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The dualisability of a quasi-variety is independent of the generating algebra

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Abstract. We prove the claim made in the title of the paper.

It is to be expected that different generating algebras $\underline{\mathbf{D}}$ and $\underline{\mathbf{M}}$ for a quasi-variety will lead to different natural dualities. Indeed, this is the case—see the examples in [2]. But it would be most unfortunate if the very existence of a natural duality for a finitely generated quasi-variety depended upon the choice of generator. Hence, the question below was posed in [2] and again in [4]. A positive solution to this question means that we may unambiguously refer to a finitely generated quasi-variety $\mathcal{A}$ as being dualisable provided $\mathcal{A} = \mathbb{ISP}(\underline{\mathbf{D}})$ for some finite dualisable algebra $\underline{\mathbf{D}}$.

- *If* $\underline{\mathbf{D}}$ *and* $\underline{\mathbf{M}}$ *are finite algebras such that* $\mathbb{ISP}(\underline{\mathbf{D}}) = \mathbb{ISP}(\underline{\mathbf{M}})$, *and if* $\underline{\mathbf{D}}$ *is dualisable, does it follow that* $\underline{\mathbf{M}}$ *is dualisable?*

THEOREM. *Yes.*[1]

Proof. We refer the reader to Clark and Davey [1] for the necessary definitions and the basic theory of natural dualities. We recall only that $\underline{\mathbf{D}}$ is a finite algebra and $\underset{\sim}{\mathbf{D}}$ is an alter ego for $\underline{\mathbf{D}}$, that is, $\underset{\sim}{\mathbf{D}}$ is a topological structure whose universe is the universe of $\underline{\mathbf{D}}$, whose relations (if any) are non-empty subuniverses of the appropriate finite powers of $\underline{\mathbf{D}}$, whose operations and partial operations (if any) are such that their graphs are non-empty subuniverses of finite powers of $\underline{\mathbf{D}}$, and whose topology is discrete.

Let $\mathcal{A} = \mathbb{ISP}(\underline{\mathbf{D}}) = \mathbb{ISP}(\underline{\mathbf{M}})$. Let $N = \{\nu_0, \ldots, \nu_{k-1}\}$ be a set of homomorphisms from $\underline{\mathbf{D}}$ to $\underline{\mathbf{M}}$ which separate the points of D. For example, an inefficient choice would be

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[1]This theorem was obtained independently, via a different proof, by M. J. Saramago [5].

$N = \mathcal{A}(\underline{\mathbf{D}}, \underline{\mathbf{M}})$. Let $\nu := \nu_0 \sqcap \cdots \sqcap \nu_{k-1} : \underline{\mathbf{D}} \to \underline{\mathbf{M}}^k$ be the natural product map and let $D_\nu := \nu(D)$. Thus D_ν is a k-ary algebraic relation on $\underline{\mathbf{M}}$ and $\nu : \underline{\mathbf{D}} \cong \mathbf{D}_\nu \leqslant \underline{\mathbf{M}}^k$. Note that $\nu_i = \pi_i \circ \nu$ for $i < k$, where $\pi_i : \underline{\mathbf{M}}^k \to \underline{\mathbf{M}}$ is the ith projection.

For $n \geqslant 1$ we construe $M^{k\times n}$ as the set of all matrices with k rows and n columns and with elements from M. If $\mathbf{A} \leqslant \underline{\mathbf{D}}^n$, then we define $\nu^n_{\mathbf{A}} : \mathbf{A} \to \underline{\mathbf{M}}^{k\times n}$ by $\nu^n_{\mathbf{A}}(a_0, \ldots, a_{n-1}) = [\nu(a_0), \ldots, \nu(a_{n-1})]$ where each $\nu(a_i) \in D_\nu$ is construed as a column vector, and we set $A_\nu = \nu^n_{\mathbf{A}}(A)$. Thus $\nu^n_{\mathbf{A}} : \mathbf{A} \cong \mathbf{A}_\nu \leqslant \underline{\mathbf{M}}^{k\times n}$, and A_ν is a $(k \times n)$-ary algebraic relation on $\underline{\mathbf{M}}$.

