequence $\{x_n\}$ of E and a point $x \in E$, the weak convergence of $\{x_n\}$ to x and the strong convergence of $\{x_n\}$ to x are denoted by $x_n \rightharpoonup x$ and $x_n \to x$, respectively. A set-valued mapping $A \subset E \times E^*$ with domain $D(A) = \{x \in E : Ax \neq \emptyset\}$ and range $R(A) = \bigcup\{Ax : x \in D(A)\}$ is said to be *monotone* if $\langle x - y, x^* - y^*\rangle \geq 0$ whenever $(x, x^*), (y, y^*) \in A$. A monotone operator $A \subset E \times E^*$ is also said to be *maximal monotone* if $A = B$ whenever $B \subset E \times E^*$ is a monotone operator such that $A \subset B$.

Let E be a Banach space. Then the *duality mapping* J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\} \tag{11}$$

for all $x \in E$. Let $S(E)$ be the unit sphere centered at the origin of E. Then the space E is said to be *smooth* if the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{12}$$

exists for all $x, y \in S(E)$. The norm of E is also said to be *uniformly Gâteaux differentiable* if for all $y \in S(E)$, the limit (12) attains uniformly in $x \in S(E)$. A Banach space E is said to be *strictly convex* if $\|(x+y)/2\| < 1$ whenever $x, y \in S(E)$ and $x \neq y$. We know the following; see, for instance, Cioranescu [13] and Takahashi [31]:

(1) If E is smooth, then J is single-valued;
(2) if E is reflexive, then J is onto;
(3) if E is strictly convex, then J is one-to-one, that is, $Jx \cap Jy \neq \emptyset$ implies that $x = y$;
(4) if E is strictly convex, then J is strictly monotone, that is, if (x, x^*), $(y, y^*) \in J$ and $\langle x - y, x^* - y^* \rangle = 0$, then $x = y$.

Let E be a smooth Banach space. Following Alber [1] and Kamimura and Takahashi [18], let $\phi : E \times E \to [0, \infty)$ be the mapping defined by (7). It is well-known that ϕ is the *Bregman distance* corresponding to $\|\cdot\|^2$; see Bregman [3], Butnariu and Iusem [8] and Censor and Lent [11]. It is known that

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle \tag{13}$$

for all $x, y, z \in E$. It is also known that

$$2\langle x - y, Jz - Jw \rangle = \phi(x, w) + \phi(y, z) - \phi(x, z) - \phi(y, w) \tag{14}$$

for all $x, y, z, w \in E$. Let C be a nonempty closed convex subset of E and let T be a mapping from C into itself. We denote the set of fixed points of T by $F(T)$, that is, $F(T) = \{z \in C : Tz = z\}$. A point $u \in C$ is said to be an *asymptotic fixed point* (Reich [23]) of T if there exists a sequence $\{x_n\}$ of C such that $x_n \rightharpoonup u$ and $x_n - Tx_n \to 0$. We denote the set of asymptotic fixed points of T by $\widehat{F}(T)$. Following Matsushita and Takahashi [21, 22], we say that a mapping T from C into itself is *relatively nonexpansive* if the following conditions are satisfied; see also [8, 9, 10, 12, 23] for similar classes of nonlinear operators:

(1) $F(T)$ is nonempty;
(2) $\phi(u, Tx) \leq \phi(u, x)$ for all $(x, u) \in C \times F(T)$;
(3) $\widehat{F}(T) = F(T)$.

We know the following lemma:

Lemma 2.1 ([22]). *Let E be a smooth and strictly convex Banach space, let C be a nonempty closed convex subset of E and let T be a mapping from C into itself such that $F(T)$ is nonempty and*

$$\phi(u, Tx) \leq \phi(u, x) \tag{15}$$

for all $u \in F(T)$ and $x \in C$. Then $F(T)$ is closed and convex.

Using Lemma 2.1, we can show the following:

Proposition 2.2. *Let E be a smooth and strictly convex Banach space, let C be a nonempty closed convex subset of E and let T be a nonspreading mapping from C into itself. Then $F(T)$ is closed and convex.*

Proof. It is sufficient to consider the case that $F(T)$ is nonempty. In this case, since T is nonspreading, we have

$$\phi(Tx, Tu) + \phi(Tu, Tx) \leq \phi(Tx, u) + \phi(Tu, x) \tag{16}$$

for all $u \in F(T)$ and $x \in C$, that is, $\phi(u, Tx) \leq \phi(u, x)$ for all $u \in F(T)$ and $x \in C$. Thus, by Lemma 2.1, $F(T)$ is closed and convex. □

3. Firmly nonexpansive type mappings and nonspreading mappings. In this section, we study some properties of firmly nonexpansive type mappings and nonspreading mappings in Banach spaces. We first show that the class of firmly nonexpansive type mappings coincides with that of resolvents of monotone operators in Banach spaces.

