Algebra univers. **48** (2002) 309–331
0002-5240/02/030309 – 23

Algebra Universalis

Centrality and connectors in Maltsev categories

Dominique Bourn and Marino Gran

Abstract. We develop a new approach to the classical property of centrality of equivalence relations. The internal notion of connector allows to clarify classical results in Maltsev varieties and to extend them in the more general context of regular Maltsev categories, hence including the important new examples of Maltsev quasivarieties and of topological Maltsev algebras. We also prove that Maltsev categories can be characterized in terms of a property of connectors.

Introduction

The notion of commutator of equivalence relations in Maltsev varieties was introduced by Smith in his 1976 book [25]; his theory was later extended to congruence modular varieties by Hagemann and Hermann [17]. Various interesting ways of developing commutator theory in congruence modular varieties are presented in the works by Gumm [12] and Freese-McKenzie [11].

An alternative approach to commutator theory, of categorical nature, was developed by Pedicchio in exact Maltsev categories with coequalizers thanks to the notion of internal pregroupoid [23]. This work was later generalized by Janelidze and Pedicchio in [20] by introducing the notion of internal pseudogroupoid in general categories.

The present paper further develops the categorical approach to centrality, and emphasizes the role of the notion of connector between two equivalence relations introduced in [7]. If R and S are two equivalence relations on the same object X, we denote by $R \times_X S$ the pullback

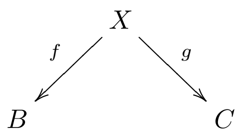
$$\begin{array}{ccc} R \times_X S & \xrightarrow{p_1} & S \\ {\scriptstyle p_0}\downarrow & & \downarrow{\scriptstyle d_0} \\ R & \xrightarrow[d_1]{} & X. \end{array}$$

Presented by K. Kearnes.
Received June 7, 2001; accepted in final form July 11, 2002.
2000 *Mathematics Subject Classification*: 18C05, 08C15, 08B05, 18E10.
Key words and phrases: Maltsev categories, centrality of equivalence relations, connector, internal groupoids.
The second author acknowledges financial support by the F.N.R.S.

A connector between R and S is an arrow $p : R \times_X S \to X$ such that

1.	$xSp(x, y, z)$	1.*	$zRp(x, y, z)$
2.	$p(x, x, y) = y$	2.*	$p(x, y, y) = x$
3.	$p(x, y, p(y, u, v)) = p(x, u, v)$	3.*	$p(p(x, y, u), u, v) = p(x, y, v)$

The notion of connector coincides with the one of pregroupoid when equivalence relations are effective. In the more general context of regular categories the main difference between the notion of pregroupoid and the notion of connector consists in the fact that the notion of pregroupoid is a global structure on a span



while the notion of connector focuses on the link between the kernel equivalence relations $R[f]$ and $R[g]$ associated with each span projection. This viewpoint allows to forget the projections f and g, and consequently resets the appropriate tool uniquely at the level of the equivalence relations.

In the context of Maltsev categories the internal notion of connector can be simplified and reduces to a partial ternary operation as above verifying only axioms 2. and 2.* in the previous list. An appropriate use of connectors allows us to get rid of the requirement of the existence of coequalizers and, mainly, of the effectiveness of equivalence relations (both conditions being usually required in order to define the commutator). In this way our theory naturally includes also the important examples of Maltsev quasivarieties and of topological groups. One of the main interests of the notion of connector is that it enables us to understand centrality even without defining the commutator of equivalence relations. Indeed, we can prove the important basic centrality properties which correspond to the classical properties of the commutator:

1. Symmetry: $[R, S] = [S, R]$
2. Inclusion of the commutator in the intersection: $[R, S] \leq R \cap S$
3. Monotonicity: if $S_1 \leq S_2$, then $[R, S_1] \leq [R, S_2]$
4. Stability with respect to products: $[R_1 \times R_2, S_1 \times S_2] \leq [R_1, S_1] \times [R_2, S_2]$
5. Stability with respect to restriction: if $i \colon Y \to X$ is a monomorphism, then $[R, S] = \Delta_X$ implies $[i^{-1}(R), i^{-1}(S)] = \Delta_Y$, where Δ_X is the smallest equivalence relation on X.
6. Stability with respect to regular images: if $f \colon X \to Y$ is a regular epimorphism, then $[R, S] = \Delta_X$ implies that $[f(R), f(S)] = \Delta_Y$
7. Stability with respect to joins: $[R, S_1 \vee S_2] = [R, S_1] \vee [R, S_2]$

(where R, R_1, R_2, S, S_1 and S_2 are equivalence relations on a given object X). A recent theorem in the categorical theory of central extensions shows that in any exact Maltsev category there is an intrinsic notion of central extension, which agrees with the one arising from commutator theory in universal algebra [14]. In the present article we show that in exact Maltsev categories the category of central extensions is equivalent to the category of connected internal groupoids, clarifying the well-known fact that any central extension can be considered as a crossed module. Finally, in the last section, we show how the notion of connector can be used to characterize Maltsev categories.

1. Connector

In this paper we shall always assume that $\mathcal{C}$ is a finitely complete category. If $f\colon X \to Y$ is an arrow in $\mathcal{C}$, $R[f]$ denotes its kernel pair. If R and S are two equivalence relations on X, we denote by $R \times_X S$ the pullback

$$\begin{array}{ccc} R \times_X S & \xrightarrow{p_1} & S \\ {\scriptstyle p_0}\downarrow & & \downarrow{\scriptstyle d_0} \\ R & \xrightarrow[d_1]{} & X. \end{array} \tag{1}$$

1.1. Definition. A *left action* of R on S is a map $p\colon R \times_X S \to X$ such that

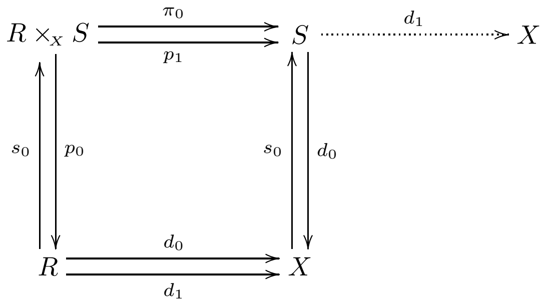
1. $xSp(x, y, z)$
2. $p(x, x, y) = y$
3. $p(x, y, p(y, u, v)) = p(x, u, v)$

1.2. Definition. A left action of R on S is a *connector* between R and S when the map $p : R \times_X S \to X$ also satisfies

1. $zRp(x, y, z)$
2. $p(x, y, y) = x$
3. $p(p(x, y, u), u, v) = p(x, y, v)$

A left action of R on S is equivalent to an action of the equivalence relation $(d_0, d_1)\colon R \rightarrowtail X \times X$ (thought of as an internal groupoid) on $d_0\colon S \to X$. The action $\pi_0\colon R \times_X S \to S$ can be defined by $\pi_0(x, y, z) = (x, p(x, y, z))$ (and, consequently, we get back the connector by setting $p = d_1 \circ \pi_0$).

