Algebra univers. 42 (1999) 61–77
0002–5240/99/020061–17 \$ 1.50 + 0.20/0

Algebra Universalis

Absolutely closed nil-2 groups

A. MAGIDIN

Abstract. Using the description of dominions in the variety of nilpotent groups of class at most two, we give a characterization of which groups are absolutely closed in this variety. We use the general result to derive an easier characterization for some subclasses; e.g., an abelian group G is absolutely closed in $\mathcal{N}_2$ if and only if G/pG is cyclic for every prime number p.

0. Introduction

The main result of this paper is a characterization of the absolutely closed groups in the variety $\mathcal{N}_2$ (definitions are recalled in Section 1 below). We obtain this result by using the description of dominions in the variety $\mathcal{N}_2$, and applying some ideas D. Saracino used in his classification of the strong amalgamation bases for the same variety [7].

In Section 1 we will recall the main definitions and review the notion of amalgam. In Section 2 we will recall the results of Saracino related to his classification of amalgamation bases of $\mathcal{N}_2$, and we will prove our main result. Finally, in Section 3 we will prove several reduction theorems, and deduce some conditions which are sufficient for a group to be absolutely closed in $\mathcal{N}_2$. We will also give easier to check conditions for special classes of groups; for example, we will show that a finitely generated abelian group is absolutely closed in $\mathcal{N}_2$ if and only if it is cyclic.

The contents of this paper are part of investigations that developed out of the author's doctoral dissertation, which was conducted at the University of California at Berkeley, under the direction of Prof. George M. Bergman. It is my very great pleasure to express my deep gratitude and indebtedness to Prof. Bergman, for his advice and encouragement throughout my graduate work and the preparation of a prior version of this paper, and for suggesting Theorem 3.17.

1. Preliminaries

Recall that Isbell [2] defines for a variety $\mathcal{C}$ of algebras (in the sense of Universal Algebra) of a fixed type Ω, and an algebra $A \in \mathcal{C}$ and subalgebra B of A, the *dominion* of *B in A* to

Presented by Bjarni Jónsson.
Received October 28, 1998; accepted in final form May 7, 1999.
1991 *Mathematics Subject Classification*: 20E06, 20F18 (primary).
Keywords: Amalgam, special amalgam, dominion, absolutely closed, nilpotent.

be the intersection of all equalizers containing B. Explicitly,

$$\mathrm{dom}_A^{\mathcal{C}}(B) = \left\{a \in A \,\middle|\, \forall f, g : A \longrightarrow C,\ \text{if } f|_B = g|_B \text{ then } f(a) = g(a)\right\}$$

where C ranges over all algebras in $\mathcal{C}$, and f, g are morphisms.

Also, Isbell calls an algebra $B \in \mathcal{C}$ *absolutely closed* (in $\mathcal{C}$) if and only if

$$\forall A \in \mathcal{C} \text{ with } B \subseteq A,\ \mathrm{dom}_A^{\mathcal{C}}(B) = B.$$

For example, in the variety of semigroups, every group (when considered as a semigroup using the forgetful functor) is absolutely closed; this follows easily from the Zigzag Lemma [2].

REMARK 1.1. Note that the property of being "absolutely closed" depends on the variety of context $\mathcal{C}$; it is common for an algebra to be absolutely closed when considered a member of $\mathcal{C}$, and not absolutely closed when considered as a member of a different variety $\mathcal{C}'$.

In the variety $\mathcal{N}_2$ of nilpotent groups of class at most 2 (i.e., groups G for which $[G, G] \subseteq Z(G)$) there are nontrivial dominions [5]. The precise description of dominions in this variety is recalled below. Given that there are nontrivial dominions, an interesting problem is to characterize all groups that are absolutely closed in $\mathcal{N}_2$.

For the remainder of this paper, every group will be assumed to lie in $\mathcal{N}_2$ unless otherwise specified, and all maps are assumed to be group morphisms, unless otherwise noted. We will write all groups multiplicatively. We will say that a group is *absolutely closed* to mean it is absolutely closed in $\mathcal{N}_2$. The identity element of the group G will be denoted e_G, omitting the subscript if there is no danger of ambiguity. For a group G and elements x and y in G, the commutator of x and y is $[x, y] = x^{-1}y^{-1}xy$. The commutator subgroup of a group G, denoted by G' or $[G, G]$, is the normal subgroup of G generated by all $[x, y]$ with x, y in G. More generally, given two subsets A and B of G (not necessarily subgroups), $[A, B]$ denotes the subgroup of G generated by all elements $[a, b]$, where $a \in A$ and $b \in B$. The center of G will be denoted by $Z(G)$. Any presentation of a group will be understood to be a presentation in $\mathcal{N}_2$; that is, the identities of $\mathcal{N}_2$ will be imposed on the group, as well as all the relations specified in the presentation. We will use Z to denote the infinite cyclic group, which we also write multiplicatively.

In $\mathcal{N}_2$, since commutators are central, the commutator bracket acts as a bilinear map from $G^{\mathrm{ab}} \times G^{\mathrm{ab}}$ onto $[G, G]$. In particular, for every $x, y, z \in G$, and $n \in \mathbb{Z}$,

$$[x, yz] = [x, y][x, z]; \quad [xy, z] = [x, z][y, z]; \quad [x^n, y] = [x, y]^n = [x, y^n].$$

Also, given $A, B \in \mathcal{N}_2$, every element of their coproduct $A \coprod^{\mathcal{N}_2} B$ has a unique expression in the form $\alpha\beta\gamma$, where $\alpha \in A$, $\beta \in B$, and $\gamma \in [A, B]$. A theorem of T. MacHenry [4] states that the subgroup $[A, B]$ of $A \coprod^{\mathcal{N}_2} B$ is isomorphic to the tensor product $A^{\mathrm{ab}} \otimes B^{\mathrm{ab}}$.

