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Strong convergence theorems for resolvents of maximal monotone operators in Banach spaces

By

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Abstract. In this paper, we prove a strong convergence theorem for resolvents of maximal monotone operators in Banach spaces by the hybrid method in mathematical programming. Using this, we consider the problem of finding a minimizer of a convex function.

1. Introduction. Let H be a Hilbert space and let $T : H \to H$ be a maximal monotone operator. Then the problem of finding a solution $z \in H$ with $0 \in Tz$ has been investigated by many researchers; see, for example, Bruck [4], Rockafellar [16], Brézis and Lions [2], Reich [12, 13], Nevanlinna and Reich [11], Bruck and Reich [5, 6], Takahashi and Ueda [19], Jung and Takahashi [8], Khang [10], and others. One popular method of solving $0 \in Tz$ is the proximal point algorithm. The proximal point algorithm generates, for any starting point $x_1 = x \in H$, a sequence $\{x_n\}$ in H by the rule

(1.1) $$x_{n+1} = J_{r_n} x_n, \qquad n \in \mathbb{N},$$

where $J_{r_n} = (I + r_n T)^{-1}$ and $\{r_n\}$ is a sequence of positive real numbers. Some of them dealt with the weak convergence of the sequence $\{x_n\}$ generated by (1.1) and others proved strong convergence theorems by imposing strong assumptions on T. Recently Kamimura and Takahashi [9] introduced the following two iterative schemes:

(1.2) $$x_{n+1} = \alpha_n x_1 + (1 - \alpha_n) J_{r_n} x_n, \qquad n \in \mathbb{N}$$

and

(1.3) $$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) J_{r_n} x_n, \qquad n \in \mathbb{N},$$

where $x_1 = x \in H$, $\{\alpha_n\}$ is a sequence in [0, 1] and $\{r_n\}$ is a sequence of positive real numbers. They showed that the sequence $\{x_n\}$ generated by (1.2) converges strongly and the sequence $\{x_n\}$ generated by (1.3) converges weakly; see also [14]. On the other

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hand, Solodov and Svaiter [17] introduced the following hybrid method in mathematical programming:

$$(1.4)\qquad \begin{cases} x_1 = x \in H, \\ y_n = J_{r_n} x_n, \\ C_n = \{z \in H : \langle y_n - z, x_n - y_n \rangle \geqq 0\}, \\ D_n = \{z \in H : \langle x_n - z, x_1 - x_n \rangle \geqq 0\}, \\ x_{n+1} = P_{C_n \cap D_n}(x_1) \end{cases}$$

for each $n \in \mathbb{N}$, where $J_{r_n} = (I + r_n T)^{-1}$ and $r_n > 0$. They showed that the sequence $\{x_n\}$ generated by (1.4) converges strongly to $P_{T^{-1}0}(x_1)$. The aim of this paper is to prove a strong convergence theorem for resolvents of maximal monotone operators in Banach spaces which generalizes the result by Solodov and Svaiter [17]. Using the result, we consider the problem of finding a minimizer of a convex function.

2. Preliminaries. A Banach space E is *uniformly convex* if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that

$$\|x_n\| = \|y_n\| = 1 \quad \text{and} \quad \lim_{n\to\infty} \|x_n + y_n\| = 2,$$

$\lim_{n\to\infty} \|x_n - y_n\| = 0$ holds. It is well known that a Banach space E is uniformly convex if and only if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that

$$\lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|y_n\| = 1 \quad \text{and} \quad \lim_{n\to\infty} \|x_n + y_n\| = 2,$$

$\lim_{n\to\infty} \|x_n - y_n\| = 0$ holds. We also know that if E is a uniformly convex Banach space, then $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ imply $x_n \to x$, where $\rightharpoonup$ means the weak convergence. Let C be a nonempty closed convex subset of a uniformly convex Banach space E and let $x_1 \in E$. Then there exists a unique element $x \in C$ such that $\|x_1 - x\| \leqq \|x_1 - y\|$ for all $y \in C$. Putting $x = P_C(x_1)$, we call P_C the metric projection on C; see [7, p.12].

