Arch. Math. 90 (2008), 193–199

0003/889X/030193-7, *published online* 2008-02-06
DOI 10.1007/s00013-007-2392-z

Archiv der Mathematik

A note on character kernels in finite groups of prime power order

Guohua Qian and Yanming Wang

Abstract. In this note, we classify the finite groups of prime power order for which all nonlinear irreducible character kernels constitute a chain with respect to inclusion.

Mathematics Subject Classification (2000). 20C15.

Keywords. Finite p-group, p-group of maximal class, character kernel.

1. Introduction. For a finite group G, we write $\mathrm{Kern}(G)$ to denote the set of kernels of nonlinear irreducible characters of G. Since every normal subgroup is an intersection of some irreducible character kernels, the set $\mathrm{Kern}(G)$ heavily influences the structure of the group G. In this note, we determine the finite p-groups G for which $\mathrm{Kern}(G)$ is just a chain with respect to inclusion. This is the first half of Research Problem 25 in [1] posed by Y. Berkovich.

Main Theorem *Let G be a finite nonablian p-group. Then the following statements are equivalent* :

(1) $\mathrm{Kern}(G)$ *is a chain with respect to inclusion.*

(2) *Whenever $N < G'$ is a normal subgroup of G, N is a member of* $\mathrm{Kern}(G)$.

(3) *G is one of the following groups* :

(3.1) *G' is a unique minimal normal subgroup of G.*

(3.2) *G is of maximal class.*

Project supported by the NNSF of China (Grant No. 10571128) and the NSF of Jiangsu Educational Committee(05KJB110002).

Remark 1.1. If a finite p-group G is of type (3.1), then $Z(G) \geq G'$ is cyclic, $G/Z(G)$ is an elementary abelian group of order p^{2m}, and all its nonlinear irreducible characters are faithful and of degree p^m (see [3, Lemma 12.3]).

Remark 1.2. For a finite p-group G, we write $G_1 = G$, and $G_n = [G_{n-1}, G]$ for $n \geq 2$. Then $G_2 = G'$. The class $c(G)$ of G is defined by an integer n such that $G_{n+1} = 1$ but $G_n > 1$. A nonabelian p-group G is called to be of maximal class provided that $c(G) = -1 + log_p|G|$. For more detailed information about p-groups of maximal class, we refer readers to [2, Chapter 3, §14].

In this note, p always denotes a prime integer. For a finite group, $\mathrm{Irr}(G)$ is the set of irreducible complex characters of G, and $\mathrm{cd}(G) := \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$.

2. Proofs.

Lemma 2.1. *Let G be a finite p-group of class 3. Suppose that $G_3 = Z(G) \cap G_2$, $|G_3| = p$, and G_2/G_3 is cyclic. Then G/G_3 has a normal abelian subgroup of index p. In particular, $cd(G/G_3) = \{1, p\}$*

Proof. Let $y_1 \in G_2 - G_3$ be such that $G_2 = \langle y_1 \rangle G_3$ and y_2 be any element in $G_2 - G_3$. Note that $|C_{G/G_3}(y_2G_3)| \leq |C_G(y_2)|$ by [3, Corollary 2.24]. We have

$$p^{-1}|G| = |C_{G/G_3}(y_2G_3)| \leq |C_G(y_2)| \leq p^{-1}|G|$$

and

$$C_G(y_1) = C_G(G_2) \leq C_G(y_2).$$

It follows that $C_G(y_2) = C_G(G_2)$ is of order $p^{-1}|G|$ for any $y_2 \in G_2 - G_3$. Take $x \in G - C_G(G_2)$. Then

$$C_G(x) \cap G_2 = G_3.$$

Since $G_2/G_3 \leq Z(G/G_3)$, we have

$$C_G(x)G_2/G_3 = C_G(x)/G_3 \times G_2/G_3.$$

Observe that $|C_G(x)/G_3| = p^{-1}|C_G(x)| \geq p^{-1}|G/G_2|$, and that $C_G(x)/G_3 \cong C_G(x)G_2/G_2 \leq G/G_2$ is abelian. Thus $C_G(x)G_2/G_3$ is abelian and of order at least $p^{-1}|G/G_2||G_2/G_3| = p^{-1}|G/G_3|$, and so $|G/G_3 : C_G(x)G_2/G_3| = p$ because G/G_3 is nonabelian. Now [3, Theorem 12.11] implies that $\mathrm{cd}(G/G_3) = \{1, p\}$. □

For any finite nonabelian p-group G, it is easy to see that $c(G) \leq 1 + log_p|G'|$. Now applying Lemma 2.1 on G/G_4, we conclude the following consequence which seems of independent interest.