Recall that an $\mathcal{N}_2$-amalgam of two groups $A, C \in \mathcal{N}_2$ with core B consists of groups A, B, and C, equipped with one to one group morphisms

$$\Phi_A : B \longrightarrow A$$
$$\Phi_C : B \longrightarrow C.$$

To simplify notation, we denote this situation by $(A, C; B)$. To say that the amalgam $(A, C; B)$ is (*weakly*) *embeddable in* $\mathcal{N}_2$ means that there exists a group M in $\mathcal{N}_2$ and one-to-one group morphisms

$$\lambda_A : A \longrightarrow M, \quad \lambda_C : C \longrightarrow M, \quad \lambda : B \longrightarrow M$$

such that

$$\lambda_A \circ \Phi_A = \lambda \qquad \lambda_C \circ \Phi_C = \lambda.$$

When we examine whether or not the amalgam $(A, C; B)$ is embeddable, the obvious candidate for M is the coproduct with amalgamation of A and C over B, denoted $A \coprod_B^{\mathcal{N}_2} C$. This coproduct is sometimes called the $\mathcal{N}_2$-free product with amalgamation. We say that $(A, C; B)$ is *weakly embeddable* (in $\mathcal{N}_2$) if no two distinct elements of A are identified with each other in the coproduct $A \coprod_B^{\mathcal{N}_2} C$, and similarly with two distinct elements of C. Note that weak embeddability does not preclude the possibility that an element x of $A \backslash B$ be identified with an element y of $C \backslash B$ in $A \coprod_B^{\mathcal{N}_2} C$. We say that $(A, C; B)$ is *strongly embeddable* (in $\mathcal{N}_2$) if there is also no identification between elements of $A \backslash B$ and elements of $C \backslash B$. By *special amalgam* we mean an amalgam $(A, A'; B)$, where there is an isomorphism ψ between A and A' over B, meaning that $\psi \circ \Phi_A = \Phi_{A'}$. In this case, we usually write $(A, A; B)$, with $\psi = \mathrm{id}_A$ being understood.

Also, we recall that a group G is said to be a *weak amalgamation base* for $\mathcal{N}_2$ if every amalgam with G as a core is weakly embeddable in $\mathcal{N}_2$; it is a *strong amalgamation base* (for $\mathcal{N}_2$) if every such amalgam is strongly embeddable (in $\mathcal{N}_2$); and it is a *special amalgamation base* for $\mathcal{N}_2$ if every special amalgam with core G is strongly embeddable in $\mathcal{N}_2$. Note that a special amalgam is always weakly embeddable.

The connection between amalgams and dominions is via special amalgams. Letting A' be an isomorphic copy of A, and $M = A \coprod_B^{\mathcal{N}_2} A'$, we have that

$$\mathrm{dom}_A^{\mathcal{N}_2}(B) = A \bigcap A' \subseteq M \tag{1}$$

where we have identified B with its common image in A and A'.

The above discussion can be done in the much more general context of an arbitrary variety $\mathcal{C}$ of algebras of a fixed type. For a more complete discussion of amalgams in general and their connection with dominions, see [1].

REMARK 1.2. It is not hard to verify that a group B is a strong amalgamation base if and only if it is both a weak amalgamation base and a special amalgamation base. For a proof we direct the reader to [1]. We also note that for a group H in a variety $\mathcal{C}$, being a special amalgamation base for $\mathcal{C}$ is equivalent to being absolutely closed in $\mathcal{C}$. Indeed, the equality given in (1) shows that H is absolutely closed in $\mathcal{C}$ if and only if for every group G containing H, the special amalgam $(G, G; H)$ is strongly embeddable, which holds if and only if H is a special amalgamation base.

2. Absolutely closed groups

In this section we recall the characterization of weak and strong amalgamation bases in the variety $\mathcal{N}_2$, due to Saracino. Then we will state the characterization of absolutely closed groups in this variety.

It will be helpful to recall a theorem about adjunction of roots to $\mathcal{N}_2$-groups:

THEOREM 2.1. (Saracino, Theorem 2.1 in [7]) *Let G be a nilpotent group of class at most two, let $m > 0$, let* **n** *be an m-tuple of positive integers, and let* **g** *be an m-tuple of elements of G. Then there exists a nilpotent group H of class two, containing G, and which contains an n_i-th root for $g_i (1 \leq i \leq m)$ if and only if for every $m \times m$ array $\{c_{ij}\}$ of integers such that $n_i c_{ij} = n_j c_{ji}$ for all i and j, and for all $y_1, \ldots, y_m \in G$,*

$$if \quad y_j^{n_j} \equiv \prod_{i=1}^{m} g_i^{c_{ij}} (\text{mod } G'),\ then\ \prod_{j=1}^{m} [y_j, g_j] = e.$$

REMARK 2.2. Note that Theorem 2.1 implies that we can always adjoin n_i-th roots to a finite family of commutators (in fact, of central elements). In particular, if $g \in G \in \mathcal{N}_2$, and $g \in G^n G'$, then there is an extension of G which contains an n-th root for g: since $g = x^n x'$, adjoin an n-th root for x', and we are done.

THEOREM 2.3. (Saracino, Theorem 3.3 in [7]) *Let $G \in \mathcal{N}_2$. The following are equivalent:*

(i) *G is a weak amalgamation base for $\mathcal{N}_2$.*
(ii) *G is a strong amalgamation base for $\mathcal{N}_2$.*
(iii) *G satisfies $G' = Z(G)$, and $\forall g \in G \forall n > 0$ $(g \in G^n G'$ or $\exists y \in G$ and $\exists k \in \mathbb{Z}$ such that $(y^n \equiv g^k \ (\text{mod } G')$ and $[y, g] \neq e))$.*
(iv) *G satisfies $G' = Z(G)$, and for all $g \in G$ and all $n > 0$, either g has an n-th root modulo G', or else g has no n-th root in any overgroup $K \in \mathcal{N}_2$ of G.*

We pause briefly to give some examples of groups that are strong amalgamation bases in $\mathcal{N}_2$.

