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Groups with isomorphic Burnside rings

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Abstract. We prove that for some families of finite groups, the isomorphism class of the group is completely determined by its Burnside ring. Namely, we prove the following: if two finite simple groups have isomorphic Burnside rings, then the groups are isomorphic; if G is either Hamiltonian or abelian or a minimal simple group, and G' is any finite group such that $B(G) \cong B(G')$, then $G \cong G'$.

1. Introduction. To any finite group G one can associate its *Burnside ring $B(G)$*, which is a very important algebraic invariant. We refer the reader to [12] for a short survey on Burnside rings.

A natural question to ask is whether there are families of groups whose Burnside rings characterize them, that is, if for some groups G we have that $B(G) \cong B(G')$ implies that G and G' are isomorphic groups. Some results have been proved along these lines, but with stronger assumptions. Namely, in [13] it is proved that if two groups G and G' have isomorphic Burnside rings and each of the two groups is either abelian or Hamiltonian (that is, all its subgroups are normal), then $G \cong G'$; and in [9] it is proved that if two finite simple groups have isomorphic tables of marks, then the groups are isomorphic. In this paper we generalize these results. We prove that if G is either Hamiltonian or abelian or a minimal simple group (a minimal simple group is a non-abelian simple group whose proper subgroups are soluble) and G' is any finite group such that $B(G) \cong B(G')$, then $G \cong G'$. In [5] Dress gives a characterization of minimal simple groups from which one could more readily prove our result about these groups. However, this characterization is incorrect. In Section 4 we explain the flaw in Dress's argument.

We also prove that if two finite simple groups have isomorphic Burnside rings (which in theory is a little weaker than having isomorphic tables of marks), then the groups are isomorphic, slightly generalizing the result in [9].

In Section 2 we define Burnside rings. We also quote Corollary 5.3 from [12] (Theorem 2.1 here), which says that when two groups have isomorphic Burnside rings,

there is a one-to-one correspondence between the families of conjugacy classes of soluble subgroups; this correspondence preserves the orders of the subgroups and their normalizers. Using this result, in Sections 3 and 4 we prove our main theorems.

2. Burnside rings. In this section we introduce the basic concepts and notation that we shall use in this paper. Our presentation is extremely terse. For a fuller account of Burnside rings, we refer the reader to [1], [2], [3], [4] or [7].

Let G be a finite group. A *G-set* is a finite set X where G acts on the left via a group homomorphism into the group of permutations of X. Two G-sets are *isomorphic* if there exists a bijection between them which preserves the action of G. The disjoint union and the Cartesian product of G-sets can be given naturally a structure of G-set. With these operations, the isomorphism classes of G-sets form a commutative half-ring, $B^+(G)$. Its associated ring is the *Burnside ring* of the group G, denoted by $B(G)$ (some authors write $\Omega(G)$ for the Burnside ring).

The most important result about Burnside rings which we shall need in this paper is the following theorem from [12].

Theorem 2.1. *Let G and G' be finite groups such that their Burnside rings are isomorphic. Then there is a one-to-one correspondence between the conjugacy classes of soluble subgroups of G and G' which preserves order of subgroup and cardinality of the conjugacy class (so we can also define a bijection between the families of soluble subgroups of G and G').*

We remark that the correspondence mentioned in the previous theorem is induced by a *normalized isomorphism* (see [12]).

3. Hamiltonian and Abelian groups. Since Hamiltonian and abelian groups are soluble, it is possible to apply Theorem 2.1 to generalize the main theorem from [13].

Theorem 3.1. *Let G and G' be finite groups whose Burnside rings are isomorphic. If G is a Hamiltonian or an abelian group, then G is isomorphic to G'.*

P r o o f. By [13], it suffices to prove that G' is Hamiltonian or abelian as well. By Theorem 2.1, we may assume that there is a one-to-one correspondence $U \mapsto U^*$ between their conjugacy classes of subgroups which preserves soluble subgroups, and for these subgroups, it also preserves their orders and the orders of their normalizers. Since all subgroups of G are soluble, it follows that all subgroups of G' are soluble. Moreover, for any subgroup U^* of G', its corresponding subgroup U of G is such that $|N_{G'}(U*)| = |N_G(U)| = |G| = |G'|$. Therefore, all subgroups of G' are normal, so G' is Hamiltonian or abelian. $\square$

4. Simple groups. The following theorem is a more general version of Proposition 2 from [9]. Note that the important invariants that are needed to distinguish one simple group

from another simple group are: their orders, the number of conjugacy classes of subgroups, and the sizes of the conjugacy classes of involutions. All these invariants must be preserved when the Burnside rings are isomorphic.