Proposition 3.1. *Let E be a smooth, strictly convex and reflexive Banach space, let C be a nonempty closed convex subset of E and let T be a mapping from C into itself. Then the following are equivalent:*

(1) *T is of firmly nonexpansive type;*
(2) *there exists a monotone operator $A \subset E \times E^*$ such that*

$$D(A) \subset C \subset J^{-1}R(J + A) \tag{17}$$

and $Tx = (J + A)^{-1}Jx$ for all $x \in C$.

Proof. By [20], we know that (2) implies (1). For the sake of completeness, we give the proof. Suppose that there exists a monotone operator $A \subset E \times E^*$ such that $D(A) \subset C \subset J^{-1}R(J + A)$ and $Tx = (J + A)^{-1}Jx$ for all $x \in C$. Since E is smooth and strictly convex, T is single-valued. By $D(A) \subset C \subset J^{-1}R(J+A)$, T is a mapping from C into itself. Let $x, y \in C$ be given. Then we have $Jx-JTx \in ATx$ and $Jy - JTy \in ATy$. Since A is monotone, we have

$$\langle Tx - Ty, Jx - JTx - (Jy - JTy)\rangle \geq 0. \tag{18}$$

This shows that T is of firmly nonexpansive type.

We next show that (1) implies (2). Suppose that T is of firmly nonexpansive type. Let $A \subset E \times E^*$ be the set-valued mapping defined by $A = JT^{-1} - J$, where T^{-1} is defined by

$$T^{-1}u = \begin{cases} \{v \in C : Tv = u\} & (u \in R(T)); \\ \emptyset & (\text{otherwise}). \end{cases} \tag{19}$$

It is obvious that $Tx = (J + A)^{-1}Jx$ for all $x \in C$. We show that A is monotone. Let $(x_i, x_i^*) \in A$ be given $(i = 1, 2)$. Then we have

$$
\begin{aligned}
x_i^* \in Ax_i &\Longleftrightarrow x_i^* \in JT^{-1}x_i - Jx_i \\
&\Longleftrightarrow x_i^* + Jx_i \in JT^{-1}x_i \\
&\Longleftrightarrow J^{-1}(x_i^* + Jx_i) \in T^{-1}x_i \\
&\Longleftrightarrow TJ^{-1}(x_i^* + Jx_i) = x_i
\end{aligned} \tag{20}
$$

for $i = 1, 2$. Putting $u_i = J^{-1}(x_i^* + Jx_i)$, we have $Tu_i = x_i$ and $Ju_i - Jx_i = x_i^*$ $(i = 1, 2)$. Since T is of firmly nonexpansive type, we have

$$
\begin{aligned}
\langle x_1 - x_2, x_1^* - x_2^* \rangle &= \langle Tu_1 - Tu_2, x_1^* - x_2^* \rangle \\
&= \langle Tu_1 - Tu_2, Ju_1 - Jx_1 - (Ju_2 - Jx_2) \rangle \\
&= \langle Tu_1 - Tu_2, Ju_1 - JTu_1 - (Ju_2 - JTu_2) \rangle \geq 0.
\end{aligned} \tag{21}
$$

Thus A is monotone. We finally show that $D(A) \subset C \subset J^{-1}R(J + A)$. It is easy to see that $D(A) = D(JT^{-1} - J) = D(T^{-1}) = R(T) \subset C$. Since $J + A = JT^{-1}$, we also have

$$
R(J + A) = R(JT^{-1}) = D((JT^{-1})^{-1}) = D(TJ^{-1}) = JD(T) = JC. \tag{22}
$$

Thus $C = J^{-1}R(J + A)$. This completes the proof. □

We next show that every nonspreading mapping with a fixed point is relatively nonexpansive in the sense of Matsushita and Takahashi [21, 22] (Theorem 3.3). Before proving it, we show the following proposition:

Proposition 3.2. *Let E be a strictly convex Banach space with a uniformly Gâteaux differentiable norm, let C be a nonempty closed convex subset of E and let T be a nonspreading mapping from C into itself. Then $F(T) = \widehat{F}(T)$.*

