Arch. Math. 89 (2007), 419–429

0003/889X/050419-11, *published online* 2007-10-19
DOI 10.1007/s00013-007-2108-4

Archiv der Mathematik

Metrical characterization of super-reflexivity and linear type of Banach spaces

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Abstract. We prove that a Banach space X is not super-reflexive if and only if the hyperbolic infinite tree embeds metrically into X. We improve one implication of J.Bourgain's result who gave a metrical characterization of super-reflexivity in Banach spaces in terms of uniform embeddings of the finite trees. A characterization of the linear type for Banach spaces is given using the embedding of the infinite tree equipped with the metrics d_p induced by the ℓ_p norms.

Mathematics Subject Classification (2000). 46B20, 51F99.

Keywords. Super-reflexivity, trees, linear type, metric embedding.

1. Introduction and Notation. We fix some notation and recall basic results.

Let (M,d) and (N,δ) be two metric spaces and an injective map $f : M \to N$. Following [11], we define the *distortion* of f to be

$$\operatorname{dist}(f) := \|f\|_{Lip}\|f^{-1}\|_{Lip} = \sup_{x\neq y\in M}\frac{\delta(f(x),f(y))}{d(x,y)}.\sup_{x\neq y\in M}\frac{d(x,y)}{\delta(f(x),f(y))}.$$

If $\operatorname{dist}(f)$ is finite, we say that f is a metric embedding, or simply an embedding of M into N.
And if there exists an embedding f from M into N, with $\operatorname{dist}(f) \le C$, we use the notation $M \overset{C}{\hookrightarrow} N$.
Denote $\Omega_0 = \{\varnothing\}$, the root of the tree. Let $\Omega_n = \{-1,1\}^n$, $T_n = \bigcup_{i=0}^{n}\Omega_i$ and $T = \bigcup_{n=0}^{\infty}\Omega_n$. Thus T_n is the finite tree with n levels and T the infinite tree.

For ε, $\varepsilon' \in T$, we note $\varepsilon \le \varepsilon'$ if ε' is an extension of ε.

Denote $|\varepsilon|$ the length of ε; i.e the numbers of nodes of ε. We define the hyperbolic distance between ε and ε' by $\rho(\varepsilon,\varepsilon') = |\varepsilon| + |\varepsilon'| - 2|\delta|$, where δ is the greatest common ancestor of ε and ε'. The metric on T_n, is the restriction of ρ.

For a Banach space X, we denote B_X its closed unit ball, and X^* its dual space.

T embeds isometrically into $\ell_1(\mathbb{N})$ in a trivial way. Actually, let $(e_\varepsilon)_{\varepsilon\in T}$ be the canonical basis of $\ell_1(T)$ (T is countable), then the embedding is given by $\varepsilon \mapsto \sum_{s\leq\varepsilon} e_s$.

Aharoni proved in [1] that every separable metric space embeds into c_0, so T does.

The main result of this article is an improvement of Bourgain's metrical characterization of super-reflexivity. Bourgain proved in [2] that X is not super-reflexive if and only if the finite trees T_n uniformly embed into X (i.e with embedding constants independent of n). Obviously if T embeds into X then the $T_n's$ embed uniformly into X and X is not super-reflexive, but if X is not super-reflexive we did not know whether the infinite tree T embeds into X. In this paper, we prove that it is indeed the case :

Theorem 1.1. *Let X be a non super-reflexive Banach space, then (T,ρ) embeds into X.*

The proof of the direct part of Bourgain's Theorem essentially uses James' characterization of super-reflexivity (see [7]) and an enumeration of the finite trees T_n. We recall James' Theorem:

Theorem 1.2 (James). *Let $0<\theta<1$ and X a non super-reflexive Banach space, then: $\forall\ n\in\mathbb{N}$, $\exists\ x_1,x_2,\ldots,x_n\in B_X$, $\exists\ x_1^*,x_2^*,\ldots,x_n^*\in B_{X^*}$ s.t:*

$$\begin{aligned} x_k^*(x_j) &= \theta \quad \forall k<j \\ x_k^*(x_j) &= 0 \quad \forall k\geq j \end{aligned}$$

2. Metrical characterization of super-reflexivity. The main obstruction to the embedding of T into any non-super-reflexive Banach space X is the finiteness of the sequences in James' characterization. How, with a sequence of Bourgain's type embedding, can we construct a single embedding from T into X?