EXAMPLE 2.4. Both the dihedral group D_8 and the quaternion group of 8 elements Q are strong amalgamation bases. It is clear that they lie in $\mathcal{N}_2$, and a routine calculation shows that they both satisfy (iii).

EXAMPLE 2.5. Analogously, any non-abelian group of order p^3, with p an odd prime, is a strong amalgamation base for $\mathcal{N}_2$.

REMARK 2.6. On the other hand, we remark that a nontrivial abelian group cannot be a strong amalgamation base in $\mathcal{N}_2$, since it never satisfies $G' = Z(G)$.

Next, we recall the description of dominions in $\mathcal{N}_2$:

LEMMA 2.7. (See [5]) *Let $G \in \mathcal{N}_2$, H a subgroup of G. Let D be the subgroup of G generated by all elements of H and all elements $[x, y]^q$, where x, y lie in G, $q \geq 0$, and $x^q, y^q \in H[G, G]$. Then $D = \mathrm{dom}_G^{\mathcal{N}_2}(H)$.*

REMARK 2.8. Lemma 2.7 also follows from B. Maier's work on amalgams of nilpotent groups; we direct the reader to [6].

We can now prove our main result:

THEOREM 2.9. *Let $G \in \mathcal{N}_2$. Then G is absolutely closed in $\mathcal{N}_2$ if and only if for all $x, y \in G$ and for all $n > 0$, one of the following holds:*
There exist $a, b, c \in \mathbb{Z}$, $g_1, g_2 \in G$ such that

$$\begin{aligned} g_1^n &\equiv x^a y^b \pmod{G'} \\ g_2^n &\equiv x^b y^c \pmod{G'} \end{aligned} \tag{2}$$

and $[g_1, x][g_2, y] \neq e$; or
There exist $a, b, c \in \mathbb{Z}$, $g_1, g_2 \in G$ such that

$$\begin{aligned} g_1^n &\equiv x^a y^b \pmod{G'} \\ g_2^n &\equiv x^{b+1} y^c \pmod{G'}. \end{aligned} \tag{3}$$

REMARK 2.10. Note that (2) is simply the statement that there is no extension of G which contains n-th roots for both x and y.

Proof. First, suppose that for all $x, y \in G$, and all $n > 0$, either (2) or (3) holds. Let K be an overgroup of G, and suppose that there exist $k_1, k_2 \in K, k_1', k_2' \in K'$ such that $k_1^n k_1', k_2^n k_2' \in G$. We want to show that $[k_1, k_2]^n$ lies in G. Let $x = k_1^n k_1'$ and $y = k_2^n k_2'$.

Note that since both x and y have n-th roots modulo the commutator in K, there is an extension of K which has n-th roots for both x and y (as in Remark 2.2 above). Therefore, (2) cannot hold in G. Hence, there exist $a, b, c \in \mathbb{Z}$, and $g_1, g_2 \in G$ such that $g_1^n \equiv x^a y^b$ and $g_2^n \equiv x^{b+1} y^c$ modulo G'.

Since $x = k_1^n k'$ and $y = k_2^n k'$, we have that xk_1^{-n} and yk_2^{-n} are central in K. In particular, $\left[g_1 k_1^{-a} k_2^{-b-1}, x k_1^{-n}\right]\left[g_2 k_1^{-b} k_2^{-c}, y k_2^{-n}\right] = e$.

On the other hand,

$$\begin{aligned}
[g_1 k_1^{-a} k_2^{-b-1}, x k_1^{-n}][g_2 k_1^{-b} k_2^{-c}, y k_2^{-n}] &= [g_1, x][g_1, k_1^{-n}][k_1^{-a} k_2^{-b-1}, x]\\
&\quad [k_1^{-a}, k_1^{-n}][k_2^{-b-1}, k_1^{-n}]\\
&\quad [g_2, y][g_2, k_2^{-n}][k_1^{-b} k_2^{-c}, y]\\
&\quad [k_1^{-b}, k_2^{-n}][k_2^{-c}, k_2^{-n}]\\
&= [g_1, x][g_2, y][g_1, k_1^{-n}][g_2, k_2^{-n}]\\
&\quad [k_1, x^{-a}][k_2, x^{-b-1}][k_1, y^{-b}][k_2, y^{-c}]\\
&\quad [k_2, k_1]^{(-n)(-b-1)}[k_1, k_2]^{(-b)(-n)}\\
&= [g_1, x][g_2, y][g_1, k_1^{-n}][g_2, k_2^{-n}]\\
&\quad [k_1, x^{-a} y^{-b}][k_2, x^{-b-1} y^{-c}]\\
&\quad [k_1, k_2]^{nb-n(b+1)}\\
&= [g_1, x][g_2, y][g_1, k_1]^{-n}[g_2, k_2]^{-n}\\
&\quad [k_1, g_1^{-n}][k_2, g_2^{-n}][k_1, k_2]^{-n}\\
&= [g_1, x][g_2, y][k_1, k_2]^{-n}.
\end{aligned}$$

Therefore, $[g_1, x][g_2, y][k_1, k_2]^{-n} = e$, so $[k_1, k_2]^n = [g_1, x][g_2, y]$; since g_1, g_2, x, and y all lie in G, it follows that $[k_1, k_2]^n \in G$, as claimed.

Therefore, if G satisfies the conditions, then G is absolutely closed.