Proof. The inclusion $\widehat{F}(T) \supset F(T)$ is obvious. Thus we show $\widehat{F}(T) \subset F(T)$. Let $u \in \widehat{F}(T)$ be given. Then we have a sequence $\{x_n\}$ of C such that $x_n \rightharpoonup u$ and $x_n - Tx_n \to 0$. Since the norm of E is uniformly Gâteaux differentiable, J is uniformly norm-to-weak* continuous on each bounded subset of E; see Takahashi [31]. Thus

$$
\lim_{n\to\infty} \langle w, JTx_n - Jx_n \rangle = 0 \tag{23}
$$

for all $w \in E$. On the other hand, since T is nonspreading, we have

$$
\phi(Tx_n, Tu) + \phi(Tu, Tx_n) \leq \phi(Tx_n, u) + \phi(Tu, x_n) \tag{24}
$$

for all $n \in \mathbb{N}$. This implies that

$$\begin{aligned} 0 &\le \phi(Tx_n, u) - \phi(Tx_n, Tu) + \phi(Tu, x_n) - \phi(Tu, Tx_n) \\ &= 2\langle Tx_n, JTu - Ju\rangle + \|u\|^2 - \|Tu\|^2 \\ &\quad + 2\langle Tu, JTx_n - Jx_n\rangle + \|x_n\|^2 - \|Tx_n\|^2 \\ &\le 2\langle Tx_n, JTu - Ju\rangle + \|u\|^2 - \|Tu\|^2 \\ &\quad + 2\langle Tu, JTx_n - Jx_n\rangle + (\|x_n\| + \|Tx_n\|)\|x_n - Tx_n\| \end{aligned} \tag{25}$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ in (25), we get from (23), $Tx_n - x_n \to 0$ and $x_n \rightharpoonup u$ that

$$\begin{aligned} 0 &\le 2\langle u, JTu - Ju\rangle + \|u\|^2 - \|Tu\|^2 \\ &\le \phi(u, u) - \phi(u, Tu) \\ &\le -\phi(u, Tu). \end{aligned} \tag{26}$$

Thus $\phi(u, Tu) \le 0$. This implies that $\phi(u, Tu) = 0$. Since E is strictly convex, we obtain $u = Tu$. This completes the proof. □

Using Proposition 3.2, we can prove the following theorem:

Theorem 3.3. *Let E be a strictly convex Banach space with a uniformly Gâteaux differentiable norm, let C be a nonempty closed convex subset of E and let T be a nonspreading mapping from C into itself such that $F(T)$ is nonempty. Then T is relatively nonexpansive.*

Proof. By assumption, $F(T)$ is nonempty. Since T is nonspreading, we have

$$\phi(u, Tx) \le \phi(u, x) \tag{27}$$

for all $u \in F(T)$ and $x \in C$. By Proposition 3.2, $\widehat{F}(T) = F(T)$. Thus T is a relatively nonexpansive mapping from C into itself. □

4. Fixed point theorems for nonspreading mappings. In this section, we obtain fixed point theorems for nonspreading mappings in a Banach space. Using the technique developed by Takahashi [29, 31], we first prove the following fixed point theorem for a single nonspreading mapping in Banach spaces:

Theorem 4.1. *Let E be a smooth, strictly convex and reflexive Banach space, let C be a nonempty closed convex subset of E and let T be a nonspreading mapping from C into itself. Then the following are equivalent:*

(1) *There exists $x \in C$ such that $\{T^n x\}$ is bounded;*
(2) *$F(T)$ is nonempty.*

Proof. Since it is obvious that (2) implies (1), we show that (1) implies (2). Suppose that there exists $x \in C$ such that $\{T^n x\}$ is bounded. Let $y \in C$, $k \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$ be given. Since T is nonspreading and (13) holds, we have

$$\begin{aligned} &\phi(T^{k+1}x, Ty) + \phi(Ty, T^{k+1}x) \\ &\le \phi(T^{k+1}x, y) + \phi(Ty, T^k x) \\ &= \phi(T^{k+1}x, Ty) + \phi(Ty, y) + 2\langle T^{k+1}x - Ty, JTy - Jy\rangle + \phi(Ty, T^k x). \end{aligned} \tag{28}$$

This implies that

$$0 \le \phi(Ty, y) + \phi(Ty, T^k x) - \phi(Ty, T^{k+1}x) + 2\langle T^{k+1}x - Ty, JTy - Jy\rangle. \tag{29}$$

Summing these inequalities with respect to $k = 0, 1, \ldots, n-1$, we have

$$\begin{aligned} 0 \le\ & n\phi(Ty, y) \\ &+ \phi(Ty, x) - \phi(Ty, T^n x) + 2\left\langle \sum_{k=0}^{n-1} T^{k+1}x - nTy, JTy - Jy \right\rangle. \end{aligned} \tag{30}$$