Let E be a Banach space and let E^* be its dual. With each $x \in E$, we associate the set

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}.$$

By the Hahn-Banach theorem, $J(x)$ is nonempty. The multivalued operator $J : E \to E^*$ is called the *duality mapping* of E. Let $S(E) = \{x \in E : \|x\| = 1\}$. Then a Banach space E is said to be *Gâteaux differentiable* provided the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in S(E)$. If E is Gâteaux differentiable, then the duality mapping J of E is single valued. We use the following theorem in the proof of our theorem.

Theorem 2.1 [18, p.196]**.** *Let E be a uniformly convex Banach space with a Gâteaux differentiable norm. Let C be a nonempty closed convex subset of E and $x_1 \in E$. Then, $x = P_C(x_1)$ if and only if*

$$\langle x - z, J(x_1 - x) \rangle \geqq 0 \quad \textit{for all} \quad z \in C,$$

where P_C is the metric projection on C and J is the duality mapping of E.

This theorem is also a special case of Proposition 3.4 on p.13 of [7]. A mapping T of E into E^* is *monotone* if for each $(x, x^*), (y, y^*) \in T$, we have

$$\langle x - y, x^* - y^* \rangle \geqq 0.$$

A monotone mapping T is said to be maximal if its graph $G(T) = \{(x, y) : y \in Tx\}$ is not properly contained in the graph of any other monotone mapping. The following theorem is due to Browder [3]; see also Barbu's book [1].

Theorem 2.2 [1, p. 39]**.** *Let E be a uniformly convex Banach space with a Gâteaux differentiable norm and let T be a monotone operator from E into E^*. Then T is maximal if and only if for any $r > 0$,*

$$R(J + rT) = E^*,$$

where $R(J + rT)$ is the range of $J + rT$.

We also know the following theorem.

Theorem 2.3 [18, p. 102]**.** *Let E be a uniformly convex Banach space with a Gâteaux differentiable norm and let $x, y \in E$. If*

$$\langle x - y, J(x) - J(y) \rangle = 0,$$

then $x = y$.

3. A strong convergence theorem. Let E be a uniformly convex Banach space with a Gâteaux differentiable norm and let T be a maximal monotone operator from E into E^* such that $T^{-1}0 \neq \emptyset$. For all $x \in E$ and $r > 0$, we consider the following equation

$$J(x_r - x) + rTx_r \ni 0.$$

By Theorems 2.2 and 2.3, this equation has a unique solution x_r; see also Corollary 1.1 in [1]. We define J_r by

$$x_r = J_r x$$

and such $J_r, r > 0$ are called *resolvents* of T. Now motivated by Solodov and Svaiter [17], we consider the sequence $\{x_n\}$ generated by

$$(3.1) \qquad \begin{cases} x_1 = x \in E, \\ y_n = J_{r_n} x_n, \\ C_n = \{z \in E : \langle y_n - z, J(x_n - y_n) \rangle \geqq 0\}, \\ D_n = \{z \in E : \langle x_n - z, J(x_1 - x_n) \rangle \geqq 0\}, \\ x_{n+1} = P_{C_n \cap D_n}(x_1) \end{cases}$$

for each $n \in \mathbb{N}$, where $J(y_n - x_n) + r_n T y_n \ni 0$, and $r_n > 0$.