Conversely, suppose that G does not satisfy the condition given. Let $x_1, x_2 \in G$, and $n > 0$, such that:

For all $a, b, c \in \mathbb{Z}$, if $g_1, g_2 \in G$ are such that

$$\begin{aligned}
g_1^n &\equiv x_1^a x_2^b \pmod{G'}\\
g_2^n &\equiv x_1^b x_2^c \pmod{G'}
\end{aligned} \tag{4}$$

then $[g_1, x_1][g_2, x_2] = e$; and

For all $a, b, c \in \mathbb{Z}$, there do not exist $g_1, g_2 \in G$ such that

$$\begin{aligned}
g_1^n &\equiv x_1^a x_2^b \pmod{G'}\\
g_2^n &\equiv x_1^{b+1} x_2^c \pmod{G'}.
\end{aligned} \tag{5}$$

Let $F = G \coprod^{\mathcal{N}_2} (Z \coprod^{\mathcal{N}_2} Z)$, and denote the generators of the two copies of Z by r_1 and r_2. Every element of $Z \coprod^{\mathcal{N}_2} Z$ has a unique expression of the form $r_1^a r_2^b [r_1, r_2]^c$. Let N be the minimal normal subgroup of F containing $x_1 r_1^{-n}$ and $x_2 r_2^{-n}$. We will show that $N \cap G = \{e\}$, and that for every $g \in G$, $g[r_1, r_2]^{-n} \notin N$. This will prove that G is not absolutely closed, by looking at F/N, which contains G as a subgroup, and where $[r_1, r_2]^n$ lies in the dominion of G but not in G. The proof is patterned after a proof of Saracino (Theorem 2.1 in [7]).

A general element of N may be written as

$$\prod_{j=1}^{2} \left(\prod_{k=1}^{s_j} (b_{jk} z_{jk})(x_j r_j^{-n})^{\varepsilon_{jk}} (b_{jk} z_{jk})^{-1} \right), \tag{6}$$

where $b_{jk} \in G$, s_j is a positive integer, $\varepsilon_{jk} = \pm 1$, and $z_{jk} = r_1^{a_{jk1}} r_2^{a_{jk2}}$. Since F is nilpotent of class two, this does indeed represent a general element of N.

We may rewrite (6) as follows:

$$\prod_{j=1}^{2} \left(\prod_{k=1}^{s_j} [(b_{jk} z_{jk})^{-1}, (x_j r_j^{-n})^{-\varepsilon_{jk}}] (x_j r_j^{-n})^{\varepsilon_{jk}} \right)$$

which, expanding the brackets bilinearly, becomes

$$\prod_{j=1}^{2} \left(\prod_{k=1}^{s_j} [b_{jk}, x_j]^{\varepsilon_{jk}} [b_{jk}, r_j^{-n}]^{\varepsilon_{jk}} [z_{jk}, x_j]^{\varepsilon_{jk}} [z_{jk}, r_j^{-n}]^{\varepsilon_{jk}} \right) (x_j r_j^{-n})^{t_j}$$

where $t_j = \sum_{k=1}^{s_j} \varepsilon_{jk}$.

Now suppose that this element is equal to an element of the form $g[r_1, r_2]^{qn}$, for some $g \in G, q \in \mathbb{Z}$; if we write the general expression in the form $\alpha\beta\gamma$, where $\alpha \in G, \beta \in Z \coprod^{\mathcal{N}_2} Z$, and $\gamma \in [G, Z \coprod^{\mathcal{N}_2} Z]$, then the β-factor is equal to $r_1^{-nt_1} r_2^{-nt_2} z$, where z is in the commutator of $Z \coprod^{\mathcal{N}_2} Z$. But on the other hand, by uniqueness $t_1 = t_2 = 0$. Again by uniqueness, and using this fact, we have:

$$\begin{aligned}
g &= \prod_{j=1}^{2} \left(\prod_{k=1}^{s_j} [b_{jk}, x_j]^{\varepsilon_{jk}} \right) \\
[r_1, r_2]^{qn} &= \prod_{j=1}^{2} \left(\prod_{k=1}^{s_j} [z_{jk}, r_j^{-n}]^{\varepsilon_{jk}} \right) \\
e &= \prod_{j=1}^{2} \left(\prod_{k=1}^{s_j} [b_{jk}, r_j^{-n}]^{\varepsilon_{jk}} [z_{jk}, x_j]^{\varepsilon_{jk}} \right).
\end{aligned}$$

Feeding in the value of z_{jk} and rearranging, we have

$$g = \left[\prod_{k=1}^{s_1} b_{1k}^{\varepsilon_{1k}}, x_1\right]\left[\prod_{k=1}^{s_2} b_{2k}^{\varepsilon_{2k}}, x_2\right] \tag{7}$$

$$[r_1, r_2]^{qn} = [r_1, r_2]^{-n\Sigma\varepsilon_{2k}a_{2k1}+n\Sigma\varepsilon_{1k}a_{1k2}} \tag{8}$$

and

$$e = \prod_{j=1}^{2}\left[\left(\prod_{k=1}^{s_j} b_{jk}^{\varepsilon_{jk}}\right)^{-n} x_1^{-\Sigma\varepsilon_{1k}a_{1kj}} x_2^{-\Sigma\varepsilon_{2k}a_{2kj}}, r_j\right] \tag{9}$$

Now define $g_j \in G$ by $g_j = \prod_k b_{jk}^{\varepsilon_{jk}}$, and define c_{ij} by $c_{ij} = -\Sigma\varepsilon_{ik}a_{ikj}$. Then (7) becomes

$$g = [g_1, x_1][g_2, x_2],$$

equation (8) becomes

$$[r_1, r_2]^{qn} = [r_1, r_2]^{n(c_{21}-c_{12})},$$

and equation (9) becomes

$$e = [g_1^{-n}x_1^{c_{11}}x_2^{c_{21}}, r_1]\,[g_2^{-n}x_1^{c_{12}}x_2^{c_{22}}, r_2].$$

Since we know that $[G, Z \coprod^{\mathcal{N}_2} Z]$ is isomorphic to $G^{\mathrm{ab}} \otimes (Z \oplus Z)$, this implies that

$$g_1^{-n}x_1^{c_{11}}x_2^{c_{21}},\ g_2^{-n}x_1^{c_{12}}x_2^{c_{22}} \in G'$$

that is,

$$\begin{aligned} g_1^n &\equiv x_1^{c_{11}}x_2^{c_{21}} \pmod{G'} \\ g_2^n &\equiv x_1^{c_{12}}x_2^{c_{22}} \pmod{G'}. \end{aligned} \tag{10}$$

Now, suppose that $q = 0$; that is, we are trying to find which elements lie in $G \cap N$. Since $q = 0$, it follows from (8) that $c_{21} - c_{12} = 0$, that is, that $c_{12} = c_{21}$. By (4) and (10), $[g_1, x_1][g_2, x_2] = e$, and therefore, $g = e$. In particular, $G \cap N = \{e\}$, as claimed.