Corollary 2.2. *Suppose that G is a finite p-group of class at least 3. Then G is of maximal class if and only if $c(G) = 1 + log_p|G'|$ and G/G_3 is an extraspecial group.*

Lemma 2.3. *Let G be a finite nonabelian group and K be the intersection of all members of* $\mathrm{Kern}(G)$*. Then $K = 1$.*

Proof. By [3, Lemma 2.21] and [3, Corollary 2.23], we have $K \cap G' = 1$. Assume the contrary, that is $K > 1$. Let λ be a nonprincipal irreducible character of K and χ be any irreducible constituent of λ^G. Then $\ker\chi \not\geq K$. This implies that χ is linear, and so χ is an extension of λ to G. Take nonlinear $\psi_0 \in \mathrm{Irr}(G/K)$. By [3, Corollary 6.17], $\chi\psi_0 \in \mathrm{Irr}(G)$ is nonlinear and $\ker(\chi\psi_0) \not\geq K$, a contradiction. $\square$

Definition 2.4. *Let K be a normal subgroup of some finite group G. We say that K is a heavy subgroup of G (or K is heavy in G) if for any normal subgroup T of G, either $T \geq K$ or $T < K$.*

Now we are ready to prove our main theorem.

Theorem 2.5. *Let G be a finite nonabelian p-group. Then the following statements are equivalent:*

(1) $\mathrm{Kern}(G)$ *is a chain with respect to inclusion.*

(2) *All normal subgroups of G contained in G' are heavy subgroups of G.*

(3) *G is one of the following groups:*

(3.1) *G' is a unique minimal normal subgroup of G.*

(3.2) *G is of maximal class.*

(4) *Whenever $N < G'$ is a normal subgroup of G, N is a member of* $\mathrm{Kern}(G)$.

Proof. (1) $\Rightarrow$ (2). Suppose that $\mathrm{Kern}(G)$ is a chain with respect to inclusion. Let $K \lhd G$ with $K \leq G'$. For any normal subgroup T of G, let us consider a quotient group $G/(T \cap K)$. If $G/(T \cap K)$ is abelian, then $T \geq T \cap K \geq G' \geq K$. Suppose that $G/(T \cap K)$ is nonabelian. Let $\chi_1, \cdots, \chi_s$ be all nonlinear irreducible characters of G with $T \cap K \leq \ker\chi_i$. Clearly, those χ_i's are just all nonlinear irreducible characters of $G/(T \cap K)$. Applying Lemma 2.3 on $G/(T \cap K)$, we conclude that $T \cap K = \ker\chi_1 \cap \cdots \cap \ker\chi_s$. By the hypothesis, we may assume that $\ker\chi_1 \leq \ker\chi_2 \leq \cdots \leq \ker\chi_s$, then we see that $G/(T \cap K)$ has a faithful irreducible character χ_1. It follows by [3, Lemma 2.27] that the center of $G/(T \cap K)$ is cyclic. Suppose that both $T/(T \cap K)$ and $K/(T \cap K)$ are nontrivial. Since $G/(T \cap K)$ is a p-group, both $Z(G/(T \cap K)) \cap T/(T \cap K)$ and $Z(G/(T \cap K)) \cap K/(T \cap K)$ are nontrivial, and this leads to a contradiction: $Z(G/(T \cap K))$ is not cyclic. Thus either $T/(T \cap K) = 1$ or $K/(T \cap K) = 1$, and so $T \leq K$ or $K \leq T$. Now any normal subgroup K of G with $K \leq G'$ is a heavy subgroup of G.

(2) $\Rightarrow$ (3). We claim first that the hypothesis (2) is inherited by any quotient group G/N whenever $N \leq G'$ and $N \lhd G$. Suppose that $K/N, T/N$ are normal subgroups of G/N with $K/N \leq (G/N)' = G'N/N = G'/N$. Then K, T are normal in G with $K \leq G'$. Since K is heavy in G, we have either $K \leq T$ or $K \geq T$, and so either $K/N \leq T/N$ or $K/N \geq T/N$. Thus K/N is heavy in G/N, as claimed.