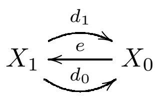
This action determines a connector when the map $\pi_1(x, y, z) = (p(x, y, z), z)$ also defines an action of $(d_0, d_1)\colon S \rightarrowtail X \times X$ on $d_1 : R \to X$. Remark that all the commutative squares in the diagram below are pullbacks, and that the definition of connector is symmetric in R and S.



Let us point out the fact that when an arrow $p\colon R \times_X S \to X$ satisfies the conditions 1. and 2. in the Definitions 1.1 and 1.2, then it satisfies 3. if and only if it satisfies the classical associativity $p(x, y, p(z, u, v)) = p(p(x, y, z), u, v)$.

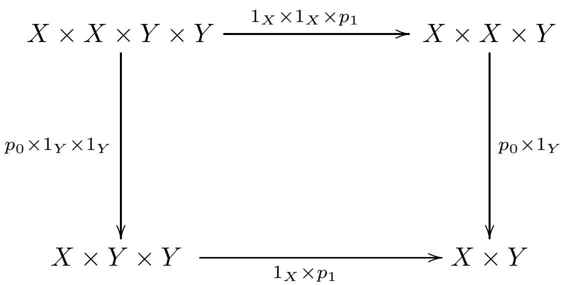
1.3. Example. If ∇_X is the largest equivalence relation on an object X, then an associative Maltsev operation $p\colon X \times X \times X \to X$ is precisely a connector between ∇_X and ∇_X.

1.4. Example. Let



be a reflexive graph. The connectors between $R[d_0]$ and $R[d_1]$ are in bijection with the groupoid structures on this reflexive graph [10].

1.5. Example. Given two objects X and Y, consider their product $X \times Y$. There is a canonical connector between $R[p_X]$ and $R[p_Y]$ (where the arrows p_X and p_Y are the product projections). Indeed, if we consider the following pullback



then the canonical connector $p\colon X \times X \times Y \times Y \to X \times Y$ is defined by

$$p(x, x', y, y') = (x', y).$$

2. Connectors and centralizing relations

Let us denote by $R \square S$ the double relation on R and S given by the following pullback:

$$\begin{array}{ccc} R \square S & \longrightarrow & R \times R \\ \big\downarrow & & \big\downarrow {\scriptstyle (d_0 \times d_0, d_1 \times d_1)} \\ S \times S & \xrightarrow[(d_0,d_1)\times(d_0,d_1)]{} & X \times X \times X \times X \end{array}$$

$R \square S$ is the subobject of X^4 consisting in the quadruples (x, y, z, u) with xSy, zSu, xRz and yRu. It determines a double equivalence relation on R and S:

$$\begin{array}{ccc} R \square S & \overset{p_0}{\underset{p_1}{\rightrightarrows}} & S \\ {\scriptstyle p_0} \downarrow\downarrow {\scriptstyle p_1} & & {\scriptstyle d_0} \downarrow\downarrow {\scriptstyle d_1} \\ R & \overset{d_0}{\underset{d_1}{\rightrightarrows}} & X \end{array}$$

A double equivalence relation C on R and S is called a centralizing relation [10]

$$\begin{array}{ccc} C & \overset{p_0}{\underset{p_1}{\rightrightarrows}} & S \\ {\scriptstyle p_0} \downarrow\downarrow {\scriptstyle p_1} & & {\scriptstyle d_0} \downarrow\downarrow {\scriptstyle d_1} \\ R & \overset{d_0}{\underset{d_1}{\rightrightarrows}} & X \end{array}$$

when the following square is a pullback:

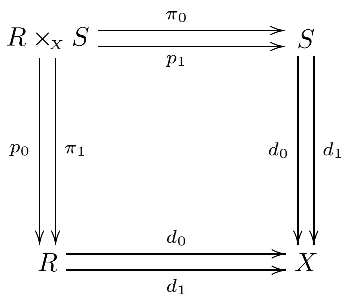
$$\begin{array}{ccc} C & \xrightarrow{p_1} & S \\ {\scriptstyle p_0} \big\downarrow & & \big\downarrow {\scriptstyle d_0} \\ R & \xrightarrow[d_1]{} & X. \end{array}$$

By the symmetry of the equivalence relations it follows that any of the commutative squares in the definition of a centralizing relation is a pullback. The following lemma gives the precise relationship between connectors and centralizing relations (for the special case of Maltsev categories see also [10]).

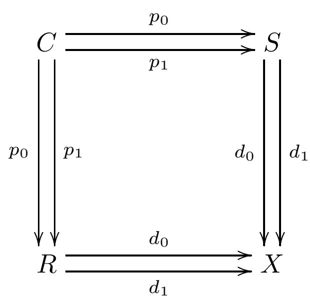
2.1. Lemma. *If $\mathcal{C}$ is a category with finite limits and R and S are two equivalence relations on the same object X, then the following conditions are equivalent:*

1. *R and S are connected;*
2. *there exists a centralizing relation on R and S.*

Proof. 1. $\Rightarrow$ 2. If $p\colon R \times_X S \to X$ is a connector between R and S, then by defining $\pi_0(x, y, z) = (x, p(x, y, z))$ and $\pi_1(x, y, z) = (p(x, y, z), z)$ one gets a centralizing relation on R and S:



2. $\Rightarrow$ 1. If



is a centralizing relation on R and S, then the arrow $d_1 \circ p_0\colon C = R \times_X S \to X$ defines a connector between R and S. $\square$

3. Connectors in Maltsev categories

In this section we show how the Maltsev assumption allows us at the same time to simplify the definitions and to strengthen the properties of connectors.

There are several equivalent properties which can be used to define Maltsev categories [9]. We shall adopt the classical

3.1. Definition. A category $\mathcal{C}$ is *Maltsev* if any reflexive relation in $\mathcal{C}$ is an equivalence relation.

There are many important algebraic examples of Maltsev categories: indeed, by a classical theorem [22], a variety of universal algebras has this property precisely when there is a ternary term $p(x, y, z)$ satisfying the identities $p(x, y, y) = x$ and $p(x, x, y) = y$. Accordingly, groups, abelian groups, modules over some fixed ring, crossed modules, quasi-groups, rings, Lie algebras and also Heyting algebras are all Maltsev categories. Any localization of a Maltsev variety is an exact Maltsev category [15]; the quasivariety of torsion-free abelian groups provides an example of a regular Maltsev category which is not exact.

Any protomodular category [2] is Maltsev, and then in particular any semi-abelian, and any abelian category [19]. The dual category of the category of sets is exact Maltsev, as is more generally the dual of any elementary topos. The category of localic groups and of topological groups are regular Maltsev categories [21].