Finally, suppose that $q = -1$. Then $c_{21} - c_{12} = -1$, so $c_{12} = c_{21} + 1$. But then (5) says that (10) cannot occur, so there is no element $g \in G$ such that $g[r_1, r_2]^{-n} \in N$. This proves the theorem. ☐

In fact, we need only verify (2) and (3) for prime powers:

COROLLARY 2.11. *Let $G \in \mathcal{N}_2$. Then G is absolutely closed if and only if for every $x, y \in G$, and every prime power p^a, G satisfies* (2) *or* (3) *with $n = p^a$.*

Proof. Necessity is immediate. To show that it is also sufficient, note that for a given n, if for all $x, y \in G$, G satisfies (2) or (3), then it follows that whenever K is an overgroup of G, and $k_1^n k_1', k_2^n k_2' \in G$, then $[k_1, k_2]^n \in G$.

Let K be an overgroup of G, and suppose that for some $n > 0$, $k_1^n k_1', k_2^n k_2'$ both lie in G. Let $n = p_1^{a_1} \cdots p_r^{a_r}$ be a prime factorization of n. Since G satisfies (2) or (3) for prime powers, it follows that

$$[k_1, k_2]^{n^2/p_i^{a_i}} = [k_1^{n/p_i^{a_i}}, k_2^{n/p_i^{a_i}}]^{p_i^{a_i}} \in G$$

for each i. Let $a = \gcd\{n^2/p_1^{a_1}, \dots, n^2/p_r^{a_r}\}$. Then $[k_1, k_2]^a \in G$. But it is not hard to see that $a = n$, so $[k_1, k_2]^n \in G$, as claimed. □

We also note the following result:

LEMMA 2.12. *Let $G \in \mathcal{N}_2$, and let $n > 0$. If $x \in G^n G'$, then for all y there exist $g_1, g_2 \in G$ and $a, b, c \in \mathbb{Z}$ such that*

$$g_1^n \equiv x^a y^b \pmod{G'}$$
$$g_2^n \equiv x^{b+1} y^c \pmod{G'}.$$

In particular, G, n, x, and y satisfy (3)*. Analogously, if $y \in G^n G'$, then for all x we have that G, n, x, and y satisfy* (3)*.*

Proof. Suppose that $x = r^n r'$, and $y \in G$. Let $a = b = c = 0$, $g_1 = e$, and $g_2 = r$. If, on the other hand, $y = s^n s'$, and $x \in G$, let $a = c = 0$, $b = -1$, $g_2 = e$, and $g_1 = s^{-1}$. □

COROLLARY 2.13. *If $G \in \mathcal{N}_2$ is such that for every $x \in G$ and every $n > 0$, either x has an n-th root in G modulo G', or else x does not have an n-th root in any $\mathcal{N}_2$-overgroup of G, then G is absolutely closed.*

Proof. Given $x, y \in G$, and $n > 0$, if either x or y has an n-th root modulo the commutator, then (3) is satisfied. Otherwise, no overgroup of G contains an n-th root for either x or y, and hence no overgroup of G contains an n-th root for *both* x and y, so G satisfies (2). □

In particular, we deduce that any group that satisfies Saracino's conditions is absolutely closed, which is in keeping with the fact that any strong amalgamation base is necessarily also a special amalgamation base.

3. Consequences and applications

First, we deduce some easy conditions from Theorem 2.9 which are sufficient for a group to be absolutely closed.

COROLLARY 3.1. *If G is a divisible nilpotent group of class at most 2, then G is absolutely closed in $\mathcal{N}_2$.*

Proof. If G is divisible, then every element has an n-th root modulo the commutator, so G satisfies (3) by Lemma 2.12. □

Note that any nontrivial divisible abelian group G is absolutely closed, even though it cannot be a strong amalgamation base, since the commutator subgroup cannot equal the center. Therefore, the class of absolutely closed groups is strictly larger than the class of strong amalgamation bases in $\mathcal{N}_2$.

Before proceeding, we will prove some reduction theorems regarding absolutely closed groups.

If π is a set of primes, we will say that a group G is π-divisible if every element of G has a p-th root in G, for every prime $p \in \pi$. We will say that G is π'-divisible if every element of G has a q-th root in G, for every prime $q \notin \pi$.

It is not hard to verify that for a nilpotent group G of class 2, being π-divisible is equivalent to asking that G^{ab} be π-divisible.

THEOREM 3.2. *Let π be a set of primes, and let $A, B \in \mathcal{N}_2$. Suppose that A is π-divisible, and B is π'-divisible. Then $G = A \oplus B$ is absolutely closed if and only if both A and B are.*

Proof. It is easy to see that, in general, if $A \oplus B$ is absolutely closed, then so are A and B.

For the converse, suppose that both A and B are absolutely closed, and let K be an overgroup of $A \oplus B$. Let $x, y \in K$, $x', y' \in K'$, and $n > 0$ be such that $x^n x', y^n y' \in A \oplus B$. We want to show that $[x, y]^n \in A \oplus B$. Write $x^n x' = a_1 \oplus b_1$, and $y^n y' = a_2 \oplus b_2$.

By Corollary 2.11, it suffices to consider the case when n is a prime power, say $n = p^\alpha$.

If $p \in \pi$, then a_1^{-1} has an n-th root in A. That is, there exists $r \in A$ such that $r^n = a_1^{-1}$. Similarly, there exists $s \in A$ such that $s^n = a_2^{-1}$.

