Arch. Math. 87 (2006) 272–279
0003–889X/06/030272–08
DOI 10.1007/s00013-006-1685-y

Archiv der Mathematik

A note on the stability of the Wulff shape

By

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Abstract. We give a new proof of Palmer's result [6] that the Wulff shapes are the only closed, oriented, stable hypersurfaces with constant anisotropic mean curvature. Our approach is based on the construction of a suitable testfunction in the anisotropic index form, thus generalizing the original proof of Barbosa, do Carmo [1].

1. Introduction. Let $X : M^n \to \mathbb{R}^{n+1}$ be a closed hypersurface smoothly immersed in euclidean $\mathbb{R}^{n+1}$. It is well-known that X has constant mean curvature if and only if X is a critical point of the area functional

$$\mathcal{A}(X) = \int_M dA$$

under a volume constraint. If additionally the second variation of area is non-negative for all volume preserving variations of X, then X is called stable.

According to a celebrated result of Barbosa, do Carmo [1], the round sphere is – up to translation and dilatation – the only closed, stable hypersurface with constant mean curvature. Their proof is based on the construction of a suitable testfunction in the index form of X, which they obtain from a systematic study of the Jacobi operator. Later, Wente [8] gave a more direct proof, by showing that the particular testfunction can be obtained from parallel translations of the hypersurface followed by a dilatation that fixes the enclosed volume. On the other hand, it turns out that the method developed by Barbosa, do Carmo is applicable to other important geometric situations, see for example Ritoré, Rosales [7] for the discussion of a free boundary problem for constant mean curvature hypersurfaces.

In the present work we will focus on a variational problem related to elliptic parametric functionals of the type

$$\mathcal{F}(X) = \int_M F(N)\, dA.$$

Mathematics Subject Classification (2000): 53C42, 49Q10, 35J60.

It is well-known, that critical points of $\mathcal{F}$ under a volume constraint can be characterized as hypersurfaces with constant anisotropic mean curvature or F-mean curvature (see section 2 for details). Moreover, there is a natural hypersurface associated with F, the so called Wulff shape, which has constant F-mean curvature and minimizes $\mathcal{F}$, cf. Taylor [5]. According to Palmer [6], the Wulff shape plays the same role for $\mathcal{F}$ as do the spheres for the area functional. In fact, by adapting Wente's [8] method to the anisotropic case, Palmer was able to prove the following

Theorem 1.1 (6)**.** *If $X : M^n \to \mathbb{R}^{n+1}$ is a closed, oriented, F-stable hypersurface with constant F-mean curvature, then up to scaling and translation $X(M)$ is the Wulff shape.*

It is the aim of this note to give a new proof of this result by a systematic investigation of the anisotropic index form associated with the second variation of $\mathcal{F}$. In particular, we carefully study the F-analogue of the Jacobi operator (see Theorem 3.1). We hope that the transparency and clarity of this approach will also be of importance in future investigations. Moreover, we refer to the recent work of Clarenz [3], where it is shown that the Wulff shape minimizes an anisotropic Willmore functional.

2. Preliminaries. In this section we set up our notation and collect the basic facts on F-stationarity and F-stability of closed hypersurfaces.

Let $X : M^n \to \mathbb{R}^{n+1}$ be a smooth immersion of an n-dimensional, oriented, compact manifold without boundary into euclidean $\mathbb{R}^{n+1}$. We denote by $N : M \to S^n$ and dA the corresponding Gauß mapping and induced measure, respectively, and consider elliptic parametric functionals of the type

$$\mathcal{F}(X) = \int_M F(N)\, dA.$$

The integrand

$$F : S^n \to \mathbb{R}^+$$

is a smooth, *positive* Lagrangian which we assume to be 1-homogeneously extended to $\mathbb{R}^{n+1} \setminus \{0\}$ by

$$F(tz) = tF(z) \quad \forall t > 0, z \in S^n. \tag{1}$$

Furthermore, we always assume F to be *elliptic*, i.e., the restriction of

$$F_{zz}(z) = (\partial_{\alpha\beta} F(z))_{\alpha,\beta=1,\dots,n+1}$$

to $z^\perp = \{V \in \mathbb{R}^{n+1} : \langle V, z\rangle = 0\}$ is a positive definite endomorphism $z^\perp \to z^\perp$ for all $z \in S^n$.

