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Statistical convergence of multiple sequences

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Abstract. We extend the concept of and basic results on statistical convergence from ordinary (single) sequences to multiple sequences of (real or complex) numbers. As an application to Fourier analysis, we obtain the following Theorem 3: (i) If $f \in L(\log^+ L)^{d-1}(\mathbb{T}^d)$, where $\mathbb{T}^d := [-\pi, \pi)^d$ is the d-dimensional torus, then the Fourier series of f is statistically convergent to $f(\mathbf{t})$ at almost every $\mathbf{t} \in \mathbb{T}^d$; (ii) If $f \in C(\mathbb{T}^d)$, then the Fourier series of f is statistically convergent to $f(\mathbf{t})$ uniformly on $\mathbb{T}^d$.

1. Introduction and Background. The concept of statistical convergence was introduced by Fast [2] in 1951. A sequence $(x_k : k = 0, 1, 2, \ldots)$ of (real or complex) numbers is said to be statistically convergent to some finite number ξ, in symbol: $\text{st} - \lim x_k = \xi$, if for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n+1} \, |\{k \leqq n : |x_k - \xi| > \varepsilon\}| = 0,$$

where by $k \leqq n$ we mean that $k = 0, 1, 2, \ldots, n$; and by $|\mathcal{S}|$ we mean the cardinality of the set $\mathcal{S} \subseteqq \mathbb{N} := \{0, 1, 2, \ldots\}$.

Some basic properties of statistical convergence were proved by Schoenberg [9] in 1959.

The following concept is due to Fridy [3]. A sequence (x_k) is said to be statistically Cauchy if for each $\varepsilon > 0$ and $\ell \geqq 0$ there exists an integer $m\ (\geqq \ell)$ such that

$$\lim_{n\to\infty} \frac{1}{n+1} \, |\{k \leqq n : |x_k - x_m| > \varepsilon\}| = 0.$$

Fridy [3] proved that a sequence (x_k) is statistically convergent if and only if it is statistically Cauchy.

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We recall (see, for example, [8, p. 290]) that the "natural (or asymptotic) density" of a set $\mathcal{S} \subseteqq \mathbb{N}$ is defined by

$$d(\mathcal{S}) := \lim_{n\to\infty} \frac{1}{n+1} |\{k \leqq n : k \in \mathcal{S}\}|,$$

provided that the limit on the right-hand side exists. It is clear that a set $\mathcal{S} \subseteqq \mathbb{N}$ has natural density 0 if and only if its complement $\mathcal{S}^c := \mathbb{N}\backslash\mathcal{S}$ has natural density 1.

Now, the concept of statistical convergence can be reformulated in terms of natural density as follows. A sequence (x_k) is statistically convergent to some number ξ if and only if for each $\varepsilon > 0$,

$$d(\{k \in \mathbb{N} : |x_k - \xi| > \varepsilon\}) = 0.$$

2. New results on double sequences. Our goal is to extend a few results known in the literature from ordinary (single) sequences to double ones. We begin with two definitions.

We say that a double sequence $(x_{jk} : j, k = 0, 1, 2, \ldots)$ of (real or complex) numbers is statistically convergent to some number ξ, in symbol: $\mathrm{st} - \lim x_{jk} = \xi$, if for each $\varepsilon > 0$,

$$\lim_{m,n\to\infty} \frac{1}{(m+1)(n+1)} |\{j \leqq m \text{ and } k \leqq n : |x_{jk} - \xi| > \varepsilon\}| = 0. \tag{2.1}$$

Here and in the sequel, m and n tend to infinity independently of one another.

It is plain that statistical convergence enjoys the property of additivity and homogeneity. Statistical convergence implies statistical boundedness, the latter being defined by the requirement that there exists a constant K such that

$$\lim_{m,n\to\infty} \frac{1}{(m+1)(n+1)} |\{j \leqq m \text{ and } k \leqq n : |x_{jk}| > K\}| = 0.$$

It follows from (2.1) that in the capacity of K we may take any number greater than $|\xi|$. Furthermore, usual convergence (in Pringsheim's sense) implies statistical convergence to the same limit; that is, if $\lim_{j,k\to\infty} x_{jk} = \xi$, then (2.1) is satisfied for each $\varepsilon > 0$.