Therefore,

$$(rx)^n \equiv r^n x^n \equiv a_1^{-1}(a_1 \oplus b_1) \equiv b_1 \pmod{K'}$$
$$(sy)^n \equiv s^n y^n \equiv a_2^{-1}(a_2 \oplus b_2) \equiv b_2 \pmod{K'}$$

so $[rx, sy]^n \in \mathrm{dom}_K^{\mathcal{N}_2}(B)$. Since B is absolutely closed, it follows that $[rx, sy]^n$ lies in B. However,

$$\begin{aligned} [rx, sy]^n &= [r, s]^n [r, y]^n [x, s]^n [x, y]^n \\ &= [r, s]^n [r, y^n y'][x^n x', s][x, y]^n \\ &= [r, s]^n [r, a_2 \oplus b_2][a_1 \oplus b_1, s][x, y]^n. \end{aligned}$$

Since r and s lie in A, the first three terms on the right hand side lie in $A \oplus B$. Since $[rx, sy]^n \in B$, it follows that $[x, y]^n \in A \oplus B$ as well.

If, on the other hand, $p \notin \pi$, then the argument proceeds as above, taking roots of b_1^{-1} and b_2^{-1}. □

COROLLARY 3.3. (Cf. Theorem 3.5 in [7]) *If $A, B \in \mathcal{N}_2$ are of relatively prime exponents, then $A \oplus B$ is absolutely closed if and only if both A and B are.*

Proof. If A is of finite exponent n, then A is π-divisible, where π is the set of all primes not occuring in the prime factorization of n. The result now follows from Theorem 3.2. □

Recall that every abelian group G may be written as $G = D \oplus G_{\mathrm{red}}$, where D is divisible and G_{red} is reduced. By letting π be the set of all primes, we obtain:

COROLLARY 3.4. *An abelian group G is absolutely closed if and only if its reduced part is absolutely closed.*

COROLLARY 3.5. *If $G \in \mathcal{N}_2$ is π-divisible, then G is absolutely closed if and only if for every $x, y \in G$ and every prime power $n = q^a$, with $q \notin \pi$, G, x, y, and n satisfy* (2) *or* (3).

COROLLARY 3.6. *If $G \in \mathcal{N}_2$ is a torsion group, then G is absolutely closed if and only if its p-parts are.*

Next we analyze what (2) and (3) mean for finitely generated abelian groups.

THEOREM 3.7. *If G is cyclic, then G is absolutely closed.*

Proof. Let $G = \langle t \rangle$, and let $x = t^r$, $y = t^s$ be any two elements. Let $n = p^\alpha$ be a prime power. We claim that G, x, y, and n satisfy (3). To see this, it will suffice to show that we can find a, b, and $c \in \mathbb{Z}$ such that $p^\alpha | ar + bs$ and $p^\alpha | (b+1)r + cs$.

If $(p, r) = 1$, set $b = -1$, $c = 0$; then we want to find an a such that $p^\alpha | ar - s$. But since r is relatively prime to p, as a ranges over $\mathbb{Z}$, ar ranges over all congruence classes modulo p^α, so there is one which is congruent to s.

If $(p, s) = 1$, we proceed similarly. Finally, suppose that $r = p^\delta$, $s = p^\gamma$; we may assume that $\delta, \gamma < \alpha$.

If $\delta \leq \gamma < \alpha$, then set $b = -1$, $c = 0$, and $a = p^{\gamma-\delta} + p^{\alpha-\delta}$.

And if $\gamma \leq \delta < \alpha$, then set $a = b = 0$, and let $c = -p^{\delta-\gamma} + p^{\alpha-\gamma}$. □

In fact, if G is a finitely generated abelian group, then being cyclic is also necessary for G to be absolutely closed. To prove this, we start with a series of examples:

EXAMPLE 3.8. $Z \oplus Z$ is not absolutely closed. Indeed, let F be the $\mathcal{N}_2$ group presented (in $\mathcal{N}_2$) by

$$F = \langle x, y \mid [x, y]^4 = e \rangle;$$

then the subgroup of F generated by x^2 and y^2 is abelian, isomorphic to $Z \oplus Z$, but $[x, y]^2$ lies in the dominion of $\langle x^2, y^2 \rangle$, and not in the subgroup.

EXAMPLE 3.9. $Z/p^{a_1}Z \oplus Z/p^{a_2}Z$ with p a prime, and $a_1, a_2 \geq 1$, is not absolutely closed. This time let F be the $\mathcal{N}_2$ group presented by

$$F = \langle x, y | x^{p^{a_1+1}} = y^{p^{a_2+1}} = [x, y]^{p^2} = e \rangle$$

and let $G = \langle x^p, y^p \rangle$. Then $G \cong Z/p^{a_1}Z \oplus Z/p^{a_2}Z$, but

$$[x, y]^p \in \mathrm{dom}_F^{\mathcal{N}_2}(G) \backslash G.$$

EXAMPLE 3.10. $Z \oplus Z/p^a Z$ is not absolutely closed, where p is a prime and $a \geq 1$. Let F be the $\mathcal{N}_2$ group presented by

$$F = \langle x, y | y^{p^{a+1}} = [x, y]^{p^2} = e \rangle$$

and let $G = \langle x^p, y^p \rangle$. Then G is isomorphic to $Z \oplus Z/p^a Z$, and

$$[x, y]^p \in \mathrm{dom}_F^{\mathcal{N}_2}(G) \backslash G.$$

THEOREM 3.11. *A finitely generated abelian group is absolutely closed in $\mathcal{N}_2$ if and only if it is cyclic.*

Proof. Sufficiency is Theorem 3.7. For necessity, let G be a finitely generated abelian group, and write

$$G \cong Z^r \oplus Z/a_1 Z \oplus \cdots \oplus Z/a_s Z$$

where each a_i is a prime power.