Suppose that G' is of order p. Since G' is a heavy subgroup of G, G' is a unique minimal normal subgroup of G, and thus G is of type (3.1).

In what follows, we always assume $|G_2| \geq p^2$ and we shall show that G is of maximal class. Write $c(G) = n$.

Claim (I). G_i/G_{i+1} is cyclic for any $i = 2, \cdots, n$.

Since the hypothesis is inherited by quotient group G/G_n, by induction we conclude that G_i/G_{i+1} is cyclic for any $i = 2, \cdots, n-1$. Since $G_n \leq Z(G)$, all subgroups in G_n are normal and so heavy in G. This implies that all subgroups of G_n constitute a chain with respect to inclusion, and hence G_n is cyclic.

Claim (II). Let T be an abelian normal subgroup of G and suppose that $T \cap G_2$ is a cyclic group of order at least p^2. Then T is cyclic.

Let us consider $\Omega_1(T) = \langle x \in T \mid x^p = 1 \rangle$. $\Omega_1(T)$ is characteristic in T, and so $\Omega_1(T) \lhd G$. Since $\Omega_1(T)$ is elementary abelian and $G_2 \cap T$ is a cyclic heavy subgroup of order at least p^2, we must have $\Omega_1(T) < G_2 \cap T$. Thus $\Omega_1(T)$ is of order p, and so T is cyclic.

Claim (III). $|G_2/G_3| = p$.

Suppose that $|G_2/G_3| \geq p^2$. To see a contradiction, by induction we may assume $G_3 = 1$. Now G_2 is cyclic by claim (I). Let x be any element outside G_2. As $G_2 = G' \leq Z(G)$, $\langle x \rangle G_2$ is an abelain normal subgroup of G. Since G_2 is a cyclic group of order at least p^2, claim (II) yields that $\langle x \rangle G_2$ is cyclic. Thus any element outside G_2 is of order at leat p^3. By [2, Chapter 3, Theorem 8.2], we conclude that G is a cyclic group or a quaternion 2-group. Since G is nonabelian and any nonabelian quaternion group has more than two cyclic subgroups of order 4, we get a contradiction.

Claim (IV). If $|G_2| = p^2$, then G is of maximal class.

By claim (III), we have $|G_2/G_3| = |G_3| = p$. It follows by Lemma 2.1 that $\mathrm{cd}(G/G_3) = \{1, p\}$. Let $Z \lhd G$ be such that $Z/G_3 = Z(G/G_3)$. Since G/G_3 is of type (3.1), [3, Lemma 12.3] yields that $|G : Z| = p^2$ and Z/G_3 is a cyclic group.

Suppose that $Z > G_2$ and Z is noncyclic . Observe that Z is abelian, it follows that $Z = A \times G_3$, where $A \cong Z/G_3$. Let $W = \langle g^p \mid g \in Z \rangle$. Then W is a characteristic subgroup of Z, so $W \lhd G$. As $1 < W < A$, neither $W \geq G_3$ nor $W < G_3$ holds, and this contradicts the assumption that G_3 is heavy in G.

Suppose that $Z > G_2$ and Z is cyclic. In this case, it is easy to see that $G_3 = Z(G)$, $Z/Z(G) = Z(G/Z(G))$. It follows by [2, Chapter 3, Theorem 7.7] that $p = 2$ and G possesses a cyclic subgroup of index 2. Now we know either $|G/G_2| = 4$ or $|G_2| = 2$ (see [2, Chapter 1, Theorem 14.9], a contradiction.

Therefore $Z = G_2$, and so G is of maximal class.

Claim (V). G is of maximal class.

Let $E \lhd G$ with $|G_2/E| = p^2$. Since our hypothesis is inherited by the quotient group G/E, it follows by claim (IV) that G/E is of maximal class. In particular, G/G_2 is an elementary abelian p-group of order p^2.

Suppose that $p = 2$. Then the result follows from [2, Chapter 3, Theorem 11.9].