Theorem 3.1. *Let E be a uniformly convex Banach space with a Gâteaux differentiable norm and let T be a maximal monotone operator from E into E^* such that $T^{-1}0 \neq \emptyset$. Suppose $\{x_n\}$ is the sequence generated by* (3.1) *and* $\liminf_{n\to\infty} r_n > 0$. *Then, $\{x_n\}$ converges strongly to $P_{T^{-1}0}(x_1)$ as $n \to \infty$.*

Proof. We first show that $\{x_n\}$ is well defined. It is obvious that $C_n \cap D_n$ is a closed convex subset of E for every $n \in \mathbb{N}$. Let $(z, 0) \in T$. Since $(y_n, \frac{1}{r_n} J(x_n - y_n)) \in T$ and T is monotone, we have

$$\left\langle y_n - z, \frac{1}{r_n} J(x_n - y_n) \right\rangle \geqq 0.$$

So, we get

$$\langle y_n - z, J(x_n - y_n)\rangle \geqq 0$$

and hence $z \in C_n$. So we have $T^{-1}0 \subset C_n$ for every $n \in \mathbb{N}$.

We show by mathematical induction that $T^{-1}0 \subset C_n \cap D_n$ for each $n \in \mathbb{N}$. Since $T^{-1}0 \subset C_1$ and $D_1 = E$, we obtain $T^{-1}0 \subset C_1 \cap D_1$. Suppose $T^{-1}0 \subset C_k \cap D_k$ for $k \in \mathbb{N}$. Then, there exists a unique element $x_{k+1} \in C_k \cap D_k$ such that $x_{k+1} = P_{C_k \cap D_k}(x_1)$. From $x_{k+1} = P_{C_k \cap D_k}(x_1)$ and Theorem 2.1, we have

$$\langle x_{k+1} - z, J(x_1 - x_{k+1})\rangle \geqq 0$$

for each $z \in C_k \cap D_k$. Since $T^{-1}0 \subset C_k \cap D_k$, we get

$$\langle x_{k+1} - z, J(x_1 - x_{k+1})\rangle \geqq 0$$

for each $z \in T^{-1}0$ and hence $T^{-1}0 \subset D_{k+1}$. Therefore we have $T^{-1}0 \subset C_{k+1} \cap D_{k+1}$. This means that $\{x_n\}$ is well defined.

Since $T^{-1}0$ is a nonempty closed convex subset of E, there exists a unique element $z_1 \in T^{-1}0$ such that $z_1 = P_{T^{-1}0}(x_1)$. From $x_{n+1} = P_{C_n \cap D_n}(x_1)$, we have

$$\|x_{n+1} - x_1\| \leqq \|z - x_1\|$$

for every $z \in C_n \cap D_n$. Since $z_1 \in T^{-1}0 \subset C_n \cap D_n$, we get

$$\|x_{n+1} - x_1\| \leqq \|z_1 - x_1\| \tag{3.2}$$

for each $n \in \mathbb{N}$. This means that $\{x_n\}$ is bounded.

Next we show $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$. By (3.1) and Theorem 2.1, we get $x_n = P_{D_n}(x_1)$. From $x_{n+1} \in D_n$, we have

$$\|x_1 - x_n\| \leqq \|x_1 - x_{n+1}\|$$

for every $n \in \mathbb{N}$. This implies that $\{\|x_1 - x_n\|\}$ is bounded and nondecreasing. So there exists the limit of $\{\|x_1 - x_n\|\}$. Put $\lim_{n\to\infty} \|x_1 - x_n\| = a$. Without loss of generality, we assume that $a > 0$. Since $x_n = P_{D_n}(x_1)$, $x_{n+1} \in D_n$ and $\frac{x_n + x_{n+1}}{2} \in D_n$, we have

$$\|x_1 - x_n\| \leqq \left\| x_1 - \frac{x_n + x_{n+1}}{2} \right\| \leqq \frac{1}{2}(\|x_1 - x_n\| + \|x_1 - x_{n+1}\|)$$

and hence

$$\lim_{n\to\infty}\left\|x_1 - \frac{x_n + x_{n+1}}{2}\right\| = a.$$

Since E is uniformly convex, we get $\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0$.

By (3.1) and Theorem 2.1, we get $y_n = P_{C_n}(x_n)$.