In [13], Ribe shows in particular, that $\bigoplus_2 l_{p_n}$ and $(\bigoplus_2 l_{p_n})\bigoplus l_1$ are uniformly homeomorphic, where $(p_n)_n$ is a sequence of numbers such that $p_n>1$, and p_n tends to 1. But T embeds into l_1, hence via the uniform homeomorphism T embeds into $\bigoplus_2 l_{p_n}$. However T does not embed into any l_{p_n}(they are super-reflexive).

The problem solved in the next theorem, inspired in part by Ribe's proof, is to construct a subspace with a Schauder decomposition $\bigoplus F_n$ where $T_{2^{n+1}}$ embeds into F_n and to repast properly the embeddings in order to obtain the desired embedding.

Proof of Theorem 1.1. Let $(\varepsilon_i)_{i\geq 0}$, a sequence of positive real numbers such that $\prod_{i\geq 0}(1+\varepsilon_i)\leq 2$, and fix $0<\theta<1$. Let $k_n=2^{2^{n+1}+1}-1$.

First we construct inductively a sequence $(F_n)_{n\geq 0}$ of subspaces of X, which is a Schauder finite dimensional decomposition of a subspace of X s.t the projection from $\bigoplus_{i=0}^{q} F_i$ onto $\bigoplus_{i=0}^{p} F_i$, with kernel $\bigoplus_{i=p+1}^{q} F_i$ (with $p < q$) is of norm at most $\prod_{i=p}^{q-1}(1+\varepsilon_i)$, and sequences

$$x_{n,1}, x_{n,2}, \ldots, x_{n,k_n} \in B_{F_n}$$
$$x_{n,1}^*, x_{n,2}^*, \ldots, x_{n,k_n}^* \in B_{X^*}$$

s.t:

$$x_{n,k}^*(x_{n,j}) = \theta \quad \forall k < j$$
$$x_{n,k}^*(x_{n,j}) = 0 \quad \forall k \geq j.$$

Denote $\Phi_n : T_n \to \{1, 2, \ldots, 2^{n+1} - 1\}$ the enumeration of T_n following the lexicographic order. It is an enumeration of T_n such that any pair of segments in T_n starting at incomparable nodes (with respect to the tree ordering $\leq$) are mapped inside disjoint intervals.

Let $\Psi_n = \Phi_{2^{n+1}}$ and $\Gamma_n = T_{2^{n+1}}$.

X is non super-reflexive, hence from James' Theorem:
$\exists\ x_{0,1}, x_{0,2}, \ldots, x_{0,7} \in B_X$, $\exists\ x_{0,1}^*, x_{0,2}^*, \ldots, x_{0,7}^* \in B_{X^*}$ s.t:

$$x_{0,k}^*(x_{0,j}) = \theta \quad \forall k < j$$
$$x_{0,k}^*(x_{0,j}) = 0 \quad \forall k \geq j.$$

$\Gamma_0 = T_2$ embeds into X via the embedding $f_0(\varepsilon) = \sum_{s \leq \varepsilon} x_{0,\Psi_0(s)}$ (see [2]).
Let $F_0 = \mathrm{Span}\{x_{0,1}, \ldots, x_{0,7}\}$, then $\dim(F_0) < \infty$.

Suppose that F_p and

$$x_{p,1}, x_{p,2}, \ldots, x_{p,k_p} \in B_{F_p}$$
$$x_{p,1}^*, x_{p,2}^*, \ldots, x_{p,k_p}^* \in B_{X^*}$$

verifying the required conditions, are constructed for all $p \leq n$.

We apply Mazur's Lemma (see [9] page 4) to the finite dimensional subspace $\bigoplus_{i=0}^{n} F_i$ of X. Thus there exists $Y_n \subset X$ such that $\dim(X/Y_n) < \infty$ and:

$$\|x\| \leq (1+\varepsilon_n)\|x+y\|, \forall (x,y) \in \bigoplus_{i=0}^{n} F_i \times Y_n.$$

But Y_n is of finite codimension in X, hence is not super-reflexive. From James' Theorem and Hahn-Banach Theorem:

$$\exists\ x_{n+1,1}, x_{n+1,2}, \ldots, x_{n+1,k_{n+1}} \in B_{Y_n},$$
$$\exists\ x_{n+1,1}^*, x_{n+1,2}^*, \ldots, x_{n+1,k_{n+1}}^* \in B_{X^*},$$

s.t:

$$x_{n+1,k}^*(x_{n+1,j}) = \theta \quad \forall k < j$$
$$x_{n+1,k}^*(x_{n+1,j}) = 0 \quad \forall k \geq j.$$

Γ_{n+1} embeds into Y_n via the embedding $f_{n+1}(\varepsilon) = \sum_{s \leq \varepsilon} x_{n+1,\Psi_{n+1}(s)}$.