We say that (x_{jk}) is statistically Cauchy if for each $\varepsilon > 0$ and $\ell \geqq 0$ there exist integers M $(\geqq \ell)$ and N $(\geqq \ell)$ such that

$$\lim_{m,n\to\infty} \frac{1}{(m+1)(n+1)} |\{j \leqq m \text{ and } k \geqq n : |x_{jk} - x_{MN}| > \varepsilon\}| = 0. \tag{2.2}$$

We prove that statistical convergence is equivalent to the property of being statistically Cauchy.

Theorem 1. *A double sequence (x_{jk}) is statistically convergent if and only if (x_{jk}) is statistically Cauchy.*

Proof. Necessity. The proof that (x_{jk}) is statistically Cauchy if (x_{jk}) is statistically convergent is trivial, as in the case of usual convergence.

S u f f i c i e n c y . Assume that (x_{jk}) is statistically Cauchy. We shall prove that (x_{jk}) is statistically convergent. To this effect, let $(\varepsilon_p : p = 1, 2, \ldots)$ be a strictly decreasing sequence of numbers converging to 0. By (2.2), there exist two strictly increasing sequences (M_p) and (N_p) of positive integers such that

$$\lim_{m,n\to\infty} \frac{1}{(m+1)(n+1)} |\{j \leqq m \text{ and } k \leqq n : |x_{jk} - x_{M_p,N_p}| > \varepsilon_p\}| = 0, \tag{2.3}$$
$$p = 1, 2, \ldots .$$

Clearly, for each pair (p, q) of positive integers, $p \neq q$, we can select a pair $(j_{pq}, k_{pq}) \in \mathbb{N}^2$ such that

$$|x_{j_{pq},k_{pq}} - x_{M_p,N_p}| \leqq \varepsilon_p \quad \text{and} \quad |x_{j_{pq},k_{pq}} - x_{M_q,N_q}| \leqq \varepsilon_q.$$

It follows that

$$|x_{M_p,N_p} - x_{M_q,N_q}| \leqq \varepsilon_p + \varepsilon_q \to 0 \quad \text{as} \quad p, q \to \infty,$$

that is, the ordinary (single) sequence $(x_{M_p,N_p} : p = 1, 2, \ldots)$ satisfies the Cauchy convergence criterion. Thus, the sequence (x_{M_p,N_p}) is convergent in the usual sense to a finite limit ξ, say. Consequently, given an arbitrary $\varepsilon > 0$, there exists $p_0 \in \mathbb{N}$ such that $\varepsilon_{p_0} < \varepsilon/2$ and

$$|x_{M_p,N_p} - \xi| < \varepsilon/2 \quad \text{if} \quad p \geqq p_0. \tag{2.4}$$

Now, consider an arbitrary pair $(j, k) \in \mathbb{N}^2$. By (2.4),

$$|x_{jk} - \xi| \leqq |x_{jk} - x_{M_{p_0},N_{p_0}}| + \varepsilon/2,$$

whence, by (2.3),

$$\begin{aligned} &\frac{1}{(m+1)(n+1)} |\{j \leqq m \text{ and } k \leqq n : |x_{jk} - \xi| > \varepsilon\}| \\ &\quad \leqq \frac{1}{(m+1)(n+1)} |\{j \leqq m \text{ and } k \leqq n : |x_{jk} - x_{M_{p_0},N_{p_0}}| > \varepsilon/2\}| \\ &\quad \leqq \frac{1}{(m+1)(n+1)} |\{j \leqq m \text{ and } k \leqq n : |x_{jk} - x_{M_{p_0},N_{p_0}}| > \varepsilon_{p_0}\}| \to 0 \\ &\qquad \text{as} \quad m, n \to \infty. \end{aligned}$$

This completes the proof of Theorem 1.

3. Decomposition Theorem. We extend the decomposition theorem of Connor [1] from ordinary (single) to double sequences as follows.