Similarly, choose homomorphisms $\omega_0, \ldots, \omega_{\ell-1} : \underline{\mathbf{M}} \to \underline{\mathbf{D}}$ which separate the points of M, let $\omega := \omega_0 \sqcap \cdots \sqcap \omega_{\ell-1} : \underline{\mathbf{M}} \to \underline{\mathbf{D}}^\ell$ be the induced embedding and define $M_\omega := \omega(M)$ so that $\omega : \underline{\mathbf{M}} \cong \mathbf{M}_\omega \leqslant \underline{\mathbf{D}}^\ell$. Finally, define $\tau : \underline{\mathbf{M}} \cong (\mathbf{M}_\omega)_\nu \leqslant \underline{\mathbf{M}}^{k\times\ell}$ by $\tau = \nu^\ell_{\mathbf{M}_\omega} \circ \omega$. Thus $\sigma := \tau^{-1} : (\mathbf{M}_\omega)_\nu \to \underline{\mathbf{M}}$ is a partial $(k \times \ell)$-ary algebraic operation on $\underline{\mathbf{M}}$.

We can now state our theorem more precisely.

Duality Transfer Theorem *Assume that $\underline{\mathbf{D}}$ and $\underline{\mathbf{M}}$ are finite algebras which generate the same quasi-variety and let $\underset{\sim}{\mathbf{D}} = \langle D; R, \mathcal{T}\rangle$ be an alter ego which dualises $\underline{\mathbf{D}}$. Then $\underset{\sim}{\mathbf{M}} := \langle M; \sigma, \{r_\nu \mid r \in R\} \cup \{D_\nu\}, \mathcal{T}\rangle$ is an alter ego which dualises $\underline{\mathbf{M}}$.*

Our assumption that $\underset{\sim}{\mathbf{D}}$ is relational, that is, that it has no operations or partial operations in its type, results in no loss of generality (see [1], Lemma 2.1.2). It should be clear that $\underset{\sim}{\mathbf{M}}$ is an alter ego for $\underline{\mathbf{M}}$ (modulo identification of $\underline{\mathbf{M}}^{k\times n}$ with $\underline{\mathbf{M}}^{kn}$). Given $\mathbf{A} \in \mathcal{A}$ we shall use $e_{\mathbf{A}}$ to denote the canonical embedding relative to the category $\mathcal{X} := \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{D}})$ and the functors $\mathcal{A}(-, \underline{\mathbf{D}})$ and $\mathcal{X}(-, \underset{\sim}{\mathbf{D}})$, while we use $e'_{\mathbf{A}}$ for the canonical embedding relative to $\mathcal{X}' := \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$ and the functors $\mathcal{A}(-, \underline{\mathbf{M}})$ and $\mathcal{X}'(-, \underset{\sim}{\mathbf{M}})$.