Let $F_{n+1} = \mathrm{Span}\{x_{n+1,j}\ ;\ 1 \leq j \leq k_{n+1}\}$, then $\dim(F_{n+1}) < \infty$, which achieves the induction.

Now define the following projections:

Let, P_n the projection from $\overline{\text{Span}}(\bigcup_{i=0}^{\infty} F_i)$ onto $F_0 \bigoplus \cdots \bigoplus F_n$ with kernel $\overline{\text{Span}}(\bigcup_{i=n+1}^{\infty} F_i)$.

It is easy to show that $\|P_n\| \leq \prod_{i=n}^{\infty}(1+\varepsilon_i) \leq 2$.

We denote now $\Pi_0 = P_0$ and $\Pi_n = P_n - P_{n-1}$ for $n \geq 1$. We have that $\|\Pi_n\| \leq 4$.

From Bourgain's construction, for all n:

$$\frac{\theta}{3}\rho(\varepsilon,\varepsilon') \leq \|f_n(\varepsilon) - f_n(\varepsilon')\| \leq \rho(\varepsilon,\varepsilon'), \tag{1}$$

where f_n denotes the Bourgain's type embedding from Γ_n in F_n, i.e $f_n(\varepsilon) = \sum_{s\leq\varepsilon} x_{n,\Psi_n(s)}$.

Note that:

$$\forall\ n, \forall\ \varepsilon \in \Gamma_n\ \ \|f_n(\varepsilon)\| \leq |\varepsilon|.$$

Now we define our embedding.

Let

$$\begin{aligned} f: \quad T \quad &\rightarrow \quad Y = \overline{\text{Span}}(\textstyle\bigcup_{i=0}^{\infty} F_i) \subset X \\ \varepsilon \quad &\mapsto \quad \lambda f_n(\varepsilon) + (1-\lambda) f_{n+1}(\varepsilon)\ ,\ \text{if } 2^n \leq |\varepsilon| \leq 2^{n+1} \end{aligned}$$

where,

$$\lambda = \frac{2^{n+1} - |\varepsilon|}{2^n}$$

We will prove that:

$$\forall \varepsilon, \varepsilon' \in T\ \frac{\theta}{24}\rho(\varepsilon,\varepsilon') \leq \|f(\varepsilon) - f(\varepsilon')\| \leq 9\rho(\varepsilon,\varepsilon'). \tag{2}$$

Remark 2.1. We have $\frac{\theta}{24}|\varepsilon| \leq \|f(\varepsilon)\| \leq |\varepsilon|$.

First of all, we show that f is 9-Lipschitz.

We can suppose that $0 < |\varepsilon| \leq |\varepsilon'|$ w.r.t remark 2.1.

If $|\varepsilon| \leq \frac{1}{2}|\varepsilon'|$ then:

$$\rho(\varepsilon,\varepsilon') \geq |\varepsilon'| - |\varepsilon| \geq \frac{|\varepsilon| + |\varepsilon'|}{3}$$

Hence,

$$\|f(\varepsilon) - f(\varepsilon')\| \leq 3\rho(\varepsilon,\varepsilon').$$

If $\frac{1}{2}|\varepsilon'| < |\varepsilon| \leq |\varepsilon'|$, we have two different cases to consider.