Theorem 2. *A double sequence (x_{jk}) is statistically convergent to some number ξ if and only if there exist two sequences (u_{jk}) and (v_{jk}) such that*

$$x_{jk} = u_{jk} + v_{jk}, \quad j,k = 0,1,2,\ldots, \tag{3.1}$$

$$\lim_{j,k\to\infty} u_{jk} = \xi, \tag{3.2}$$

and

$$\lim_{m,n\to\infty} \frac{1}{(m+1)(n+1)} |\{j \leqq m \text{ and } k \leqq n : v_{jk} \neq 0\}| = 0. \tag{3.3}$$

Moreover, if (x_{jk}) is bounded, then (u_{jk}) and (v_{jk}) are also bounded.

Proof. Necessity. By (2.1), we can select a sequence $(N_p : p = 1,2,\ldots)$ of positive integers such that

$$2N_p \leqq N_{p+1}, \quad p = 1,2,\ldots, \tag{3.4}$$

and

$$\frac{1}{(m+1)(n+1)}|\{j \leqq m \text{ and } k \leqq n : |x_{jk} - \xi| > 2^{-p}\}| < 2^{-2p} \quad \text{if} \quad m,n \geqq N_p. \tag{3.5}$$

We shall define the sequence (u_{jk}) in the following way: If $\min(j,k) < N_1$, then we set $u_{ij} := x_{ij}$; while if $N_p \leqq j < N_{p+1}$ and $N_q \leqq k < N_{q+1}$ for some positive integers p an q, then we set

$$u_{jk} := \begin{cases} x_{jk} & \text{if} \quad |x_{jk} - \xi| \leqq 2^{-\min(p,q)}, \\ \xi & \text{if} \quad |x_{jk} - \xi| > 2^{-\min(p,q)}. \end{cases} \tag{3.6}$$

Finally, we set $v_{jk} := x_{jk} - u_{jk}$. Condition (3.1) is clearly satisfied.

The proof of (3.2) is almost trivial. Indeed, given any $\varepsilon > 0$, we choose p_0 so large that $2^{-p_0} < \varepsilon$. By (3.6), if $(j,k) \in \mathbb{N}^2$ is such that $N_p \leqq \min(j,k) < N_{p+1}$ for some $p \geqq p_0$, then

$$|u_{jk} - \xi| = \begin{cases} |x_{jk} - \xi| \leqq 2^{-p} < \varepsilon & \text{if} \quad |x_{jk} - \xi| \leqq 2^{-p}, \\ |\xi - \xi| = 0 & \text{if} \quad |x_{jk} - \xi| > 2^{-p}. \end{cases}$$

In any case, we have

$$|u_{jk} - \xi| < \varepsilon \quad \text{if} \quad j,k \geqq N_{p_0}.$$

So, the sequence (u_{jk}) converges to ξ in Pringsheim's sense.

It remains to prove (3.3). Since $v_{jk} = 0$ if $\min(j,k) < N_1$, we may assume that for some $p,q \geqq 1$,

$$N_p \leqq m < N_{p+1} \quad \text{and} \quad N_q \leqq n < N_{q+1}. \tag{3.7}$$

Let $r := \min(p, q)$. By definition (3.6), we have

$$\begin{aligned}
&\{j \leqq m \quad \text{and} \quad k \leqq n : v_{jk} \neq 0\} \\
&\quad = \{N_r \leqq j \leqq m \quad \text{and} \quad N_r \leqq k \leqq n : |x_{jk} - \xi| > 2^{-r}\} \\
&\qquad \cup \bigcup_{s=1}^{r-1} [\{N_s \leqq j \leqq m \quad \text{and} \quad N_s \leqq k < N_{s+1} : |x_{jk} - \xi| > 2^{-s}\} \\
&\qquad \cup \{N_s \leqq j < N_{s+1} \quad \text{and} \quad N_s \leqq k \leqq n : |x_{jk} - \xi| > 2^{-s}\}],
\end{aligned}$$