Geometrically speaking, the ellipticity of F implies that F is the support function of some convex body

$$\bigcap_{z\in S^n} \{y \in \mathbb{R}^{n+1} : \langle y, z\rangle \leqq F(z)\},$$

the boundary $\mathcal{W}_F$ of which is the convex hypersurface parametrized by

$$\Phi : S^n \to \mathbb{R}^{n+1},\ \Phi(z) := F_z(z).$$

In the terminology of Taylor [5], $\mathcal{W}_F = \Phi(S^n)$ is called the *Wulff shape*.

Let us now consider an arbitrary variation X_ε of $X = X_0$ with variation vector field $Y := \frac{dX_\varepsilon}{d\varepsilon}|_{\varepsilon=0}$. Decomposing $Y = \varphi N +$ tangential terms, it is well-known that the first variation of $\mathcal{F}$ is given by

$$\delta\mathcal{F}(X, Y) := \frac{d}{d\varepsilon}\mathcal{F}(X_\varepsilon)|_{\varepsilon=0} = -\int_M H_F \varphi \, dA, \tag{2}$$

see e.g. [2]. Here, H_F is the *F-mean curvature* or *anisotropic mean curvature* of X, which is defined as follows: Let

$$N_F : M \to \mathcal{W}_F, \quad N_F := \Phi(N)$$

denote the generalized Gauß mapping into the Wulff shape. Then

$$S_F := -dX^{-1} \circ dN_F$$

is called *F-Weingarten operator* and

$$H_F := \mathrm{tr}(S_F).$$

We remark that for technical reasons it is convenient to write

$$S_F = A_F \circ S,$$

where $S := -dX^{-1} \circ dN$ denotes the classical Weingarten operator and A_F is the symmetric positive definite $(1, 1)$-tensor given by

$$A_F := dX^{-1} \circ F_{zz}(N) \circ dX.$$

Clearly, these definitions coincide with their classical counterparts in case $F(z) = |z|$ is the area-integrand.

Let us now introduce the volume functional

$$\mathcal{V}(X) = \frac{1}{n+1}\int_M \langle X, N\rangle \, dA.$$

It is well-known, that the first variation of $\mathcal{V}$ is given by

$$\delta\mathcal{V}(X, Y) := \frac{d}{d\varepsilon}\mathcal{V}(X_\varepsilon)|_{\varepsilon=0} = \int_M \varphi \, dA. \tag{3}$$

We say that a variation X_ε of X is *volume preserving*, if $\mathcal{V}(X_\varepsilon) = \text{const}$, and we say that X is *F-stationary*, if $\delta\mathcal{F}(X, Y) = 0$ for all volume preserving variations. Due to a well-known reasoning of Barbosa, do Carmo [1], it follows from (2) and (3) that X is F-stationary if and only if X has constant F-mean curvature.

An F-stationary immersion X is called *F-stable*, if the second variation $\delta^2\mathcal{F}(X, Y) := \frac{d^2}{d\varepsilon^2}\mathcal{F}(X_\varepsilon)|_{\varepsilon=0}$ is non-negative for all volume preserving variations. We recall from Clarenz, von der Mosel [4], that for an arbitrary variation X_ε the F-mean curvature $H_F(\varepsilon)$ of X_ε satisfies the equation

$$\frac{d}{d\varepsilon}H_F(\varepsilon)|_{\varepsilon=0} = \Delta_F\varphi + \mathrm{tr}(A_F S^2)\varphi,$$

where Δ_F is the second order elliptic operator given by

$$\Delta_F\varphi := \mathrm{div}(A_F\nabla\varphi).$$

Here, $\nabla\varphi$ denotes the gradient of φ with respect to the induced metric g. In particular, this implies that the second variation of $\mathcal{F}$ is given by

$$\delta^2\mathcal{F}(X, Y) = -\int_M \varphi(\Delta_F\varphi + \mathrm{tr}(A_F S^2)\varphi)\, dA. \tag{4}$$

Let us now define the *anisotropic index form I* of X by

$$\begin{aligned} I[\varphi] &:= -\int_M \varphi(\Delta_F\varphi + \mathrm{tr}(A_F S^2)\varphi)\, dA \\ &= \int_M (g(A_F\nabla\varphi, \nabla\varphi) - \mathrm{tr}(A_F S^2)\varphi^2)\, dA. \end{aligned} \tag{5}$$

We recall from Barbosa, do Carmo [1], that for any smooth function φ satisfying $\int_M \varphi\, dA = 0$ there exists a volume preserving variation with variation vector field $Y = \varphi N$. Hence, on account of (4) we deduce the following characterization of F-stable immersions:

Lemma 2.1. *$X : M \to \mathbb{R}^{n+1}$ is F-stable if and only if $H_F = \mathrm{const}$ and*

$$I[\varphi] \geqq 0$$

for all $\varphi \in C^\infty(M)$ satisfying

$$\int_M \varphi\, dA = 0.$$

3. Main results. Given $X : M \to \mathbb{R}^{n+1}$, we denote by $g := \langle X, N\rangle$ the *support function* and we abbreviate $F = F(N)$. In order to construct a suitable testfunction valid in the anisotropic index form, we need the following identities, which in case of the area integrand have been proved by Barbosa, do Carmo [1].