Theorem 4.1. *Let G and G' be finite simple groups such that $B(G) \cong B(G')$. Then $G \cong G'$.*

P r o o f. It is known that groups with isomorphic Burnside rings must have the same order. In [8] they prove that the only cases of non-isomorphic finite simple groups with the same order are the following:

- A_8 and $L_3(4)$: A quick run in GAP (see [6]) shows that these groups have a different number of conjugacy classes of subgroups, so their Burnside rings cannot be isomorphic.
- $B_n(q)$ and $C_n(q)$ with $n \geq 3$ and q odd: In [11] and [14] they prove that the conjugacy classes of the involutions in these groups do not match; more precisely, there is an involution in the former group whose conjugacy class has size $\gamma = \frac{q^n(q^n+\epsilon)}{2}$ (where $\epsilon = \pm 1$), and no involution in the latter group has a conjugacy class of the same size. However, if the Burnside rings of the two groups are isomorphic, by Theorem 2.1 there exists a correspondence between their families of soluble subgroups, and this correspondence preserves order and order of normalizer. In particular, the involutions in both groups, (which represent subgroups of order two), would correspond, in a bijection which also preserves the size of their conjugacy classes. This is a contradiction. □

We can strengthen this result when we restrict our attention to a *minimal simple group*, that is, a non-abelian simple group whose proper subgroups are soluble.

Theorem 4.2. *Let G and G' be finite groups whose Burnside rings are isomorphic. If G is a minimal simple group, then G is isomorphic to G'.*

P r o o f. By Theorem 4.1, it suffices to prove that G' is a simple group as well. By Theorem 2.1 we may assume that there is a one-to-one correspondence $U \mapsto U^*$ between their conjugacy classes of subgroups which preserves soluble subgroups, and for these subgroups, it also preserves their orders and the orders of their normalizers. Since all subgroups of G are soluble except G, it follows that all (conjugacy classes of) subgroups of G' are soluble except one, which must be G' (otherwise all its subgroups would be soluble). Note in particular that $G^* = G'$. Moreover, for any proper subgroup U^* of G', we have that U^* is soluble, and its corresponding subgroup U of G is such that $|N_{G'}(U*)| = |N_G(U)| < |G| = |G'|$. Therefore, G' is simple. □

In [5], Dress claims that a group G is minimal simple if and only if its Burnside ring has two indecomposable components and one of them is isomorphic to the ring of integers. If this were true, it would then follow that any group whose Burnside ring is isomorphic to that of a minimal simple group, must also be a minimal simple group, considerably simplifying our proof of Theorem 4.2. However, there is a slight flaw in Dress's claim. The Burnside

ring of a minimal simple group must indeed be as described above. In fact, we claim that the Burnside ring of a group has the form described above if and only if the group is perfect and all its proper subgroups are soluble. Let us give a short proof of this fact. We refer the reader to [3] for a description of the primitive idempotents and their indecomposable components of the Burnside ring, and to [10] for the formula that establishes that the order of a primitive idempotent e_H of the ghost ring is $[H : D(H)]_0[N_G(H) : H]$, where n_0 is the product of all the prime divisors of the integer n, and $D(H)$ denotes the derived subgroup of H. Note also that an indecomposable component of the Burnside ring is isomorphic to the ring of integers if and only if its corresponding perfect subgroup H coincides with its normalizer in G.

If G is perfect and all its subgroups are soluble, then G has exactly two conjugacy classes of perfect subgroups, namely itself and the trivial subgroup, and therefore its Burnside ring $B(G)$ has two primitive idempotents. Moreover, since G is perfect it follows that its indecomposable component is isomorphic to the ring of integers.

Conversely, if G is a finite group whose Burnside ring has two indecomposable components one of which is isomorphic to the ring of integers, then G has exactly two conjugacy classes of perfect subgroups, so G is not soluble. Let H denote the nontrivial perfect subgroup of G. The indecomposable component corresponding to H must be the ring of integers, so by the previous remark $H = N_G(H)$. Since G is not soluble, we must have that $H = O^s(G)$, but this is a normal subgroup of G, so $H = N_G(H) = G$ is perfect. Furthermore, any proper subgroup of G must be in the other component (the soluble component, the one corresponding to the trivial subgroup), so all proper subgroups of G are soluble. However, this does not imply that G should be simple.

Consider the following counterexample, which can be constructed using GAP, or by taking a representative of the nontrivial element in $H^2(A_5, \mathbb{F}_2)$, where $\mathbb{F}_2$ is the field with two elements. Let G be the special linear group $SL_2(\mathbb{F}_5)$. Note that G is the only perfect group of order 120. We have that the centre of G has order 2, and its quotient group is isomorphic to A_5, so the composition factors of G are the alternating group A_5 and the cyclic group of order 2. Note in particular that G is not simple. Since G is perfect, G has no subgroups isomorphic to A_5. Therefore G has exactly two conjugacy classes of perfect subgroups, namely itself and the trivial subgroup, so its Burnside ring has two indecomposable components, one of which is isomorphic to the ring of integers.

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