From $x_{n+1} \in C_n$, we also have

$$\|x_n - y_n\| \leqq \|x_n - x_{n+1}\|.$$

So we get $\lim_{n\to\infty}\|x_n - y_n\| = 0$. Using $\liminf_{n\to\infty} r_n > 0$, we have

$$\lim_{n\to\infty}\left\|\frac{1}{r_n}J(x_n - y_n)\right\| = \lim_{n\to\infty}\frac{1}{r_n}\|x_n - y_n\| = 0.$$

On the other hand, since E is reflexive and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converging weakly to w. Then $\{y_{n_i}\}$ also converges weakly to w because $\lim_{n\to\infty}\|x_n - y_n\| = 0$. Let $(u, v) \in T$. Since $(y_{n_i}, \frac{1}{r_{n_i}}J(x_{n_i} - y_{n_i})) \in T$ and T is monotone, we have

$$\left\langle y_{n_i} - u, \frac{1}{r_{n_i}}J(x_{n_i} - y_{n_i}) - v\right\rangle \geqq 0.$$

Letting i tend to infinity, we get

$$\langle w - u, 0 - v\rangle \geqq 0.$$

Since T is maximal, we obtain

$$(w, 0) \in T.$$

From $z_1 = P_{T^{-1}0}(x_1)$, lower semicontinuity of the norm and (3.2), we have

$$\begin{aligned}\|x_1 - z_1\| &\leqq \|x_1 - w\| \leqq \liminf_{i\to\infty}\|x_1 - x_{n_i}\| \\ &\leqq \limsup_{i\to\infty}\|x_1 - x_{n_i}\| \leqq \|x_1 - z_1\|.\end{aligned}$$

So, we get

$$\lim_{i\to\infty}\|x_1 - x_{n_i}\| = \|x_1 - w\| = \|x_1 - z_1\|.$$

Since E is uniformly convex, we have $x_1 - x_{n_i} \to x_1 - w$ and hence

$$x_{n_i} \to w = z_1.$$

Therefore, we obtain $x_n \to z_1$. □

4. Application. Using Theorem 3.1, we consider the problem of finding a minimizer of a convex function.

Theorem 4.1. *Let E be a uniformly convex Banach space with a Gâteaux differentiable norm and let $f : E \to (-\infty, \infty]$ be a proper lower semicontinuous convex function. Assume $\liminf_{n\to\infty} r_n > 0$ and let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} x_1 = x \in E, \\ y_n = \operatorname*{argmin}_{z\in E} \{f(z) + \frac{1}{2r_n}\|z - x_n\|^2\}, \\ C_n = \{z \in E : \langle y_n - z, J(x_n - y_n)\rangle \geqq 0\}, \\ D_n = \{z \in E : \langle x_n - z, J(x_1 - x_n)\rangle \geqq 0\}, \\ x_{n+1} = P_{C_n \cap D_n}(x_1) \end{cases}$$

for each $n \in \mathbb{N}$. If $(\partial f)^{-1}0 \neq \emptyset$, then $\{x_n\}$ converges strongly to the minimizer of f nearest to x_1.

Proof. Since $f : E \to (-\infty, \infty]$ is a proper lower semicontinuous convex function, by Rockafellar [15], the subdifferential ∂f of f defined by

$$\partial f(z) = \{x^* \in E^* : f(y) \geqq f(z) + \langle y - z\rangle x^*, \forall y \in E\}, \quad \forall z \in E$$

is a maximal monotone operator. We also know that

$$y_n = \operatorname*{argmin}_{z\in E} \left\{ f(z) + \frac{1}{2r_n}\|z - x_n\|^2 \right\}$$

is equivalent to

$$(\partial f)y_n + \frac{1}{r_n}J(y_n - x_n) \ni 0.$$

So, we have

$$J(y_n - x_n) + r_n(\partial f)y_n \ni 0.$$

Using Theorem 3.1, we get the conclusion. □

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