whence, making use of (3.4) and (3.5), we obtain

$$\begin{aligned}
&\frac{1}{(m+1)(n+1)} |\{j \leqq m \quad \text{and} \quad k \leqq n : v_{jk} \neq 0\}| \\
&\quad \leqq 2^{-2r} + \sum_{s=1}^{r-1} \left[\frac{N_{s+1}}{n+1} 2^{-2s} + \frac{N_{s+1}}{m+1} 2^{-2s} \right] \\
&\quad \leqq 2^{-2r} + \left[\frac{N_r}{n+1} + \frac{N_r}{m+1} \right] \sum_{s=1}^{r-1} 2^{-2s-(r-1-s)} \\
&\quad < 2^{-2r} + 2^{-r+1} \to 0 \quad \text{as} \quad r := \min(p, q) \to \infty,
\end{aligned}$$

or equivalently, $m, n \to \infty$ (cf. (3.7)). This proves (3.3).

S u f f i c i e n c y . It is immediate. By (3.3), we have

$$\text{st} - \lim v_{jk} = 0. \tag{3.8}$$

Now, (2.1) follows from (3.2) and (3.8), via additivity. The proof of Theorem 2 is complete.

Theorem 2 makes it possible to characterize the concept of statistical convergence in a more transparent form. To this end, we define the "natural (or asymptotic) density" of a set $\mathcal{S} \subseteqq \mathbb{N}^2$ as follows:

$$d(\mathcal{S}) := \lim_{m,n \to \infty} \frac{1}{(m+1)(n+1)} |\{j \leqq m \ \text{ and } \ k \leqq n : (j, k) \in \mathcal{S}\}|,$$

provided that this limit exists.

The following Proposition 1 is a reformulation of Theorem 2 if we set

$$\mathcal{S} := \{(j, k) \in \mathbb{N}^2 : u_{jk} = x_{jk}\}$$

(to see Necessity); while $u_{jk} := x_{jk}$ if $(j, k) \in \mathcal{S}$ and $u_{jk} := \xi$ if $(j, k) \notin \mathcal{S}$ (to see Sufficiency).

Proposition 1. *A double sequence (x_{jk}) is statistically convergent to some number ξ if and only if there exists a set $\mathcal{S} \subseteqq \mathbb{N}^2$ such that the natural density of $\mathcal{S}$ is* 1 *and*

$$\lim_{j,k \to \infty \ \text{and} \ (j,k) \in \mathcal{S}} x_{jk} = \xi. \tag{3.9}$$

By (3.9) we mean that for each $\epsilon > 0$ there exists an integer N such that

$$|x_{jk} - \xi| \leqq \epsilon \quad \text{if} \quad j, k \geqq N \quad \text{and} \quad (j, k) \in \mathcal{S}.$$

4. Multiple sequences and an application. The concepts and results of the preceding two sections can be extended with ease to d-multiple sequences. Let $\mathbb{N}^d$ be the set of d-tuples $\mathbf{k} := (k_1, k_2, \ldots, k_d)$ with nonnegative integers for coordinates k_j, where d is a fixed positive integer. Two tuples $\mathbf{k}$ and $\mathbf{n} := (n_1, n_2, \ldots, n_d)$ are distinct if and only if $k_j \neq n_j$ for at least on j. $\mathbb{N}^d$ is partially ordered by agreeing that $\mathbf{k} \leqq \mathbf{n}$ if and only if $k_j \leqq n_j$ for each j.

We say that a d-multiple sequence $(x_{\mathbf{k}} : \mathbf{k} \in \mathbb{N}^d)$ of (real or complex) numbers is statistically convergent to some number ξ if for each $\varepsilon > 0$,

$$\lim_{\min n_j \to \infty} \frac{1}{|\mathbf{n}+\mathbf{1}|} |\{\mathbf{k} \leqq \mathbf{n} : |x_{\mathbf{k}} - \xi| > \varepsilon\}| = 0, \tag{4.1}$$

where

$$|\mathbf{n}+\mathbf{1}| := \prod_{j=1}^{d} (n_j + 1).$$

Furthermore, we say that $(x_{\mathbf{k}})$ is statistically Cauchy if for each $\varepsilon > 0$ and $\ell \geqq 0$ there exists $\mathbf{m} := (m_1, m_2, \ldots, m_d) \in \mathbb{N}^d$ such that $\min m_j \geqq \ell$ and

$$\lim_{\min n_j \to \infty} \frac{1}{|\mathbf{n}+\mathbf{1}|} |\{\mathbf{k} \leqq \mathbf{n} : |x_{\mathbf{k}} - x_{\mathbf{m}}| > \varepsilon\}| = 0.$$