In the following we shall always assume that the category $\mathcal{C}$ has finite limits. We denote by $Pt(\mathcal{C})$ the category whose objects are the split epimorphisms in $\mathcal{C}$ and morphisms the commutative squares between these data. The functor associating its codomain with any split epimorphism is denoted by $\pi\colon Pt(\mathcal{C}) \to \mathcal{C}$. This functor π is a fibration, called the *fibration of pointed objects.* This fibration has many important classification properties [3]: as many other kinds of categories, Maltsev categories can be also characterized by a property of this fibration.

Before recalling this property, let us first fix some notations. A category $\mathcal{C}$ is pointed when it has a zero object 0. In a pointed finitely complete category we denote by $i_X\colon X \to X \times Y$ and by $i_Y\colon Y \to X \times Y$ the arrows $(1_X, 0)$ and $(0, 1_Y)$ respectively.

3.2. Definition. A pointed finitely complete category $\mathcal{C}$ is *unital* if, for each pair of objects X and Y, the canonical pair of arrows $X \xrightarrow{i_X} X \times Y \xleftarrow{i_Y} X$ is jointly strongly epimorphic.

Now, any fiber of $\pi\colon Pt(\mathcal{C}) \to \mathcal{C}$ is clearly pointed and finitely complete.

3.3. Proposition. [3] *A category with pullbacks is Maltsev if and only if the fibration π is unital, i.e., if and only if each fiber of π is unital.*

This property has some strong consequences, as we shall see in the Lemma 3.5 here below.

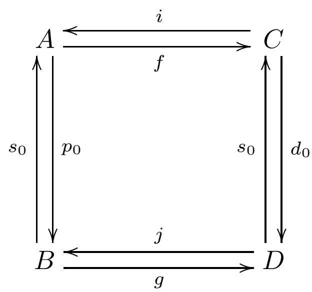
3.4. Definition. A *double zero sequence* in a pointed category is a diagram

$$X \underset{f}{\overset{s}{\rightleftarrows}} Z \underset{t}{\overset{g}{\rightleftarrows}} Y$$

with $f \circ s = 1_X$, $g \circ t = 1_Y$, $f \circ t = 0$ and $g \circ s = 0$.

In any unital category, the factorization $(f,g)\colon Z \to X \times Y$ induced by a double zero sequence as above is a strong epi [5]: indeed, if $j\colon R \to X \times Y$ is a mono whose pullback along (f,g) is an isomorphism, then its pullbacks along $(f,g) \circ s = i_X$ and $(f,g) \circ t = i_Y$ are isomorphisms and, consequently, the map j is an isomorphism.

3.5. Lemma. *If $\mathcal{C}$ is Maltsev, then any split epimorphism (f,g) in the category $Pt(\mathcal{C})$ from (p_0, s_0) to (d_0, s_0)*



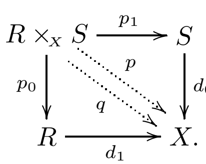
has the property that the induced arrow $\alpha\colon A \to B \times_D C$ to the corresponding pullback is a strong epimorphism.

Proof. This is a consequence of the fact that any fiber of π is unital. □

When $\mathcal{C}$ is a Maltsev category, then the Definition 1.2 of connector can be simplified. Indeed, if $p\colon R \times_X S \to X$ is a map in a Maltsev category $\mathcal{C}$ satisfying the Maltsev type axioms $p(x,x,y) = y$ and $p(x,y,y) = x$, then p is necessarily a connector [7]. Moreover, one has the following important

3.6. Lemma. [7] *Let $\mathcal{C}$ be a Maltsev category. If there is a connector between R and S, it is necessarily unique.*

Proof. Let p, q be two connectors between R and S.



Then, if $s_0\colon R \to R \times_X S$ is the splitting of p_0 induced by $s_0\colon X \to S$, one certainly has that $p \circ s_0 = q \circ s_0$. Similarly, if $\sigma_0\colon S \to R \times_X S$ is the splitting of p_1, one has that $p \circ \sigma_0 = q \circ \sigma_0$. But the pair s_0, σ_0 is jointly epimorphic by the Maltsev assumption, and then $p = q$. □

Accordingly, the existence of a connector between two equivalence relations R and S on X becomes a property; if there is a connector p between R and S, we shall then say that R and S are *connected*, or that R and S *centralize each other.*

3.7. Remark. In the category of groups the existence of a connector between the equivalence relations R_H and R_K on a group G is equivalent to the fact that the normal subgroups H and K (canonically associated with R_H and R_K) centralize each other. Indeed, suppose R_H and R_K connected by the homomorphism $p\colon R_H \times_G R_K \to G$, then the restriction homomorphism $\alpha\colon H \times K \to G$ defined by $\alpha(h,k) = p(h,1,k) = h \cdot k$ gives $H \cdot K = K \cdot H$, since we have

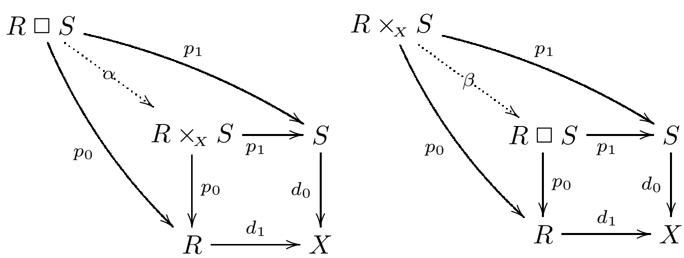
$$(h,1)\cdot(1,k) = (h,k) = (1,k)\cdot(h,1).$$

Conversely, suppose that $H \cdot K = K \cdot H$, then $p\colon R_H \times_G R_K \to G$ defined by $p(h,k,l) = h \cdot k^{-1} \cdot l$ is easily seen to be a group homomorphism.

3.8. Lemma. *If $\mathcal{C}$ is a Maltsev category and R and S are two equivalence relations on the same object X, then the following conditions are equivalent:*

1. *R and S are connected,*
2. *there is an arrow $\beta\colon R \times_X S \to R \,\square\, S$ of double zero sequences in $Pt_X(\mathcal{C})$ that splits the canonical arrow $\alpha\colon R \,\square\, S \to R \times_X S$.*

Proof. 1. $\Rightarrow$ 2. If there is a connector between R and S, let us denote by C the associated centralizing double relation on R and S as in Lemma 2.1. Then, by the universal property of the pullback $R \times_X S = C$ and by definition of $R \,\square\, S$ there are factorizations α and β as in the diagrams



where $\beta(x,y,z) = (x, p(x,y,z), y, z)$ Accordingly, the equalities

$$p_0 \circ \alpha \circ \beta = p_0 \circ \beta = p_0$$

and

$$p_1 \circ \alpha \circ \beta = p_1 \circ \beta = p_1$$

show that $\alpha \circ \beta = 1_{R \times_X S}$, as desired.