1) if $2^n \leq |\varepsilon| \leq |\varepsilon'| < 2^{n+1}$.
Then, let

$$\lambda = \frac{2^{n+1} - |\varepsilon|}{2^n} \quad \text{and} \quad \lambda' = \frac{2^{n+1} - |\varepsilon'|}{2^n}.$$

$$\begin{aligned}
\|f(\varepsilon) - f(\varepsilon')\| &= \|\lambda f_n(\varepsilon) - \lambda' f_n(\varepsilon') + (1-\lambda) f_{n+1}(\varepsilon) - (1-\lambda') f_{n+1}(\varepsilon')\| \\
&\leq \lambda \|f_n(\varepsilon) - f_n(\varepsilon')\| + |\lambda - \lambda'|(\|f_n(\varepsilon')\| + \|f_{n+1}(\varepsilon')\|) \\
&\quad + (1-\lambda)\|f_{n+1}(\varepsilon) - f_{n+1}(\varepsilon')\| \\
&\leq \rho(\varepsilon,\varepsilon') + 2\rho(\varepsilon,\varepsilon') + 2\rho(\varepsilon,\varepsilon') \\
&\leq 5\rho(\varepsilon,\varepsilon'),
\end{aligned}$$

because $\|f_n(\varepsilon')\| < 2^{n+1}$, $\|f_{n+1}(\varepsilon')\| < 2^{n+1}$ and,

$$|\lambda - \lambda'| = \frac{|\varepsilon'| - |\varepsilon|}{2^n} \leq \frac{\rho(\varepsilon,\varepsilon')}{2^n}.$$

2) if $2^n \leq |\varepsilon| \leq 2^{n+1} \leq |\varepsilon'| < 2^{n+2}$.
Then, let

$$\lambda = \frac{2^{n+1} - |\varepsilon|}{2^n} \quad \text{and} \quad \lambda' = \frac{2^{n+2} - |\varepsilon'|}{2^{n+1}}.$$

$$\begin{aligned}
\|f(\varepsilon) - f(\varepsilon')\| &= \|\lambda f_n(\varepsilon) + (1-\lambda) f_{n+1}(\varepsilon) - \lambda' f_{n+1}(\varepsilon') - (1-\lambda') f_{n+2}(\varepsilon')\| \\
&\leq \lambda(\|f_n(\varepsilon)\| + \|f_{n+1}(\varepsilon)\|) + (1-\lambda')(\|f_{n+1}(\varepsilon')\| + \|f_{n+2}(\varepsilon')\|) \\
&\quad + \|f_{n+1}(\varepsilon) - f_{n+1}(\varepsilon')\| \\
&\leq \rho(\varepsilon,\varepsilon') + 2\lambda|\varepsilon| + 2(1-\lambda')|\varepsilon'| \\
&\leq 9\rho(\varepsilon,\varepsilon'),
\end{aligned}$$

because,

$$\lambda \leq \frac{\rho(\varepsilon,\varepsilon')}{2^n}, \quad \text{so} \quad \lambda|\varepsilon| \leq 2\rho(\varepsilon,\varepsilon').$$

Similarly

$$1 - \lambda' = \frac{|\varepsilon'| - 2^{n+1}}{2^{n+1}} \leq \frac{\rho(\varepsilon,\varepsilon')}{2^{n+1}} \quad \text{and} \quad (1-\lambda')|\varepsilon'| \leq 2\rho(\varepsilon,\varepsilon').$$

Finally, f is 9-Lipschitz.

Now we deal with the minoration.

In our next discussion, whenever $|\varepsilon|$ (respectively $|\varepsilon'|$) will belong to $[2^n, 2^{n+1})$, for some integer n, we shall denote

$$\lambda = \frac{2^{n+1} - |\varepsilon|}{2^n} \quad \text{(respectively} \quad \lambda' = \frac{2^{n+1} - |\varepsilon'|}{2^n}).$$

We can suppose that ε is smaller than ε' in the lexicographic order. Denote δ the greatest common ancestor of ε and ε'. And let $d = |\varepsilon| - |\delta|$ (respectively $d' = |\varepsilon'| - |\delta|$).

1) if $2^n \le |\varepsilon|, |\varepsilon'| \le 2^{n+1}$.

We have,

$$x^*_{n,\Psi_n(\delta)}\Pi_n(f(\varepsilon) - f(\varepsilon')) = \theta(\lambda d - \lambda' d')$$
$$x^*_{n+1,\Psi_{n+1}(\delta)}\Pi_{n+1}(f(\varepsilon) - f(\varepsilon')) = \theta((1-\lambda)d - (1-\lambda')d').$$

Hence,

$$\|f(\varepsilon) - f(\varepsilon')\| \ge \frac{\theta(d-d')}{8}.$$

And,

$$-x^*_{n,\Psi_n(\varepsilon)}\Pi_n(f(\varepsilon) - f(\varepsilon')) = \theta\lambda' d'$$
$$-x^*_{n+1,\Psi_{n+1}(\varepsilon)}\Pi_{n+1}(f(\varepsilon) - f(\varepsilon')) = \theta(1-\lambda')d'.$$