Suppose now that $p > 2$. By induction and claim (III) and claim (IV), G/G_n is of maximal class. In particular, $|G_i/G_{i+1}| = p$ for any $i = 2, \cdots n-1$. Now it suffices to show that $|G_n| = p$. Suppose that $|G_n| \geq p^2$. Since G_n is cyclic, we can find an integer s minimal subject to $2 \leq s \leq n$ and G_s is cyclic. Let T/G_s be any chief factor of G.

Assume T is nonabelian for some chief factor T/G_s. Then T is nonabelian but possess a cyclic subgroup G_s of index p. It follows by [2, Chapter 3, Lemma 8.7] that T possesses a unique noncyclic subgroup A of index p (note that $p > 2$). Thus A is characteristic in T and so normal in G. Clearly neither $A \geq G_s$ nor $A < G_s$ holds, this contradicts the assumption that G_s is heavy in G.

Assume T is abelian for any chief factor T/G_s. Then T is cyclic by claim (III). Suppose that $s \geq 3$. Since $|G_{s-1}/G_s| = p$, we may choose $T = G_{s-1}$. Then G_{s-1} is cyclic, which contradicts the minimality of s. Therefore $G_2 = G_s$. Now for any element x outside G_2, since G/G_2 is elementary abelian, we may take $T = \langle x \rangle G_2$. As T is cyclic, x is of order at least p^3. This implies that G possesses a unique subgroup of order p^2, which contradicts [2, Chapter 3,Theorem 8.3].

Thus $|G_n| = p$, and G is of maximal class.

(3) $\Rightarrow$ (1) and (3) $\Rightarrow$ (4). Let G be a finite p-group satisfying (3.1) or (3.2). Then for any $p^i = p^0, p^1, \cdots, p^t = |G'|$, G' possesses unique normal subgroup N_i (of G) of order p^i. Also, $1 = N_0 < N_1 < \cdots < \cdots < N_t = G'$.

For any member K of Kern(G), we may choose $E \lhd G$ maximal subject to G/E is nonabelian with $E \geq K$. By [3, Lemma 12.3], $G'E/E$ is of prime order and hence $|G'/(G' \cap E)| = p$. It follows that $G' \cap E = N_{t-1}$. Since G/N_{t-1} is of type (3.1), G'/N_{t-1} is a unique minimal normal subgroup of G/N_{t-1}. Thus $E/N_{t-1} = 1$, so $E = N_{t-1} = G' \cap E$, and then $K \leq E < G'$. Thus Kern(G) is a subset of $\{N_i \,|\, 0 \leq i \leq t-1\}$, and so Kern($G$) is a chain with respect to inclusion, (1) holds.

By Lemma 2.3, we have

$$N_i = \cap \{\ker\chi \mid \chi \in \mathrm{Irr}(G), N_i \leq \ker\chi, \chi(1) > 1\}.$$

Since Kern(G) is a chain with respect to inclusion ((3)$\Rightarrow$ (1)), $N_i = \ker\chi$ for some nonlinear $\chi \in \mathrm{Irr}(G)$, (4) holds.

(4) $\Rightarrow$ (2). Let E, K be normal subgroups of G with $E \leq G'$. Then either $E \leq K$ or $E \cap K < E \leq G'$. Suppose that $E \cap K < E \leq G'$. Then $E \cap K$ is a member of Kern(G), and this implies by [3, Lemma 2.27] that $Z(G/(E \cap K))$ is cyclic. Arguing as in the proof of ((1) $\Rightarrow$ (2)), we conclude that $K = E \cap K$

($E = E \cap K$ is not the case), that is $K \leq E$. Thus any normal subgroup E of G contained in G' is heavy in G, (2) holds. □

Remark 2.6. Let G be a finite group without the assumption that G is a p-group. If all normal subgroups of G contained in G' are heavy subgroups of G, then Kern(G) is a chain with respect to inclusion; the converse is not true.

Proof. Let $1 = N_0, N_1, \cdots, N_s = G'$ be all normal subgroups of G contained in G', and suppose that these N_i's are heavy subgroups of G. Then we may assume $1 = N_0 < N_1 < \cdots < N_s = G'$. Let K be any member of Kern(G). Since G' is a heavy subgroup of G, we have either $G' > K$ or $G' \leq K$. This implies that $G' > K$ because G/K is nonabelian. Thus Kern(G) is a subset of $\{N_i \mid 1 \leq i \leq s-1\}$, and so Kern($G$) is a chain with respect to inclusion.