2. $\Rightarrow$ 1. If $\beta\colon R \times_X S \to R \,\square\, S$ has the property that $\alpha \circ \beta = 1_{R \times_X S}$, one can define the connector $p\colon R \times_X S \to X$ between R and S by taking the composite

$$p = d_1 \circ p_0 \circ \beta\colon R \times_X S \to R \,\square\, S \to S \to X.$$

$\square$

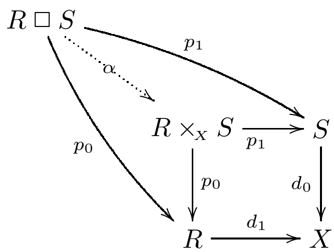
We can now prove some basic centrality properties. *From now on we shall always assume that the category $\mathcal{C}$ is finitely complete and Maltsev.*

We begin with the property corresponding to the fact that the commutator is contained in the intersection:

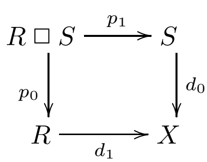
3.9. Lemma. [5]

1. *$R \cap S = \Delta_X$ if and only if $R \square S$ is a centralizing relation on R and S.*
2. *If $R \cap S = \Delta_X$, then the equivalence relations R and S are connected.*

Proof. 1. Let us denote by $\alpha\colon R \square S \to R \times_X S$ the induced factorization from the largest double equivalence relation $R \square S$ on R and S towards the corresponding pullback $R \times_X S$.



The category $\mathcal{C}$ being Maltsev, the fiber $Pt_X(\mathcal{C})$ is unital and the factorization α is then a strong epimorphism. It is a monomorphism precisely when $R \cap S = \Delta_X$.
2. If $R \cap S = \Delta_X$, then the square



is a pullback, and the arrow $d_1 \circ p_0\colon R \square S \to S \to X$ is the expected connector between R and S. □

We then have the monotonicity of the centrality property:

3.10. Proposition. *Let R, S_1 and S_2 be equivalence relations on X, with $S_1 \leq S_2$. If R and S_2 are connected, then R and S_1 are connected.*

Proof. Let $j\colon S_1 \to S_2$ denote the inclusion of S_1 in S_2. This arrow induces a factorization $k\colon R \times_X S_1 \to R \times_X S_2$ such that $p_1^{S_2} \circ k = j \circ p_1^{S_1}$ and $p_0^{S_2} \circ k = p_0^{S_1}$. If $p\colon R \times_X S_2 \to X$ is the connector between R and S_2, then it is easy to check that $p \circ k\colon R \times_X S_1 \to X$ is the connector between R and S_1. □

3.11. Corollary. *Let (R_1, S_1) and (R_2, S_2) be pairs of connected equivalence relations on X. Then $R_1 \cap R_2$ and $S_1 \cap S_2$ are connected.*

Stability with respect to products:

3.12. Proposition. *Let R_1, S_1 and R_2, S_2 be connected equivalence relations on X and Y respectively. Then the equivalence relations $R_1 \times R_2$ and $S_1 \times S_2$ on $X \times Y$ are connected.*

Proof. Let $p_1 \colon R_1 \times_X S_1 \to X$ be the connector between R_1 and S_1, let $p_2 \colon R_2 \times_Y S_2 \to Y$ be the connector between R_2 and S_2. Then the arrow $p_1 \times p_2 \colon (R_1 \times R_2) \times_{X \times Y} (S_1 \times S_2) \to X \times Y$ is the connector between $R_1 \times R_2$ and $S_1 \times S_2$. □

Stability by restriction:

3.13. Proposition. *If R and S are two connected equivalence relations on Y and $i \colon X \to Y$ is a monomorphism, then $i^{-1}(R)$ and $i^{-1}(S)$ are connected.*

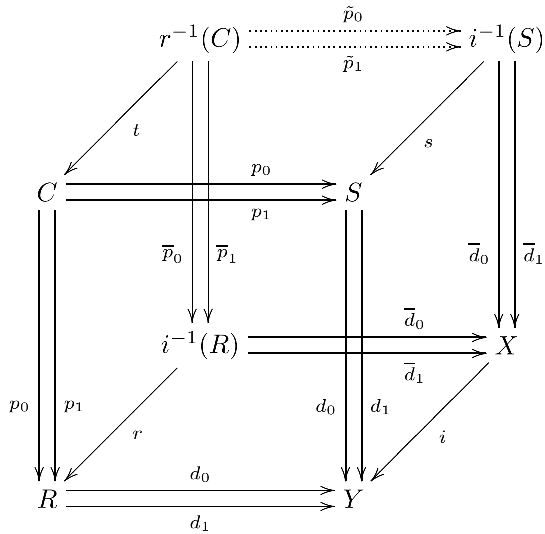
Proof. Let C be the centralizing relation on R and S:

$$\begin{array}{ccc} C & \overset{p_0}{\underset{p_1}{\rightrightarrows}} & S \\ {\scriptstyle p_0}\downarrow\downarrow{\scriptstyle p_1} & & {\scriptstyle d_0}\downarrow\downarrow{\scriptstyle d_1} \\ R & \overset{d_0}{\underset{d_1}{\rightrightarrows}} & Y \end{array}$$

We write $r \colon i^{-1}(R) \to R$ and $s \colon i^{-1}(S) \to S$ for the induced arrows in the pullbacks

$$\begin{array}{ccc} i^{-1}(R) & \xrightarrow{r} & R \\ {\scriptstyle (\overline{d}_0, \overline{d}_1)}\downarrow & & \downarrow{\scriptstyle (d_0, d_1)} \\ X \times X & \xrightarrow[i \times i]{} & Y \times Y \end{array} \qquad \begin{array}{ccc} i^{-1}(S) & \xrightarrow{s} & S \\ {\scriptstyle (\overline{d}_0, \overline{d}_1)}\downarrow & & \downarrow{\scriptstyle (d_0, d_1)} \\ X \times X & \xrightarrow[i \times i]{} & Y \times Y \end{array}$$

We want to prove that the equivalence relation $r^{-1}(C)$ determines a centralizing relation on $i^{-1}(R)$ and $i^{-1}(S)$.



There are clearly induced arrows $\tilde{p}_0, \tilde{p}_1$ and ε defining a reflexive relation on $i^{-1}(S)$:

$$r^{-1}(C) \underset{\tilde{p}_1}{\overset{\tilde{p}_0}{\underset{\longleftarrow}{\overset{\longrightarrow}{\underset{\varepsilon}{\longrightarrow}}}}} i^{-1}(S)$$

This reflexive relation actually is an equivalence relation on $i^{-1}(S)$. In order to prove that $r^{-1}(C)$ is a centralizing relation on $i^{-1}(R)$ and $i^{-1}(S)$ the only condition we still have to check is that the square

$$\begin{array}{ccc} r^{-1}(C) & \xrightarrow{\tilde{p}_1} & i^{-1}(S) \\ \Big\downarrow{\scriptstyle \overline{p}_0} & & \Big\downarrow{\scriptstyle \overline{d}_0} \\ i^{-1}(R) & \xrightarrow{\overline{d}_1} & X \end{array}$$

is a pullback. This follows by the Maltsev assumption, by Lemma 3.5 and the fact that r and s are monomorphisms. □

3.14. Corollary. *Let R and S be two connected equivalence relations on Y and let $i\colon X \to Y$ be a monomorphism. Then if $(i, i_{R'})\colon R' \to R$ and $(i, i_{S'})\colon S' \to S$ are arrows in the category of equivalence relations in $\mathcal{C}$, then R' and S' are connected.*

Proof. It follows by Propositions 3.10 and 3.13. □

4. Regular Maltsev categories

A finitely complete category is regular if kernel pairs have coequalizers and regular epimorphisms are stable under pullbacks. *In this section $\mathcal{C}$ will always denote a regular Maltsev category.* In order to establish two important centrality properties we first need to recall the following strong stability property of regular epimorphisms in regular Maltsev categories.