So,

$$\|f(\varepsilon) - f(\varepsilon')\| \ge \frac{\theta d'}{8}.$$

Finally if we distinguish the cases $\frac{d}{2} \le d'$, and $d' < \frac{d}{2}$ we obtain:

$$\|f(\varepsilon) - f(\varepsilon')\| \ge \frac{\theta(d+d')}{24} = \frac{\theta}{24}\rho(\varepsilon, \varepsilon').$$

2) if $2^n \le |\varepsilon| \le 2^{n+1} \le 2^{q+1} \le |\varepsilon'| \le 2^{q+2}$,
or $2^n \le |\varepsilon'| \le 2^{n+1} \le 2^{q+1} \le |\varepsilon| \le 2^{q+2}$.

If $n < q$,

$$|x^*_{q+1,\Psi_{q+1}(\delta)}\Pi_{q+1}(f(\varepsilon) - f(\varepsilon')) + x^*_{q+2,\Psi_{q+2}(\delta)}\Pi_{q+2}(f(\varepsilon) - f(\varepsilon'))| = \theta Max(d, d')$$

Hence,

$$\|f(\varepsilon) - f(\varepsilon')\| \ge \frac{\theta}{16}\rho(\varepsilon, \varepsilon').$$

If $n = q$ and $|\varepsilon| \le |\varepsilon'|$,

$$|x^*_{n+1,\Psi_{n+1}(\varepsilon)}\Pi_{n+1}(f(\varepsilon) - f(\varepsilon')) + x^*_{n+2,\Psi_{n+2}(\delta)}\Pi_{n+2}(f(\varepsilon) - f(\varepsilon'))| \ge \theta d'.$$

So,

$$\|f(\varepsilon) - f(\varepsilon')\| \ge \frac{\theta}{16}\rho(\varepsilon, \varepsilon').$$

If $n = q$ and $|\varepsilon'| < |\varepsilon|$,

$$x^*_{n+1,\Psi_{n+1}(\delta)}\Pi_{n+1}(f(\varepsilon) - f(\varepsilon')) - x^*_{n+1,\Psi_{n+1}(\varepsilon)}\Pi_{n+1}(f(\varepsilon) - f(\varepsilon'))$$
$$+ x^*_{n+2,\Psi_{n+2}(\delta)}\Pi_{n+2}(f(\varepsilon) - f(\varepsilon')) = \theta d.$$

Hence,

$$\|f(\varepsilon) - f(\varepsilon')\| \ge \frac{\theta}{24}\rho(\varepsilon, \varepsilon').$$

Finally $T \overset{\frac{216}{\theta}}{\hookrightarrow} X$. □

Corollary 2.2. *X is non super-reflexive if and only if (T, ρ) embeds into X.*

Proof. It follows clearly from Bourgain's result [2] and Theorem 1.1. $\square$

Remark 2.3. We deduce from the last corollary that the free group with two elements $\mathbb{F}_2$ viewed as a metric space through its Cayley graph equipped with the word metric embeds into any non super-reflexive space.

3. Metric characterization of the linear type. First we identify canonicaly $\{-1,1\}^n$ with $K_n = \{-1,1\}^n \times \prod_{k>n}\{0\}$.

Let $p \in [1,\infty)$.
Then we define an other metric on $T = \bigcup K_n$ as follows :
$\forall\ \varepsilon, \varepsilon' \in T$,

$$d_p(\varepsilon, \varepsilon') = \left(\sum_{i=0}^{\infty} |\varepsilon_i - \varepsilon'_i|^p\right)^{\frac{1}{p}}.$$

The length of $\varepsilon \in T$ can be viewed as $|\varepsilon| = (d_p(\varepsilon, 0))^p$.
The norm $\|.\|_p$ on ℓ_p coincides with d_p for the elements of T.

We recall now two classical definitions:

Let X and Y be two Banach spaces. If X and Y are linearly isomorphic, the *Banach-Mazur distance* between X and Y, denoted by $d_{BM}(X,Y)$, is the infimum of $\|T\|\,\|T^{-1}\|$, over all linear isomorphisms T from X onto Y.

For $p \in [1,\infty]$, we say that a Banach space X uniformly contains the ℓ_p^n's if there is a constant $C \geq 1$ such that for every integer n, X admits an n-dimensional subspace Y so that $d_{BM}(\ell_p^n, Y) \leq C$.