To see the converse is not true, let $G = H \times U$, where H is a nonabelian group of order 8 and U is a cyclic group of order 3. Then Kern(G) $= \{1, U\}$, but $G' = H'$ is not a heavy subgroup of G because neither $H' \leq U$ nor $H' \geq U$ holds. □

Remark 2.7. Let G be a finite group without the assumption that G is a p-group. If Kern(G) is a chain with respect to inclusion, then N is a member of Kern(G) for any normal subgroup N of G with $N < G'$; the converse is not true.

Proof. Suppose that Kern(G) is a chain with respect to inclusion and let N be a normal subgroup of G with $N < G'$. By Lemma 2.3, we have

$$N = \cap \{\ker\chi \mid \chi \in \mathrm{Irr}(G), N \leq \ker\chi, \chi(1) > 1\},$$

and then $N = \ker\chi$ for some nonlinear $\chi \in \mathrm{Irr}(G)$.

To see the converse is not true, let $G = H \times U$, where H is a quaternion group of order 16 and U is a cyclic group of order 3. Then $H' = G'$, and $1, Z(H)$ are all proper subgroups of G' which are also normal in G. Let $\chi_1, \chi_2 \in \mathrm{Irr}(H)$ be of degree 2 and 4 respectively, and let σ be a faithful linear character of U. Then $\chi_1 \times \sigma, \chi_2 \times \sigma$ are nonlinear irreducible characters of G, and their kernels are just $Z(H), 1$. However, U is the kernel of the irreducible character $\chi_1 \times 1_U$, where 1_U is the principal character of U, and so Kern(G) is not a chain with respect to inclusion. □

Corollary 2.8. *Let G be a finite nonabelian nilpotent group. If* Kern(G) *is a chain with respect to inclusion, then either G is one of the p-groups stated in our main theorem, or $G = P \times U$, where P is of type (3.1) in our main theorem, and p'-group U is a cyclic group of prime power order.*

Proof. Let P be a nonabelian Sylow p-subgroup of G. Then $G = P \times U$, where U is a nilpotent p'-group. Clearly Kern(G/U) is also a chain with respect to inclusion. Thus by our main theorem P is one of the p-groups stated in our main theorem.

Suppose that $G > P$. Let χ be a nonlinear irreducible character of P and λ be a linear character of U. Observe that $\psi_\lambda := \chi \times \lambda \in \mathrm{Irr}(G)$ and that $\ker\psi_\lambda =$

$\ker\chi \times \ker\lambda$. Since all members of the set $\{\ker(\psi_\lambda) \mid \lambda \in \mathrm{Irr}(U), \lambda(1) = 1\}$ constitute a chain, $\{\ker\lambda \mid \lambda \in \mathrm{Irr}(U), \lambda(1) = 1\}$ is also a chain with respect to inclusion. It follows that U/U' is a cyclic group of prime power order, and so is U. Now it suffices to show that P is of type (3.1) in our main theorem. Otherwise, P is of type (3.2), and we may take nonlinear $\chi_1, \chi_2 \in \mathrm{Irr}(P)$ such that $\ker\chi_1 = 1, P_1 := \ker\chi_2 > 1$. Then $P_1 = \ker(\chi_2 \times \lambda)$ for a faithful linear character λ of U, and $U = \ker(\chi_1 \times 1_U)$. Clearly P_1, U are members of $\mathrm{Kern}(G)$, hence $\mathrm{Kern}(G)$ is not a chain with respect to inclusion, a contradiction. Now the proof is complete. □

Acknowledgement. We are very grateful to the referee and Prof. Dr. Gabriele Nebe for their valuable suggestions.

References

[1] Y. Berkovich, Groups of Prime Power Order. Book I. Unpublished, 2004.

[2] B. Huppert, Endliche Gruppen I. Springer-Verlag, Berlin, 1967.

[3] I. M. Isaacs, Character Theory of Finite Groups. Academic Press, New York, 1976.

Guohua Qian, Dept. Math., Zhongshan University, Guangzhou, 510275, P.R. China
Dept. Math., Changshu Institute of Technology, Changshu, Jiangsu, 215500, P.R.China
e-mail: ghqian2000@yahoo.com.cn

Yanming Wang, Dept. Math., Zhongshan University, Guangzhou, 510275, P.R. China

Received: 6 April 2007