Both Theorem 1 and Theorem 2 are valid for d-multiple sequences. The "natural (or asymptotic) density" of a set $\mathcal{S} \subseteqq \mathbb{N}^d$ can be defined as follows:

$$d(\mathcal{S}) := \lim_{\min n_j \to \infty} \frac{1}{|\mathbf{n}+\mathbf{1}|} |\{\mathbf{k} \leqq \mathbf{n} : \mathbf{k} \in \mathcal{S}\}|,$$

provided that this limit exists. Proposition 1 remains true for d-multiple sequences, as well.

Before application, we recall the concept of strong Cesàro summability (see, for example, [1] or [10, p. 180]). Let p be a positive real number. A d-multiple sequence $(x_{\mathbf{k}})$ is said to be strongly p-Cesàro summable to some number ξ if

$$\lim_{\min n_j \to \infty} \frac{1}{|\mathbf{n}+\mathbf{1}|} \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} \ldots \sum_{k_d=0}^{n_d} |x_{\mathbf{k}} - \xi|^p = 0. \tag{4.2}$$

The following Proposition 2 is almost evident.

Proposition 2. (i) *If* $(x_{\mathbf{k}})$ *is strongly* p*-Cesàro summable to* ξ *for some* $0 < p < \infty$*, then* $(x_{\mathbf{k}})$ *is statistically convergent to* ξ.

(ii) *If* (x_k) *is statistically convergent to* ξ *and bounded, then* $(x_{\mathbf{k}})$ *is strongly* p*-Cesàro summable to* ξ *for each* $0 < p < \infty$.

Proof. (i) It follows from the Markov type inequality

$$\varepsilon^p |\{\mathbf{k} \leqq \mathbf{n} : |x_{\mathbf{k}} - \xi| > \varepsilon\}| \leqq \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} \ldots \sum_{k_d=0}^{n_d} |x_{\mathbf{k}} - \xi|^p.$$

(ii) Let $B := K + |\xi|$, where $|x_{\mathbf{k}}| \leqq K$ for all $\mathbf{k} \in \mathbb{N}^d$. Then by (4.1), we have

$$\frac{1}{|\mathbf{n}+\mathbf{1}|} \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} \dots \sum_{k_d=0}^{n_d} |x_{\mathbf{k}} - \xi|^p = \frac{1}{|\mathbf{n}+\mathbf{1}|} \left\{ \sum_{|x_{\mathbf{k}}-\xi| \leqq \varepsilon} + \sum_{|x_{\mathbf{k}}-\xi| > \varepsilon} \right\}$$

$$\leqq \varepsilon^p + \frac{B^p}{|\mathbf{n}+\mathbf{1}|} |\{\mathbf{k} \leqq \mathbf{n} : |x_{\mathbf{k}} - \xi| > \varepsilon\}| < 2\varepsilon^p,$$

if $\min n_j$ is large enough. This proves (4.2). □

Now, we present an application to Fourier analysis. Let $\mathbf{t} := (t_1, t_2, \dots, t_d) \in \mathbb{R}^d$, $\mathbf{kt} := k_1 t_1 + k_2 t_2 + \dots + k_d t_d$, and f a periodic, Lebesgue integrable function over the d-dimensional torus $\mathbb{T}^d := [-\pi, \pi)^d$. We recall that the Fourier series of f is defined by

$$f(\mathbf{t}) \sim \sum_{\mathbf{k} \in \mathbb{Z}^d} \hat{f}(\mathbf{k}) e^{i\mathbf{kt}}, \tag{4.3}$$

where $\mathbb{Z}^d$ is the set of d-tuples $\mathbf{k} := (k_1, k_2, \dots, k_d)$ with (positive, zero, or negative) integer coordinates, and the Fourier coefficient $\hat{f}(\mathbf{k})$ is defined by

$$\hat{f}(\mathbf{k}) := \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(\mathbf{t}) e^{-i\mathbf{kt}} d\mathbf{t}, \quad \mathbf{k} \in \mathbb{Z}^d.$$

The reader is referred to [10, Ch. 17] for more details.