Dividing this inequality by n, we have

$$0 \le \phi(Ty, y) + \frac{1}{n}\{\phi(Ty, x) - \phi(Ty, T^n x)\} + 2\langle S_n(Tx) - Ty, JTy - Jy\rangle, \tag{31}$$

where $S_n(z) = (1/n)\sum_{k=0}^{n-1} T^k z$ for all $z \in C$. Since $\{T^n x\}$ is bounded by assumption, $\{S_n(Tx)\}$ is also bounded. Thus we have a subsequence $\{S_{n_i}(Tx)\}$ of $\{S_n(Tx)\}$ such that $S_{n_i}(Tx) \rightharpoonup u \in C$. Letting $n_i \to \infty$ in (31), we obtain

$$0 \le \phi(Ty, y) + 2\langle u - Ty, JTy - Jy\rangle. \tag{32}$$

Putting $y = u$ in (32), we have from (14) that

$$\begin{aligned} 0 &\le \phi(Tu, u) + 2\langle u - Tu, JTu - Ju\rangle \\ &= \phi(Tu, u) + \big\{\phi(u, u) + \phi(Tu, Tu) - \phi(u, Tu) - \phi(Tu, u)\big\} \\ &= -\phi(u, Tu) \end{aligned} \tag{33}$$

Hence we have $\phi(u, Tu) \le 0$ and hence $\phi(u, Tu) = 0$. Since E is strictly convex, we obtain $u = Tu$. Therefore, $F(T)$ is nonempty. This completes the proof. □

As direct consequences of Theorem 4.1, we have the following:

Corollary 4.2. *Every bounded closed convex subset of a smooth, strictly convex and reflexive Banach space has the fixed point property for nonspreading self mappings.*

Corollary 4.3 ([20])**.** *Let E be a smooth, strictly convex and reflexive Banach space, let C be a nonempty closed convex subset of E and let T be a firmly nonexpansive type mapping from C into itself. Then the following are equivalent:*

(1) *There exists $x \in C$ such that $\{T^n x\}$ is bounded;*
(2) *$F(T)$ is nonempty.*

Corollary 4.4. *Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let T be a mapping from C into itself such that*

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2 \tag{34}$$

for all $x, y \in C$. Then the following are equivalent:

(1) *There exists $x \in C$ such that $\{T^n x\}$ is bounded;*
(2) *$F(T)$ is nonempty.*

Proof. In a Hilbert space H, we know that $\phi(u, v) = \|u - v\|^2$ for all $u, v \in H$. So, the mapping T in Corollary 4.4 is a nonspreading mapping from C into itself. By Theorem 4.1, we obtain the desired result. □

To prove a common fixed point theorem (Theorem 4.6), we need the following lemma:

Lemma 4.5. *Let E be a smooth, strictly convex and reflexive Banach space, let C be a nonempty bounded closed convex subset of E and let $\{T_1, T_2, \dots, T_N\}$ be a commutative finite family of nonspreading mappings from C into itself. Then $\{T_1, T_2, \dots, T_N\}$ has a common fixed point.*

Proof. The proof is given by induction with respect to N. We first show the case that $N = 2$. By Proposition 2.2 and Corollary 4.2, $F(T_1)$ is nonempty, bounded, closed and convex. It follows from $T_1T_2 = T_2T_1$ that $F(T_1)$ is T_2-invariant. In fact, if $u \in F(T_1)$, then we have $T_1T_2u = T_2T_1u = T_2u$.

Thus we have $T_2u \in F(T_1)$. Hence the restriction of T_2 to $F(T_1)$ is a nonspreading self mapping. By Corollary 4.2, T_2 has a fixed point in $F(T_1)$, that is, we have $v \in F(T_1)$ such that $T_2v = v$. Consequently, $v \in F(T_1) \cap F(T_2)$.

Suppose that for some $n \geq 2$, $X = \bigcap_{k=1}^{n} F(T_k)$ is nonempty. Then X is a nonempty bounded closed convex subset of C and the restriction of T_{n+1} to X is a nonspreading self mapping. By Corollary 4.2, T_{n+1} has a fixed point in X. This shows that $X \cap F(T_{n+1})$ is nonempty, that is, $\bigcap_{k=1}^{n+1} F(T_k)$ is nonempty. This completes the proof. □

Using Lemma 4.5, we can finally prove the following common fixed point theorem for a commutative family of nonspreading mappings in a Banach space:

Theorem 4.6. *Let E be a smooth, strictly convex and reflexive Banach space, let C be a nonempty bounded closed convex subset of E and let $\{T_\alpha\}_{\alpha \in A}$ be a commutative family of nonspreading mappings from C into itself. Then $\{T_\alpha\}_{\alpha \in A}$ has a common fixed point.*

Proof. By Proposition 2.2, we know that each $F(T_\alpha)$ is a closed convex subset of C. Since E is reflexive and C is bounded, closed and convex, C is weakly compact. Thus, to show that $\bigcap_{\alpha \in A} F(T_\alpha)$ is nonempty, it is sufficient to show that $\{F(T_\alpha)\}_{\alpha\in A}$ has the finite intersection property. By Lemma 4.5, $\{F(T_\alpha)\}_{\alpha\in A}$ has this property. Thus the proof is completed. □

Acknowledgement. The authors would like to express their sincere appreciation to the anonymous referee for valuable comments on the original version of the manuscript.

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Received: 10 August 2007