4.1. Lemma. [5] *Given two commutative squares of vertical split epimorphisms*

$$\begin{array}{ccc} A & \xrightarrow{\;g\;} & B \\ {\scriptstyle s_A}\uparrow\downarrow{\scriptstyle \alpha} & & {\scriptstyle s_B}\uparrow\downarrow{\scriptstyle \beta} \\ X & \xrightarrow[\;f\;]{} & Y \end{array} \qquad\qquad \begin{array}{ccc} C & \xrightarrow{\;h\;} & D \\ {\scriptstyle s_C}\uparrow\downarrow{\scriptstyle \gamma} & & {\scriptstyle s_D}\uparrow\downarrow{\scriptstyle \delta} \\ X & \xrightarrow[\;f\;]{} & Y \end{array}$$

when the arrows f, g and h are regular epimorphisms, then the induced factorization $k\colon A \times_X C \to B \times_Y D$ is a regular epimorphism.

Proof. The kernel pair $R[k]$ of the induced factorization k is given by $R[g] \times_{R[f]} R[h]$. We write $\rho\colon A \times_X C \to Q$ for the quotient of the equivalence relation $R[k]$, and $i\colon Q \to B \times_Y D$ for the monomorphic factorization of k. When the arrows f, g and h are regular epimorphisms, the double zero sequence in $Pt_X(\mathcal{C})$

$$A \rightleftarrows A \times_X C \rightleftarrows C$$

can be extended to a double sequence in $Pt_Y(\mathcal{C})$

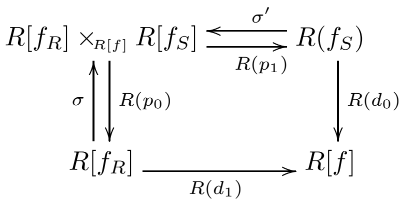
$$B \rightleftarrows Q \rightleftarrows D.$$

The fiber $Pt_Y(\mathcal{C})$ is unital by the Maltsev assumption, so that the induced factorization $i\colon Q \to B \times_Y D$ of this double zero sequence is a regular epimorphism. Thus i is an isomorphism and k a regular epimorphism. □

We are now in the position to prove the stability with respect to regular images:

4.2. Proposition. *Let R and S be equivalence relations on X, and let $f\colon X \to Y$ be a regular epimorphism. If R and S are connected, then the images $f(R)$ and $f(S)$ are connected.*

Proof. We write $f_R\colon R \to f(R)$ and $f_S\colon S \to f(S)$ for the induced arrows. The induced factorization $\phi\colon R \times_X S \to f(R) \times_Y f(S)$ is a regular epimorphism by the previous lemma. Let p denote the connector between R and S. We first prove that $f \circ p\colon R \times_X S \to Y$ factorizes through the arrow ϕ. For this, we must show that $f \circ p$ coequalizes the kernel pair $(R[\phi] = R[f_R] \times_{R[f]} R[f_S], l_0, l_1)$ of ϕ. Now the following commutative square is a pullback of split epimorphisms, and the Maltsev assumption implies that the induced sections σ and σ' of $R(p_0)$ and $R(p_1)$ are jointly strongly epimorphic:



It is then sufficient to check that $(f \circ p \circ l_0) \circ \sigma = (f \circ p \circ l_1) \circ \sigma$ and $(f \circ p \circ l_0) \circ \sigma' = (f \circ p \circ l_1) \circ \sigma'$. One clearly has that $f \circ p \circ l_0 \circ \sigma = f \circ d_0 \circ p_0$ because $(f \circ p)(x, y, y) = f(x)$. Accordingly:

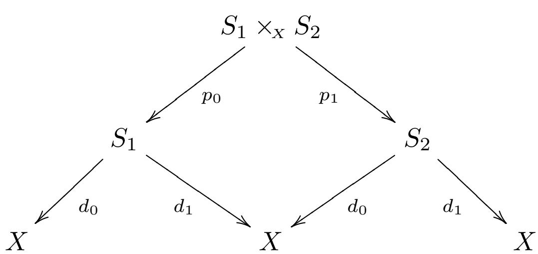
$$f \circ p \circ l_0 \circ \sigma = f \circ d_0 \circ p_0 = d_0 \circ f_R \circ p_0 = d_0 \circ f_R \circ p_1 = f \circ d_0 \circ p_1 = f \circ p \circ l_1 \circ \sigma.$$

Similarly, one has that $(f \circ p \circ l_0) \circ \sigma' = (f \circ p \circ l_1) \circ \sigma'$. Let $p_f \colon f(R) \times_Y f(S) \to Y$ be the unique arrow induced by the universal property of the coequalizer ϕ, which is such that $p_f \circ \phi = f \circ p$. It is not difficult to check that p_f is a connector between $f(R)$ and $f(S)$. □

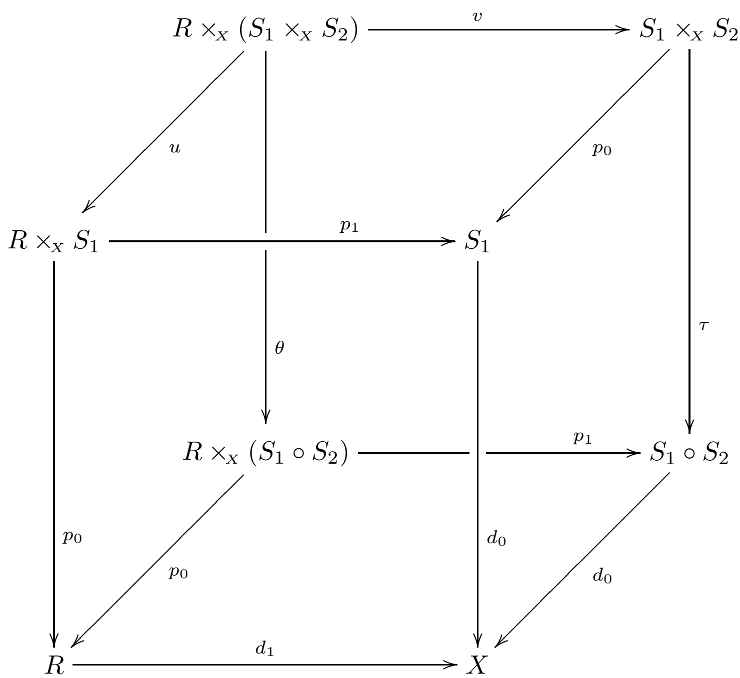
Let us then recall that in any regular Maltsev category one has that the join $S_1 \vee S_2$ of two equivalence relations S_1 and S_2 on the same object X is given by the composite $S_1 \circ S_2$ [9]. This fact will be used to prove the stability with respect to binary joins:

4.3. Proposition. *If the equivalence relations R and S_1 and the equivalence relations R and S_2 are connected, then R and $S_1 \vee S_2$ are connected.*

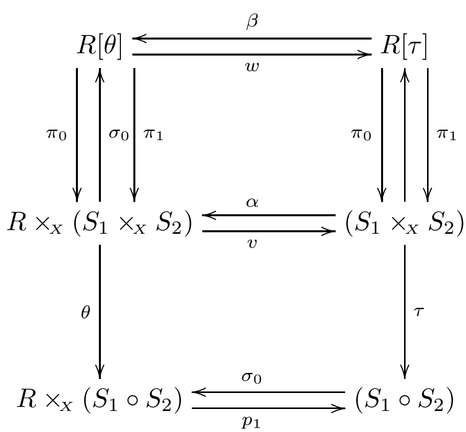
Proof. Let $\varepsilon_1 \colon R \times_X S_1 \to X$ and $\varepsilon_2 \colon R \times_X S_2 \to X$ be the connectors on R, S_1 and R, S_2 respectively. The composite equivalence relation $S_1 \circ S_2$ is the regular image factorization of the arrow $(d_0 \circ p_0, d_1 \circ p_1) \colon S_1 \times_X S_2 \to X \times X$ in the diagram



We write $S_1 \times_X S_2 \xrightarrow{\tau} S_1 \circ S_2 \xrightarrow{(d_0,d_1)} X \times X$ for the regular epi-mono factorization defining $S_1 \circ S_2$. Let us then consider the following commutative diagram:



The universal property of the bottom pullback induces a unique arrow $\theta\colon R \times_X (S_1 \times_X S_2) \to R \times_X (S_1 \circ S_2)$, which is a regular epimorphism since τ is a regular epi and the back vertical square is a pullback. Consider the kernel pairs $R[\tau]$ and $R[\theta]$ of τ and θ, respectively:



where $\alpha(x,y,z) = (x,x,y,z)$. We then define the arrow $\pi\colon R \times_X (S_1 \times_X S_2) \to X$ as follows: $\pi(x,y,z,t) = \varepsilon_2(\varepsilon_1(x,y,z),z,t)$. Let us first show that this arrow π coequalizes $\pi_0, \pi_1\colon R[\theta] \to R \times_X (S_1 \times_X S_2)$. Since $\mathcal{C}$ is Maltsev, it is enough to show that $\pi \circ \pi_0 \circ \sigma_0 = \pi \circ \pi_1 \circ \sigma_0$ and $\pi \circ \pi_0 \circ \beta = \pi \circ \pi_1 \circ \beta$. The first equality is trivial, while the second one follows from

$$\pi \circ \pi_0 \circ \beta = \pi \circ \alpha \circ \pi_0 = d_1 \circ p_1 \circ \pi_0 = d_1 \circ p_1 \circ \pi_1 = \pi \circ \alpha \circ \pi_1 = \pi \circ \pi_1 \circ \beta,$$

where $\pi \circ \alpha = d_1 \circ p_1$ is easy to check, and $d_1 \circ p_1 \circ \pi_0 = d_1 \circ p_1 \circ \pi_1$ follows from $R[\tau] = R[d_0 \circ p_0, d_1 \circ p_1]$. By the universal property of the coequalizer θ there exists a unique arrow $p\colon R \times_X (S_1 \circ S_2) \to X$ with $p \circ \theta = \pi$. This arrow is a connector between R and $S_1 \circ S_2$:

$$\begin{array}{ccc} R \times_X (S_1 \circ S_2) & \underset{p_1}{\overset{\sigma_0}{\leftrightarrows}} & S_1 \circ S_2 \\ {\scriptstyle s_0}\uparrow\downarrow{\scriptstyle p_0} & & {\scriptstyle s_0}\uparrow\downarrow{\scriptstyle d_0} \\ R & \underset{d_1}{\leftrightarrows} & X \end{array}$$

on the one hand one has

$$p \circ \sigma_0 \circ \tau = p \circ \theta \circ \alpha = \pi \circ \alpha = d_1 \circ p_1 = d_1 \circ \tau$$

and then $p \circ \sigma_0 = d_1$. On the other hand, if $\gamma\colon R \times_X S_1 \to R \times_X (S_1 \times_X S_2)$ is the arrow defined by $\gamma(x,y,z) = (x,y,z,z)$, one certainly has $\pi \circ \gamma = \varepsilon_1$. It follows that

$$p \circ s_0 = p \circ \theta \circ \gamma \circ s_0 = \pi \circ \gamma \circ s_0 = \varepsilon_1 \circ s_0 = d_0. \qquad \square$$

The previous result together with Proposition 3.10 immediately gives the following

4.4. Corollary. *The equivalence relations R, S_1 and R, S_2 are connected if and only if R and $S_1 \circ S_2$ are connected.*

5. Exact Maltsev categories

A regular epimorphism $f\colon A \to B$ in a Maltsev category is a *central extension* if there is a connector between its kernel equivalence R and the largest equivalence relation ∇_A on A [14]. We shall denote by $Centr(B)$ the category of central extensions of B. It turns out that in any exact Maltsev category with a zero object these central extensions correspond to the normalizations (in the sense of [6]) of internal connected groupoids.

Let $ConnGrpd(B)$ denote the category of internal connected groupoids in $\mathcal{C}$ with B as object of objects. These are those internal groupoids

$$X_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} B$$

with the property that the induced arrow $(d_0, d_1)\colon X_1 \to B \times B$ is a regular epimorphism. Let us recall from [6] that there is a normalization functor $N\colon ConnGrpd(B) \to Centr(B)$ associating the arrow

$$K[d_0] \xrightarrow{Ker(d_0)} X_1 \xrightarrow{d_1} B,$$

with any internal groupoid as above (where $Ker(d_0)\colon K[d_0] \to X_1$ is the kernel of d_0). When the groupoid is connected, this normalization $d_1 \circ Ker(d_0)$ of the given internal connected groupoid is a regular epi; indeed, if $s_1 = (0, 1_B)$ is the canonical arrow (where $0\colon B \to B$ is the zero arrow), then $d_1 \circ Ker(d_0)$ can be also obtained by the pullback

$$\begin{array}{ccc} K[d_0] & \longrightarrow & X_1 \\ {\scriptstyle d_1 \circ Ker(d_0)}\downarrow & & \downarrow{\scriptstyle (d_0, d_1)} \\ B & \underset{s_1}{\longrightarrow} & B \times B. \end{array}$$