The following facts are known in the literature. Gogoladze [4] proved that if $f \in L(\log^+ L)^{d-1}(\mathbb{T}^d)$, then the d-multiple sequence of the rectangular partial sums

$$s_{\mathbf{n}}(f, \mathbf{t}) := \sum_{k_1=-n_1}^{n_1} \sum_{k_2=-n_2}^{n_2} \dots \sum_{k_d=-n_d}^{n_d} \hat{f}(k) e^{i\mathbf{kt}}, \quad \mathbf{n} \in \mathbb{N}^d,$$

of the Fourier series (4.3) is strongly p-Cesàro summable to $f(\mathbf{t})$ at almost every point $\mathbf{t} \in \mathbb{T}^d$ for any $0 < p < \infty$. Analogous results are valid for the so-called conjugate series to (4.3). More precisely, if $j_1, j_2, \dots, j_s$ are natural numbers, $1 \leqq j_1 < j_2 < \dots < j_s \leqq d$, and $1 \leqq s \leqq d$, then the series

$$\sum_{\mathbf{k} \in \mathbb{Z}^d} (-i \operatorname{sign} k_{j_1})(-i \operatorname{sign} k_{j_2}) \cdots (-i \operatorname{sign} k_{j_s}) \hat{f}(\mathbf{k}) e^{i\mathbf{kt}} \tag{4.4}$$

is strongly p-Cesàro summable to the so-called conjugate function $\widetilde{f}^{(j_1, j_2, \dots, j_s)}(\mathbf{t})$ at almost every point $\mathbf{t} \in \mathbb{T}^d$ for any $0 < p < \infty$. (Altogether, there are $2^d - 1$ conjugate series to Fourier series (4.3).)

We note that in case $d = 1$, these results were proved by Marcinkiewicz ($p = 2$) and Zygmund ($0 < p < \infty$). (See, for example, [10, p. 182].)

Leindler [5] (see also [6, pp. 12 and 20]) proved the following inequality to periodic continuous functions.

Proposition 3. *For each* $0 < p < \infty$, *there exists a constant* K_p *such that if* $f \in C(\mathbb{T})$, *then*

$$\frac{1}{n+1}\sum_{k=0}^{n}|s_k(f,t) - f(t)|^p \leqq \frac{K_p}{n+1}\sum_{k=0}^{n}[E_k(f)]^p \quad for\ all \quad t \in \mathbb{T}, \tag{4.5}$$

where $E_k(f)$ *is the best approximation of* f *by trigonometric polynomials of degree at most* k *in the "maximum" norm of the space* $C(\mathbb{T})$.

In the joint paper [7] with Xianliang Shi, the present author proved the two-dimensional analogue of inequality (4.5), and the proving method there clearly indicates the straightforward way of the extension to the d-dimensional case, $d \geqq 3$. From these it follows immediately that if $f \in C(\mathbb{T}^d)$, then the rectangular partial sums $s_{\mathbf{k}}(f, \mathbf{t})$ of the Fourier series (4.3) are strongly p-Cesàro summable to $f(\mathbf{t})$ uniformly on $\mathbb{T}^d$, for any $0 < p < \infty$.

Combining these results with Proposition 2 yields the following

Theorem 3. (i) *If* $f \in L(\log^+ L)^{d-1}(\mathbb{T}^d)$, *then the Fourier series of* f *is statistically convergent to* $f(\mathbf{t})$ *at almost every point* $\mathbf{t} \in \mathbb{T}^d$. *Furthermore, each of the conjugate series* (4.4) *is statistically convergent to the corresponding conjugate function almost everywhere on* $\mathbb{T}^d$.

(ii) *If* $f \in C(\mathbb{T}^d)$, *then the Fourier series of* f *is statistically convergent to* $f(\mathbf{t})$ *uniformly on* $\mathbb{T}^d$.

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