Theorem 3.1. *If $X : M \to \mathbb{R}^{n+1}$ has constant F-mean curvature, then the following identities hold true:*

$$\Delta_F N + tr(A_F S^2)N = 0, \tag{6}$$

$$\Delta_F g + tr(A_F S^2)g = -H_F \quad \textit{and} \tag{7}$$

$$\Delta_F F + tr(A_F S^2)F = tr(S_F^2). \tag{8}$$

P r o o f. The first identity was already derived in [4]. For the convenience of the reader, we roughly sketch a proof using Ricci calculus:

First, note that for any smooth function φ we locally have

$$\Delta_F \varphi = \nabla_i (A^{ij} \nabla_j \varphi).$$

Here, ∇ denotes the covariant derivative on (M, g), and $A^{ij} = g^{ik} A_{kl} g^{kj}$, where $A_{ij} = g(A_F \partial_i, \partial_j)$ denotes the coefficients of A_F and g_{ij}, g^{ij} stands for the coefficients of the first fundamental form and its inverse, respectively. Moreover, we employ Einstein's summation convention in that we sum over repeated latin indices from $1, \dots, n$.

We now use one of the Gauß-Weingarten relations,

$$\nabla_j N = -g^{kl} h_{lj} \nabla_k X,$$

where h_{ij} are the coefficients of the second fundamental form, and compute

$$\begin{aligned}
\Delta_F N &= \nabla_i (A^{ij} \nabla_j N) \\
&= -\nabla_i (A^{ij} g^{kl} h_{lj} \nabla_k X) \\
&= -\nabla_i A^{ij} g^{kl} h_{lj} \nabla_k X - A^{ij} g^{kl} \nabla_i h_{lj} \nabla_k X - A^{ij} g^{kl} h_{lj} \nabla_i \nabla_k X.
\end{aligned}$$

Here, $\nabla_i \nabla_k X = \partial_{ik} X - \Gamma^m_{ik} \partial_m X$ denotes the second covariant derivative of X, and $\nabla_i h_{lj}$ stands for the covariant derivative of the second fundamental form. By virtue of Gauß-Weingarten and Codazzi, we have $\nabla_i \nabla_k X = h_{ik} N$ and $\nabla_i h_{lj} = \nabla_l h_{ij}$, respectively. Moreover, $H_F = A^{ij} h_{ij}$. Thus, we obtain

$$\Delta_F N = -g^{kl} \nabla_l H_F \nabla_k X + (\nabla_l A^{ij} h_{ij} - \nabla_i A^{ij} h_{lj}) g^{lk} \nabla_k X - \mathrm{tr}(A_F S^2) N.$$

Since H_F is assumed to be constant, the first term on the right hand side vanishes and (6) will follow, if we can show that

$$\nabla_l A^{ij} h_{ij} = \nabla_i A^{ij} h_{lj}. \tag{9}$$

To accomplish this, we start with $A_{ij} = \partial_{\alpha\beta} F(N) \nabla_i X^\alpha \nabla_j X^\beta$, where α, β are summed over $1, \dots, n+1$, and obtain

$$\nabla_k A_{ij} = -g^{rs} h_{sk} \partial_{\alpha\beta\gamma} F(N) \nabla_i X^\alpha \nabla_j X^\beta \nabla_r X^\gamma.$$

Note, that we have used the fact that $F_{zz}(z)z = 0$ for all $z \neq 0$, which is a consequence of the homogeneity relation (1). Hence,

$$\nabla_k A_{ij} = T_{ijr} g^{rs} h_{sk}$$

with some tensor T which is symmetric in all of its indices, and from this the desired identity (9) follows easily.

In order to prove the second identity, we note that $\partial_i X$ is tangential to X and obtain

$$\begin{aligned}\Delta_F g &= \nabla_i (A^{ij} \langle X, \nabla_j N\rangle)\\ &= A^{ij}\langle \nabla_i X, \nabla_j N\rangle + \langle X, \nabla_i (A^{ij} \nabla_j N)\rangle\\ &= -A^{ij} h_{ij} + \langle X, \Delta_F N\rangle\\ &= -H_F + \langle X, \Delta_F N\rangle.\end{aligned}$$

Using (6), the identity (7) follows.