We state and prove now the following result.

Theorem 3.1. *Let $p \in [1,\infty)$.*
If X uniformly contains the ℓ_p^n's then (T, d_p) embeds into X.

Proof. We first recall a fundamental result due to Krivine (for $1 < p < \infty$ in [8]) and James (for $p = 1$ and ∞ in [7]).

Theorem 3.2 (James-Krivine). *Let $p \in [1,\infty]$ and X be a Banach space uniformly containing the ℓ_p^n's. Then, for any finite codimensional subspace Y of X, any $\epsilon > 0$ and any $n \in \mathbb{N}$, there exists a subspace F of Y such that $d_{BM}(\ell_p^n, F) < 1 + \epsilon$.*

Using Theorem 3.2 together with the fact that each ℓ_p^n is finite dimensional, we can build inductively finite dimensional subspaces $(F_n)_{n=0}^{\infty}$ of X and $(R_n)_{n=0}^{\infty}$ so that for every $n \geq 0$, R_n is a linear isomorphism from ℓ_p^n onto F_n satisfying

$$\forall u \in \ell_p^n \qquad \frac{1}{2}\|u\| \leq \|R_n u\| \leq \|u\|$$

and also such that $(F_n)_{n=0}^{\infty}$ is a Schauder finite dimensional decomposition of its closed linear span Z. More precisely, if P_n is the projection from Z onto $F_0 \oplus ... \oplus F_n$

with kernel $\overline{\text{Span}}\,(\bigcup_{i=n+1}^{\infty} F_i)$, we will assume as we may, that $\|P_n\| \leq 2$. We denote now $\Pi_0 = P_0$ and $\Pi_n = P_n - P_{n-1}$ for $n \geq 1$. We have that $\|\Pi_n\| \leq 4$.

We now consider $\varphi_n : T_n \to \ell_p^n$ defined by

$$\forall \varepsilon \in T_n, \quad \varphi_n(\varepsilon) = \sum_{i=1}^{|\varepsilon|} \varepsilon_i e_i,$$

where (e_i) is the canonical basis of ℓ_p^n. The map φ_n is clearly an isometric embedding of T_n into ℓ_p^n.

Then we set :

$$\forall \varepsilon \in T_n, \quad f_n(\varepsilon) = R_n(\varphi_n(\varepsilon)) \in F_n.$$

Finally we construct a map $f : T \to X$ as follows:

$$\begin{array}{rccl} f: & T & \to & X \\ & \varepsilon & \mapsto & \lambda f_m(\varepsilon) + (1-\lambda) f_{m+1}(\varepsilon) \ , \ \text{if } 2^m \leq |\varepsilon| < 2^{m+1}, \end{array}$$

where,

$$\lambda = \frac{2^{m+1} - |\varepsilon|}{2^m}.$$

Remark 3.3. We have $\frac{1}{16}|\varepsilon|^{\frac{1}{p}} \leq \|f(\varepsilon)\| \leq |\varepsilon|^{\frac{1}{p}}$.

Like in the proof of Theorem 1.1, we prove that f is 9-Lipschitz using exactly the same computations.

We shall now prove that f^{-1} is Lipschitz. We consider $\varepsilon, \varepsilon' \in T$ and assume again that $0 < |\varepsilon| \leq |\varepsilon'|$. We need to study two different cases. Again, whenever $|\varepsilon|$ (respectively $|\varepsilon'|$) will belong to $[2^m, 2^{m+1})$, for some integer m, we shall denote

$$\lambda = \frac{2^{m+1} - |\varepsilon|}{2^m} \quad \text{(respectively} \quad \lambda' = \frac{2^{m+1} - |\varepsilon'|}{2^m}).$$

1) if $2^m \leq |\varepsilon|, |\varepsilon'| < 2^{m+1}$.

$$\begin{aligned} d_p(\varepsilon, \varepsilon') &\leq \|\lambda \textstyle\sum_{i=1}^{|\varepsilon|} \varepsilon_i e_i - \lambda' \sum_{i=1}^{|\varepsilon'|} \varepsilon'_i e_i\|_p + \|(1-\lambda) \sum_{i=1}^{|\varepsilon|} \varepsilon_i e_i \\ &\quad -(1-\lambda') \textstyle\sum_{i=1}^{|\varepsilon'|} \varepsilon'_i e_i\|_p \\ &\leq 2\|\Pi_m(f(\varepsilon) - f(\varepsilon'))\| + 2\|\Pi_{m+1}(f(\varepsilon) - f(\varepsilon'))\| \\ &\leq 16\|f(\varepsilon) - f(\varepsilon')\|. \end{aligned}$$