If $r > 1$, or $r = 1$ and $s > 0$, then G has a direct summand which is not absolutely closed by the examples above, hence G is not absolutely closed. If $s > 1$ and there exist i and j such that a_i and a_j are not relatively prime, then G also has a direct summand which is not absolutely closed. All other cases (namely, $r = 1$ and $s = 0$; or $r = 0$ and all a_i relatively prime) are cyclic groups. $\square$

We can also prove an analogue of a result of Saracino. Recall the following:

THEOREM 3.12. (Saracino, Theorems 3.4 and 3.6 in [7]) *Let G be a nilpotent group of class 2 and exponent n, where n is the product of distinct primes, or twice such a product. Then G is a strong amalgamation base for* $\mathcal{N}_2$ *if and only if* $G' = Z(G)$.

We obtain a similar result here:

THEOREM 3.13. *Let G be a nilpotent group of class two and exponent n, where n is a product of distinct primes. Then G is absolutely closed if and only if* $Z(G)/G'$ *is cyclic.*

Proof. By Corollary 3.6, we may assume that G is a p-group, that is, $n = p$ with p a prime. Denote the image of an element $x \in G$ in G^{ab} by $\overline{x}$.

Since G^{ab} is a $\mathbb{Z}/p\mathbb{Z}$ vector space, and $Z(G)/G'$ is a subspace, there exist elements $\{z_i\}_{i\in I}$ and $\{b_j\}_{j\in J}$ such that each z_i lies in $Z(G)$, $\{\overline{z_i}\}$ is a basis for $Z(G)/G'$, and $\{\overline{z_i}, \overline{b_j}\}$ is a basis for G^{ab}. Since G is of exponent p, it follows that $\langle z_i | i \in I\rangle$ is a direct summand of G; hence, if $|I| > 1$, then G is not absolutely closed. Thus, we may assume that $|I| \leq 1$, which proves necessity.

To see sufficiency, note that if K is an overgroup of G, and $g \in G$ has a p-th root in K modulo K', then g is central in G; for if $g = r^p r'$ in K, and $h \in G$, then

$$[g, h] = [r^p r', h] = [r^p, h] = [r, h^p] = [r, e] = e$$

since G is of exponent p.

Also note that G is q-divisible for any prime $q \neq p$, so it suffices to check p^a-th roots. Let K be any overgroup of G, and suppose that $r_1^{p^a}, r_1', r_2^{p^a} r_2' \in G$, where $r_1, r_2 \in K, r_1', r_2' \in K'$. Write $g_1 = r_1^{p^a} r_1'$, $g_2 = r_2^{p^a} r_2'$. In particular, g_1 and g_2 must be central in G, hence they lie in $\langle z_1\rangle G'$ (or in G' if $|I| = 0$). But then there exist $x', y' \in G'$ such that $r_1^{p^a} r_1' x', r_2^{p^a} r_2' y' \in \langle z_1\rangle$. Therefore,

$$[r_1, r_2]^{p^a} \in \text{dom}_K^{\mathcal{N}_2}(\langle z_1\rangle) = \langle z_1\rangle$$

since cyclic groups are absolutely closed. In particular, $[r_1, r_2]^{p^a} \in G$, and so G is absolutely closed. $\square$

Although we have proven an analogue of the “square-free” case of Theorem 3.12, the “twice a square-free number” version does not hold. A counterexample is:

EXAMPLE 3.14. A group $G \in \mathcal{N}_2$ of exponent four, with $Z(G)/G'$ cyclic, which is not absolutely closed. Let G be presented by

$$G = \langle x, y, z \mid x^4 = y^2 = z^2 = [x, y]^2 = [x, z]^2 = [y, z] = e \rangle.$$

Clearly, G is of exponent four, and $G^{\mathrm{ab}} \cong Z/4Z \oplus Z/2Z \oplus Z/2Z$. Also, the center of G is generated, modulo G', by x^2, so $Z(G)/G'$ is cyclic.

Let $F \in \mathcal{N}_2$ be presented by

$$F = \langle a, b, c \mid a^4 = b^4 = c^4 = [a, b]^4 = [a, c]^4 = [b, c]^4 = e \rangle.$$

Then $\langle a, b^2, c^2 \rangle \cong G$; yet

$$[b, c]^2 \in \mathrm{dom}_F^{\mathcal{N}_2}(G) \backslash G$$

so G is not absolutely closed.

In fact, we may generalize this example to show that $Z(G)/G'$ being cyclic is no longer sufficient for finitely generated torsion groups of exponent p^n, with $n > 1$. Simply set

$$G = \langle x, y, z \mid x^{p^n} = y^p = z^p = [y, z] = [x, y]^p = [x, z]^p = e \rangle$$

and

$$F = \langle a, b, c \mid a^{p^n} = b^{p^2} = c^{p^2} = [a, b]^{p^2} = [a, c]^{p^2} = [b, c]^{p^2} = e \rangle$$

and identify G with the subgroup generated by a, b^p, and c^p.

Nevertheless, the condition that $Z(G)/G'$ be cyclic is necessary for finitely generated torsion groups:

THEOREM 3.15. *Let $G \in \mathcal{N}_2$ be a finitely generated* (*not necessarily abelian*) *torsion group. If G is absolutely closed in $\mathcal{N}_2$, then $Z(G)/G'$ is cyclic.*

Proof. We may assume that G is a p-group; suppose that $Z(G)/G'$ is not cyclic. We want to show that G is not absolutely closed. It will suffice to show that G does not satisfy (2) or (3) for n a power of p.

Since G is finitely generated, it is of exponent p^α for some $\alpha > 0$. Since $Z(G)/G'$ is not cyclic, there exist $x, y \in Z(G)\backslash G'$ with the property that if $x^a y^b \in G'$ for some integers $a, b \in \mathbb{Z}$, then $x^a \in G'$ and $y^b \in G'$; simply write $Z(G)/G'$ as a sum of cyclic groups, and let x and y be central elements which project to generators of distinct cyclic summands.