To prove that $\underset{\sim}{\mathbf{M}}$ dualises $\underline{\mathbf{M}}$, fix $\mathbf{A} \in \mathcal{A}$ and an $\mathcal{X}'$-morphism $\alpha : \mathcal{A}(\mathbf{A}, \underline{\mathbf{M}}) \to \underset{\sim}{\mathbf{M}}$. It must be shown that $\alpha = e'_{\mathbf{A}}(a)$ for some $a \in A$. Note that if $x \in \mathcal{A}(\mathbf{A}, \underline{\mathbf{D}})$ then $\nu_i \circ x \in \mathcal{A}(\mathbf{A}, \underline{\mathbf{M}})$ for $i < k$, and $((\nu_0 \circ x)(a), \ldots, (\nu_{k-1} \circ x)(a)) = (\nu \circ x)(a) \in D_\nu$ for all $a \in A$. Since D_ν is in the type of $\underset{\sim}{\mathbf{M}}$ and α is an $\mathcal{X}'$-morphism, it follows that $(\alpha(\nu_0 \circ x), \ldots, \alpha(\nu_{k-1} \circ x)) \in D_\nu$. Thus we can define $\hat{\alpha} : \mathcal{A}(\mathbf{A}, \underline{\mathbf{D}}) \to D$ by $\hat{\alpha}(x) = \nu^{-1}(\alpha(\nu_0 \circ x), \ldots, \alpha(\nu_{k-1} \circ x))$. For each relation r in the type of $\underset{\sim}{\mathbf{D}}$, it can be easily checked that $\hat{\alpha}$ preserves r, using the fact that α preserves r_ν. Since α is continuous, there exists a finite subset $S \subseteq A$ such that if $y_1, y_2 \in \mathcal{A}(\mathbf{A}, \underline{\mathbf{M}})$ and $y_1{\restriction}_S = y_2{\restriction}_S$, then $\alpha(y_1) = \alpha(y_2)$. It easily follows that if $x_1, x_2 \in \mathcal{A}(\mathbf{A}, \underline{\mathbf{D}})$ and $x_1{\restriction}_S = x_2{\restriction}_S$, then $\hat{\alpha}(x_1) = \hat{\alpha}(x_2)$, which proves that $\hat{\alpha}$ is continuous. Hence $\hat{\alpha}$ is an $\mathcal{X}$-morphism from $\mathcal{A}(\mathbf{A}, \underline{\mathbf{D}})$ to $\underset{\sim}{\mathbf{D}}$.

Since $\underset{\sim}{\mathbf{D}}$ dualises $\underline{\mathbf{D}}$, by assumption, there exists $a \in A$ such that $\hat{\alpha} = e_{\mathbf{A}}(a)$. We shall show that $\alpha = e'_{\mathbf{A}}(a)$. First observe that if $x \in \mathcal{A}(\mathbf{A}, \underline{\mathbf{D}})$ and $i < k$, then

$$\alpha(\nu_i \circ x) = \pi_i(\nu(\hat{\alpha}(x))) = \nu_i(\hat{\alpha}(x)) = \nu_i(x(a)) = e'_{\mathbf{A}}(\nu_i \circ x).$$

Now let $y \in \mathcal{A}(\mathbf{A}, \underline{\mathbf{M}})$ be fixed, and consider the $k \times \ell$ matrix $[\nu_i \circ \omega_j \circ y]_{k\times\ell}$ of elements from $\mathcal{A}(\mathbf{A}, \underline{\mathbf{M}})$. Note that each entry in this matrix is of the form $\nu_i \circ x$ for some $x \in \mathcal{A}(\mathbf{A}, \underline{\mathbf{D}})$ and some $i < k$, so the previous observation applies to it. Secondly, note that for each $c \in A$

we have $[\nu_i(\omega_j(y(c)))]_{k\times\ell} = \tau(y(c)) \in (M_\omega)_\nu$. Thus $\sigma^{\mathcal{A}(\mathbf{A},\underline{\mathbf{M}})}([\nu_i \circ \omega_j \circ y]_{k\times\ell})$ is defined and equals y. Hence

$$\begin{aligned} \alpha(y) &= \alpha(\sigma^{\mathcal{A}(\mathbf{A},\underline{\mathbf{M}})}([\nu_i \circ \omega_j \circ y]_{k\times\ell})) \\ &= \sigma([\alpha(\nu_i \circ \omega_j \circ y)]_{k\times\ell}) \quad \text{as } \alpha \text{ preserves } \sigma \\ &= \sigma([\nu_i(\omega_j(y(a)))]_{k\times\ell}) \quad \text{by an earlier observation} \\ &= y(a), \end{aligned}$$

proving that $\alpha = e'_{\mathbf{A}}(a)$.