Finally, using (6) again, we compute

$$\begin{aligned}\Delta_F F &= \nabla_i (A^{ij}\langle F_z(N), \nabla_j N\rangle)\\ &= A^{ij}\langle F_{zz}(N)\nabla_i N, \nabla_j N\rangle + \langle F_z(N), \nabla_i (A^{ij}\nabla_j N)\rangle\\ &= A^{ij} h_{jk} A^{kl} h_{li} + \langle F_z(N), \Delta_F N\rangle\\ &= \mathrm{tr}(S_F^2) - \mathrm{tr}(A_F S^2)\langle F_z(N), N\rangle.\end{aligned}$$

Since $\langle F_z(z), z\rangle = F(z)$ by homogeneity, (8) follows. □

Proof of Theorem 1.1. Define $\varphi := F + \frac{H_F}{n} g$. Then φ is admissible in the anisotropic index form. In fact, choosing $Y = X$ in the first variation formula (2) yields

$$\delta\mathcal{F}(X, Y) = -\int_M H_F g \, dA.$$

On the other hand, the choice $Y = X$ corresponds to the radial variation $X_\varepsilon = (1 + \varepsilon)X$, and since $\mathcal{F}(X_\varepsilon) = (1 + \varepsilon)^n \mathcal{F}(X)$ by scaling, we find

$$\delta\mathcal{F}(X, Y) = n\mathcal{F}(X).$$

Hence, we obtain

$$n \int_M F(N)\, dA = -\int_M H_F g \, dA,$$

which is an analogue of Minkowski's integral formula. In particular,

$$\int_M \varphi \, dA = 0.$$

We now insert φ into (5) and apply Theorem 3.1. This gives

$$I[\varphi] = -\int_M \varphi\left(\operatorname{tr}(S_F^2) - \frac{H_F^2}{n}\right) dA$$

(10) $$= -\int_M \varphi \operatorname{tr}(S_F^2)\, dA,$$

where the last line follows since φ has mean value zero.

Multiplying (7) and (8) by F and g, respectively, and performing a partial integration yields

$$\int_M \frac{H_F}{n} g \operatorname{tr}(S_F^2)\, dA = -\int_M F(N)\frac{H_F^2}{n}\, dA.$$

Combining this with (10), we arrive at the identity

$$I[\varphi] = \int_M F(N)\left(\frac{H_F^2}{n} - \operatorname{tr}(S_F^2)\right) dA.$$

From here we can proceed as in Palmer's [6] paper. Choose an orthonormal basis $\{e_i\}_{i=1,\dots,n}$ such that $A_F(e_i) = \alpha_i e_i$ for $i = 1, \dots, n$. Then, $S_F(e_i) = \sum_j h_{ij}\alpha_j e_j$, where $h_{ij} = g(Se_i, e_j)$. Thus, by virtue of the Cauchy-Schwarz inequality we infer

$$\begin{aligned} \frac{H_F^2}{n} - \operatorname{tr}(S_F^2) &\leqq \sum_i (\alpha_i h_{ii})^2 - \sum_{i,j} \alpha_i\alpha_j h_{ij}^2 \\ &= -\sum_{i\neq j} \alpha_i\alpha_j h_{ij}^2 \leqq 0, \end{aligned}$$

and equality holds if and only if $\alpha_i h_{ii} = \alpha_j h_{jj}$ for all $i, j = 1, \dots, n$ and $h_{ij} = 0$ for all $i \neq j$. Hence,

$$I[\varphi] \leqq 0,$$

and equality holds if and only if

(11) $$S_F = c\,\mathrm{id} \quad \text{with} \quad c = \frac{H_F}{n}.$$

In particular, if X is F-stable, then $I[\varphi] = 0$ and (11) holds. Since M is compact, we deduce that $X(M) = -\frac{1}{c}\mathcal{W}_F + C$ for some vector $C \in \mathbb{R}^{n+1}$, cf. [6], [3], and this is precisely the statement of Theorem 1.1. □

A c k n o w l e d g e m e n t. This research was financially supported by the Deutsche Forschungsgemeinschaft through the project “Singuläre geometrische Variationsprobleme und partielle Differentialgleichungen” at Universität Duisburg-Essen.

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Received: 3 August 2005

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