It turns out that in any exact Maltsev categories, this functor actually gives rise to an equivalence of categories:

5.1. Proposition. *Let $\mathcal{C}$ be an exact Maltsev category with a zero object. The categories $ConnGrpd(B)$ and $Centr(B)$ are equivalent.*

Proof. To see that the normalization functor described above takes its values in the category of central extensions consider the following diagram:

$$\begin{array}{ccccccc} R[\sigma \circ k] & \underset{\pi_1}{\overset{\pi_0}{\rightrightarrows}} & K[d_0] \times K[d_0] & \xrightarrow{k} & R[d_0] & \xrightarrow{\sigma} & X_1 \\ {\scriptstyle d_0}\downdownarrows{\scriptstyle d_1} & & {\scriptstyle pr_0}\downdownarrows{\scriptstyle pr_1} & & {\scriptstyle p_0}\downdownarrows{\scriptstyle p_1} & & {\scriptstyle d_0}\downdownarrows{\scriptstyle d_1} \\ R[d_1 \circ Ker(d_0)] & \underset{p_1}{\overset{p_0}{\rightrightarrows}} & K[d_0] & \underset{Ker(d_0)}{\longrightarrow} & X_1 & \underset{d_1}{\longrightarrow} & B. \end{array}$$

The map σ corresponds to the operation which associates the arrow $\beta \circ \alpha^{-1}$ with any pair of arrows (α, β) with same domain in the groupoid X_1. The equivalence relations $R[\sigma \circ k]$ and $R[d_1 \circ Ker(d_0)]$ are determined by the obvious kernel pairs, while $d_0, d_1\colon R[\sigma \circ k] \to R[d_1 \circ Ker(d_0)]$ are induced by the universal property of kernel pairs. The central and the right hand commutative squares are discrete fibrations of internal groupoids: from this it easily follows that the $R[\sigma \circ k]$ is a centralizing relation on $R[d_1 \circ Ker(d_0)]$ and $K[d_0] \times K[d_0]$. This shows that the extension $d_1 \circ Ker(d_0)$ is central.

Conversely, there is a functor $V\colon Centr(B) \to ConnGrpd(B)$ associating with any central extension $f\colon A \to B$ an internal connected groupoid: let C be the centralizing relation associated with the kernel equivalence R of f and ∇_A. By taking the coequalizer $q\colon A \times A \to Q$ of the projections π_0 and π_1 of C on $A \times A$ one obtains a reflexive graph with B as objects of objects:

$$\begin{array}{ccccc} C & \underset{\pi_1}{\overset{\pi_0}{\rightrightarrows}} & A\times A & \xrightarrow{q} & Q \\ {\scriptstyle d_0}\downarrow\downarrow{\scriptstyle d_1} & & {\scriptstyle a_0}\downarrow\downarrow{\scriptstyle a_1} & & {\scriptstyle d_0}\vdots\vdots{\scriptstyle d_1} \\ R & \underset{p_1}{\overset{p_0}{\rightrightarrows}} & A & \xrightarrow[f]{} & B \end{array}$$

the category $\mathcal{C}$ being regular Maltsev, this reflexive graph actually is an internal groupoid in $\mathcal{C}$ [12]. It is connected, since $(d_0, d_1) \circ q = f \times f$, and $f \times f$ is a regular epi. The functor V is naturally defined on arrows, the category of internal groupoids in $\mathcal{C}$ being full in the category of internal reflexive graphs in $\mathcal{C}$ [10].

One then verifies that the functors N and V determine an equivalence of categories between $Centr(B)$ and $ConnGrpd(B)$. □

By analogy with the definition of normal monomorphism [2], one could say that an extension $f\colon A \to B$ is *normal* to a groupoid $X\colon\ X_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} B$ if there exists a discrete fibration (q, f) from ∇_A to X. A different way to state the previous result consists in saying that an extension is normal to an internal groupoid if and only if it is a central extension.

Remark also that if $ConnGrpd(\mathcal{C})$ denotes the category of internal connected groupoids in $\mathcal{C}$ and $Centr(\mathcal{C})$ the category of central extensions, the argument above also shows that

5.2. Corollary. *$ConnGrpd(\mathcal{C})$ and $Centr(\mathcal{C})$ are equivalent.*

An equivalence relation R on an object A is *central* when there is a connector between R and ∇_A. In any exact Maltsev category there is a nice presentation of central relations:

5.3. Proposition. *Let $\mathcal{C}$ be an exact Maltsev category. Any central equivalence R on A is canonically isomorphic to a product $Q \times A$, with Q an abelian object.*

Proof. Let C be the centralizing relation on R and ∇_A:

$$\begin{array}{ccc} C & \overset{p_0}{\underset{p_1}{\rightrightarrows}} & A\times A \\ {\scriptstyle\pi_0}\downarrow\downarrow{\scriptstyle\pi_1} & & {\scriptstyle a_0}\downarrow\downarrow{\scriptstyle a_1} \\ R & \overset{r_0}{\underset{r_1}{\rightrightarrows}} & A \end{array}$$

Now, by taking the coequalizer q of π_0 and π_1 we obtain the commutative diagram

$$\begin{array}{ccccc} C & \overset{\pi_0}{\underset{\pi_1}{\rightrightarrows}} & R & \xrightarrow{q} & Q \\ {\scriptstyle p_0}\downarrow & & {\scriptstyle r_0}\downarrow & (1) & \downarrow{\scriptstyle d_0} \\ A\times A & \overset{a_0}{\underset{a_1}{\rightrightarrows}} & A & \longrightarrow & 1 \end{array}$$

Since the category $\mathcal{C}$ is exact, the equivalence relation

$$C \overset{\pi_0}{\underset{\pi_1}{\rightrightarrows}} R$$

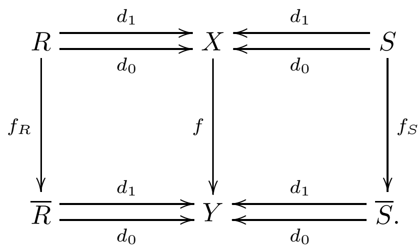
is the kernel pair of its coequalizer q, and this latter is a pullback stable regular epi. By assumption the arrow $(p_0, r_0)\colon (C, R) \to (A \times A, A)$ determines a discrete fibration of internal equivalence relations, so that by Corollary 2 in [4], the square (1) is a pullback; then R is isomorphic to $Q \times A$. By Proposition 4.2 the fact that the equivalences $R[r_0]$ and $R[r_1]$ are connected implies that $\nabla_Q = q(R[r_0])$ and $\nabla_Q = q(R[r_1])$ are connected. Accordingly, the object Q is abelian. □

6. Characterization of Maltsev categories

In this last section we prove that connectors have a property that characterizes Maltsev categories. Let us first fix some notations. If $\mathcal{C}$ is a finitely complete category, we write 2-$Eq(\mathcal{C})$ for the category whose objects are pairs of equivalence relations (R, S, X) on the same object X

$$R \overset{d_1}{\underset{d_0}{\rightrightarrows}} X \overset{d_1}{\underset{d_0}{\leftleftarrows}} S\ ,$$

and arrows in 2-$Eq(\mathcal{C})$ are triples of arrows (f_R, f_S, f) making the following diagram commutative:

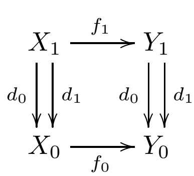


Let $Conn(\mathcal{C})$ be the category whose objects are pairs of connected equivalence relations (R, S, X, p) with a given connector $p\colon R \times_X S \to X$; arrows in $Conn(\mathcal{C})$ are arrows in 2-$Eq(\mathcal{C})$ with the property that they must respect the connectors. There is a forgetful functor $U\colon Conn(\mathcal{C}) \to$ 2-$Eq(\mathcal{C})$ which simply associates (R, S, X) with (R, S, X, p). We shall write $V\colon Grpd(\mathcal{C}) \to RG(\mathcal{C})$ for the forgetful functor from the category of internal grupoids in $\mathcal{C}$ to the category of internal reflexive graphs in $\mathcal{C}$. Clearly both functors $U\colon Conn(\mathcal{C}) \to$ 2-$Eq(\mathcal{C})$ and $V\colon Grpd(\mathcal{C}) \to RG(\mathcal{C})$ are faithful. We say that a faithful functor $F\colon \mathcal{C} \to \mathcal{D}$ is closed under subobjects if, for any monomorphism $j\colon X \to FC$ in $\mathcal{D}$, there exists a unique (up to isomorphism) monomorphism $i\colon \overline{C} \to C$ in $\mathcal{C}$ with $F(i) = j$. We can now state our characterization of Maltsev categories (the equivalence between 1. and 3. is already known and was proved in [3]):

6.1. Proposition. *Let $\mathcal{C}$ be a finitely complete category. Then the following conditions are equivalent:*

1. *$\mathcal{C}$ is a Maltsev category*
2. *$U\colon Conn(\mathcal{C}) \to$ 2-$Eq(\mathcal{C})$ is closed under subobjects*
3. *$V\colon Grpd(\mathcal{C}) \to RG(\mathcal{C})$ is closed under subobjects*

Proof. 1. $\Rightarrow$ 2. If (R, S, X, p) and $(\overline{R}, \overline{S}, Y, \overline{p})$ are objects in $Conn(\mathcal{C})$, then it is easy to check that in any Maltsev category an arrow $(f_R, f_S, f)\colon (R, S, X, p) \to (\overline{R}, \overline{S}, Y, \overline{p})$ in 2-$Eq(\mathcal{C})$ necessarily preserves the connector. The result then follows by Corollary 3.14.
2. $\Rightarrow$ 3. Let



be a monomorphism in $RG(\mathcal{C})$, with

$$Y_1 \underset{d_0}{\overset{d_1}{\rightrightarrows}} Y_0$$

an internal groupoid in $\mathcal{C}$. It determines a monomorphism $(\overline{f}, \tilde{f}, f_1)$ in 2-$Eq(\mathcal{C})$

$$\begin{array}{ccccc} R[d_0] & \rightrightarrows & X_1 & \leftleftarrows & R[d_1] \\ \downarrow{\scriptstyle \overline{f}} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle \tilde{f}} \\ R[d_0] & \rightrightarrows & Y_1 & \leftleftarrows & R[d_1] \end{array}$$

where $(R[d_0], R[d_1], Y_1)$ belongs to $Conn(\mathcal{C})$. It follows that $(R[d_0], R[d_1], X_1)$ also belongs to $Conn(\mathcal{C})$ and

$$X_1 \underset{d_0}{\overset{d_1}{\rightrightarrows}} X_0$$

is an internal groupoid, as desired.

3. $\Rightarrow$ 1. Any reflexive relation

$$R \underset{d_0}{\overset{d_1}{\rightrightarrows}} X$$

is a subobject of the largest equivalence relation

$$X \times X \underset{p_0}{\overset{p_1}{\rightrightarrows}} X$$

on X. Accordingly, R is a groupoid, hence an equivalence relation. $\square$

In order to show that connectors can be used to characterize regular Maltsev categories, we first need the following

6.2. Lemma. *Let $\mathcal{D}$ be a regular category. If $\mathcal{C}$ is a full subcategory of $\mathcal{D}$ closed in $\mathcal{D}$ under finite limits and subobjects, then $\mathcal{C}$ is regular.*

Proof. The regular epimorphism - monomorphism factorization in $\mathcal{D}$ of any arrow in $\mathcal{C}$ is also the factorization in $\mathcal{C}$ (by the assumption that $\mathcal{C}$ is closed in $\mathcal{D}$ under subobjects). Since finite limits in $\mathcal{C}$ are calculated as in $\mathcal{D}$, regular epimorphisms are stable under pullbacks and the proof is complete. $\square$

6.3. Proposition. *Let $\mathcal{C}$ be a finitely complete category. Then the following conditions are equivalent:*

1. *$\mathcal{C}$ is regular Maltsev*
2. *$Conn(\mathcal{C})$ is regular Maltsev*
3. *$Grpd(\mathcal{C})$ is regular Maltsev*

Proof. The equivalence between 1. and 3. was proved in [12].

1. $\Rightarrow$ 2. If $\mathcal{C}$ is regular Maltsev, then $Grpd(\mathcal{C})$ is regular Maltsev; this implies that $Eq(\mathcal{C})$ is regular Maltsev, being a full subcategory of $Grpd(\mathcal{C})$ closed under finite limits and subobjects. It follows that 2-$Eq(\mathcal{C})$ is regular Maltsev and, by Lemma 6.2, so is the category $Conn(\mathcal{C})$.

2. $\Rightarrow$ 1. There is a discrete functor $D\colon \mathcal{C} \to Conn(\mathcal{C})$ associating the pair of connected equivalences

$$X \underset{1}{\overset{1}{\rightrightarrows}} X \underset{1}{\overset{1}{\leftleftarrows}} X$$

with any object X in $\mathcal{C}$. This functor $D\colon \mathcal{C} \to Conn(\mathcal{C})$ is fully faithful, so $\mathcal{C}$ is Maltsev. The category $\mathcal{C}$ is closed in $Conn(\mathcal{C})$ under finite limits and subobjects, then by Lemma 6.2 the category $\mathcal{C}$ is regular since $Conn(\mathcal{C})$ is so. □

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Dominique Bourn
Université du Littoral Côte d'Opale, Laboratoire de Mathématiques Pures et Appliquées, Bt. H. Poincaré, 50 Rue F. Buisson BP 699, 62228 Calais - France
e-mail: bourn@lmpa.univ-littoral.fr

Marino Gran
Université du Littoral Côte d'Opale, Laboratoire de Mathématiques Pures et Appliquées, Bt. H. Poincaré, 50 Rue F. Buisson BP 699, 62228 Calais - France
e-mail: gran@lmpa.univ-littoral.fr