□

When applying the Duality Transfer Theorem, we typically start with a 'minimal' generator of a quasi-variety $\mathcal{A}$, say $\underline{\mathbf{D}}$, and find an alter ego $\underset{\sim}{\mathbf{D}}$ for $\underline{\mathbf{D}}$ which yields a duality on $\mathcal{A}$. The Duality Transfer Theorem then provides an alter ego for any finite algebra $\underline{\mathbf{M}}$ which generates $\mathcal{A}$. An important special case occurs when we have an embedding $\nu : \underline{\mathbf{D}} \to \underline{\mathbf{M}}$ of the 'minimal' generator $\underline{\mathbf{D}}$ into $\underline{\mathbf{M}}$. (This occurs, for example, if $\underline{\mathbf{D}}$ is subdirectly irreducible and generates the same quasi-variety as $\underline{\mathbf{M}}$.) In this case, we have $k = 1$ in the construction given above. To simplify the notation, we shall assume that $\underline{\mathbf{D}}$ is actually a subalgebra of $\underline{\mathbf{M}}$ so that ν is the inclusion map. Let r be an algebraic relation on $\underline{\mathbf{D}}$. Since $\mathbf{r} \leqslant \underline{\mathbf{D}}^n$ and $\underline{\mathbf{D}} \leqslant \underline{\mathbf{M}}$, the relation r_ν is simply r regarded as an algebraic relation on $\underline{\mathbf{M}}$. Following [2] (see also Section 7.7 of [1]), we shall denote the relation r_ν by r_D. The homomorphisms $\omega_i : \underline{\mathbf{M}} \to \underline{\mathbf{D}} \leqslant \underline{\mathbf{M}}$, which separate the points of M, may now be viewed as endomorphisms of $\underline{\mathbf{M}}$, the map $\omega := \omega_0 \sqcap \cdots \sqcap \omega_{\ell-1} : \underline{\mathbf{M}} \to \underline{\mathbf{D}}^\ell \leqslant \underline{\mathbf{M}}^\ell$ is an embedding of $\underline{\mathbf{M}}$ into $\underline{\mathbf{M}}^\ell$ and $\sigma := \omega^{-1} : \omega(\underline{\mathbf{M}}) \to \underline{\mathbf{M}}$ is an ℓ-ary algebraic partial operation on $\underline{\mathbf{M}}$. Thus, we have the following corollary of the Duality Transfer Theorem.

Subalgebra Duality Transfer Theorem *Let $\underline{\mathbf{D}}$ and $\underline{\mathbf{M}}$ be finite and assume that $\underline{\mathbf{D}}$ is a subalgebra of $\underline{\mathbf{M}}$ and that $\underline{\mathbf{M}} \in \mathbb{ISP}(\underline{\mathbf{D}})$. If $\underset{\sim}{\mathbf{D}} = \langle D; R, \mathcal{T}\rangle$ is an alter ego which dualises $\underline{\mathbf{D}}$, then $\underset{\sim}{\mathbf{M}} := \langle M; \sigma, \{r_D \mid r \in R\} \cup \{D\}, \mathcal{T}\rangle$ is an alter ego which dualises $\underline{\mathbf{M}}$.*

It is interesting to compare this with the corresponding result in Saramago [5] which states that, under the same assumptions on $\underline{\mathbf{D}}$ and $\underline{\mathbf{M}}$, a dualising alter ego for $\underline{\mathbf{M}}$ is given by $\underset{\sim}{\mathbf{M}}' := \langle M; \{\omega_j \mid j < \ell\}, \{r_D | r \in R\} \cup \{D\}, \mathcal{T}\rangle$.

Recently, Davey and Haviar [3] have shown that the algebraic partial operation $\sigma : \omega(\underline{\mathbf{M}}) \to \underline{\mathbf{M}}$ plays a vital role in the transfer of a strong duality from $\underline{\mathbf{D}}$ to $\underline{\mathbf{M}}$. They prove that, under the assumptions of the Subalgebra Duality Transfer Theorem, an alter ego which strongly dualises $\underline{\mathbf{D}}$ may be lifted to an alter ego which strongly dualises $\underline{\mathbf{M}}$ by simply adding the endomorphisms $\omega_0, \ldots, \omega_{\ell-1}$ along with partial operation σ.

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