2) if $2^m \leq |\varepsilon| \leq 2^{m+1} \leq 2^{q+1} \leq |\varepsilon'| < 2^{q+2}$.
if $m < q$,

$$\begin{aligned} d_p(\varepsilon,\varepsilon') &\leq 2d_p(\varepsilon',0) \\ &\leq 2((1-\lambda')d_p(\varepsilon',0)+\lambda' d_p(\varepsilon',0)) \\ &\leq 2(2\|\Pi_{q+2}(f(\varepsilon)-f(\varepsilon'))\|+2\|\Pi_{m+1}(f(\varepsilon)-f(\varepsilon'))\|) \\ &\leq 32\|f(\varepsilon)-f(\varepsilon')\|. \end{aligned}$$

if $m = q$,

$$\begin{aligned} d_p(\varepsilon,\varepsilon') &\leq \lambda d_p(\varepsilon,0)+\|(1-\lambda)\textstyle\sum_{i=1}^{|\varepsilon|}\varepsilon_i e_i-\lambda'\sum_{i=1}^{|\varepsilon'|}\varepsilon'_i e_i\|_p+(1-\lambda')d_p(\varepsilon',0) \\ &\leq 2\|\Pi_m(f(\varepsilon)-f(\varepsilon'))\|+2\|\Pi_{m+1}(f(\varepsilon)-f(\varepsilon'))\| \\ &\quad +2\|\Pi_{m+2}(f(\varepsilon)-f(\varepsilon'))\| \\ &\leq 24\|f(\varepsilon)-f(\varepsilon')\|. \end{aligned}$$

Finally we obtain that f^{-1} is 32-Lipschitz, and $T \overset{288}{\hookrightarrow} X$. □

In the sequel a Banach space X is said to have *type* $p > 0$ if there exists a constant $T < \infty$ such that for every n and every $x_1, \dots, x_n \in X$,

$$\mathbb{E}_\varepsilon \|\sum_{j=1}^n \varepsilon_j x_j\|_X^p \leq T^p \sum_{j=1}^n \|x_j\|_X^p,$$

where the expectation $\mathbb{E}_\varepsilon$ is with respect to a uniform choice of signs $\varepsilon_1, \dots, \varepsilon_n \in \{-1,1\}^n$.

The set of p's for which X contains ℓ_p^n's uniformly is closely related to the type of X according to the following result due to Maurey, Pisier [10] and Krivine [8], which clarifies the meaning of these notions.

Theorem 3.4 (Maurey-Pisier-Krivine). *Let X be an infinite-dimensional Banach space. Let*

$$p_X = \sup\{p\ ; \text{X is of type p}\},$$

Then X contains ℓ_p^n's uniformly for $p = p_X$.
Equivalently, we have

$$p_X = \inf\{p\ ; \text{X contains } \ell_{\text{p}}^{\text{n}}\text{\textit{'s} uniformly}\}.$$

We deduce from Theorem 3.1 two corollaries.

Corollary 3.5. *Let X a Banach space and $1 \leq p < 2$.*
The following assertions are equivalent :

i) $p_X \leq p$.

ii) *X uniformly contains the* ℓ_p^n*'s.*
iii) *the* (T_n, d_p)*'s uniformly embed into* X *.*
iv) (T, d_p) *embeds into* X*.*

Proof. *ii*) implies *i*) is obvious.

i) implies *ii*) is due to Theorem 3.2 and the work of Bretagnolle, Dacunha-Castelle and Krivine [4].

For the equivalence between *ii*) and *iii*) see the work of Bourgain, Milman and Wolfson [3] and Krivine [8].

iv) implies *iii*) is obvious.

And *ii*) implies *iv*) is Theorem 3.1. □

Corollary 3.6. *Let* X *be an infinite dimensional Banach space, then* (T, d_2) *embeds into* X*.*

Proof. This corollary is a consequence of the Dvoretsky's Theorem [6] and Theorem 3.1. □

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Received: 2 August 2006

Revised: 10 April 2007