Since x and y are both central, then (2) cannot hold for them. Suppose then that (3) holds, for $n = p^\alpha$. Then there exist elements $g_1, g_2 \in G$, and integers $a, b, c \in \mathbb{Z}$, such that

$$g_1^{p^\alpha} \equiv x^a y^b \pmod{G'}$$
$$g_2^{p^\alpha} \equiv x^{b+1} y^c \pmod{G'}.$$

However, $g_1^{p^\alpha} = g_2^{p^\alpha} = e$, hence by choice of x and y, we have that x^a, y^b, x^{b+1}, and y^c all lie in G'.

Since G is a p-group, the orders of x and y modulo G' are nontrivial powers of p. Therefore, we must have that $p|b$ (since $y^b \in G'$), and that $p|b+1$ (since $x^{b+1} \in G'$). This is clearly impossible, so G does not satisfy (3). Therefore, G is not absolutely closed, as claimed. □

Using the ideas above, we can extend Theorem 3.11 to an easy to state characterization for all abelian groups. We start with a technical lemma. Recall that if G is an abelian group, we denote by nG the subgroup of all elements x^n with $x \in G$. For an arbitrary group G, nG denotes the subgroup generated by all such elements.

LEMMA 3.16. *For an abelian group G and a prime number p, the following are equivalent:*

(i) *G/pG is cyclic.*
(ii) *G/p^aG is cyclic for some integer $a > 0$.*
(iii) *G/p^aG is cyclic for all integers $a > 0$.*

Proof. Clearly (iii) implies (ii). Since p^aG is a subgroup of pG, it follows that G/pG is a quotient of G/p^aG, so (ii) implies (i). Finally, note that for any integer $a > 0$, G/p^aG is an abelian group of exponent p^a, hence is a direct sum of cyclic groups of orders p^b, with $1 \leq b \leq a$. Hence G/pG is a direct sum of cyclic groups of order p, with one direct summand for each direct summand in G/p^aG, hence if G/pG is cyclic, then so is G/p^aG for each $a > 0$; so (i) implies (iii). □

The following result was suggested by George Bergman:

THEOREM 3.17. *Let G be an abelian group (not necessarily finitely generated). Then G is absolutely closed in $\mathcal{N}_2$ if and only if for every prime p, G/pG is cyclic.*

Proof. First, suppose that G/pG is cyclic for each prime p. Let K be any over-group of G, and let $x, y \in K$ be such that for some prime p and integer $a > 0$, x^{p^a} and y^{p^a} both lie in $G[K, K]$. We want to show that $[x, y]^{p^a}$ lies in G.

By Lemma 3.16, G/p^aG is cyclic. Let $t \in G$ be such that its image in G/p^aG is a generator for G/p^aG. Let $x', y' \in [K, K]$ be such that $x^{p^a}x', y^{p^a}y' \in G$.

Therefore, there exist $g_1, g_2 \in G$, and $r, s \in \mathbb{Z}$ such that $x^{p^a}x' = t^r g_1^{-p^a}$ and $y^{p^a}y' = t^s g_2^{-p^a}$. In particular, the elements xg_1 and yg_2 of K are such that their p^a-th powers lie in $G[K, K]$; in fact, they lie in $\langle t\rangle[K, K]$. By Lemma 2.7, $[xg_1, yg_2]^{p^a}$ lies in the dominion of $\langle t\rangle$. But by Theorem 3.7, the cyclic subgroup generated by t is absolutely closed, hence

$[xg_1, yg_2]^{p^a}$ lies in $\langle t\rangle$, and so in G. However,

$$\begin{aligned}[xg_1, yg_2]^{p^a} &= [x, y]^{p^a}[x, g_2]^{p^a}[g_1, y]^{p^a}[g_1, g_2]^{p^a}\\ &= [x, y]^{p^a}[x^{p^a}x', g_2][g_1, y^{p^a}y'][g_1, g_2]^{p^a},\end{aligned}$$

and since $g_1, g_2, x^{p^a}x', y^{p^a}y'$, and $[xg_1, yg_2]^{p^a}$ all lie in G, it follows that $[x, y]^{p^a}$ also lies in G, as claimed. This shows that G is absolutely closed.

Conversely, suppose that there exists a prime p such that G/pG is not cyclic. Therefore, G/pG is a direct sum of more than one cyclic group of order p. Let $x, y \in G$ be elements which project to generators of distinct cyclic summands of G/pG. We will show that G, x, y, and $n = p$ do not satisfy (2) nor (3).

Note that neither x nor y have p-th roots in G, and that if a product $x^a y^b$ has a p-th root in G, then necessarily $p|a$ and $p|b$.

Since G is abelian, (2) cannot be satisfied. Suppose, however, that x, y, and p satisfy (3). Therefore, there exist $a, b, c \in \mathbb{Z}$, $g_1, g_2 \in G$ such that

$$\begin{aligned}g_1^p &= x^a y^b\\ g_2^p &= x^{b+1}y^c.\end{aligned}$$

In particular, since $x^a y^b$ and $x^{b+1}y^c$ have p-th roots, $p|b$ and $p|b+1$, which is clearly impossible. Therefore, G, x, y, and n do not satisfy (3) either, so G cannot be absolutely closed.

This proves the theorem. ☐

As in the case of Theorem 3.15, when passing to a more general class of groups, we lose one of the implications:

COROLLARY 3.18. *Let $G \in \mathcal{N}_2$ be a group (not necessarily abelian). If $G/(pG)G'$ is cyclic for all primes p, then G is absolutely closed.*

Proof. The argument above goes through, noting that instead of having equalities $x^{p^a}x' = t^r g_1^{-p^a}$ and $y^{p^a}y' = t^s g_2^{-p^a}$, we obtain congruences modulo $[K, K]$, which is enough for the argument to hold. ☐

Finally, we show that the converse of Corollary 3.18 does not hold:

EXAMPLE 3.19. A group $G \in \mathcal{N}_2$ which is absolutely closed, and for which $G/(3G)G'$ is not cyclic. Let G be the $\mathcal{N}_2$ group presented by

$$G = \langle x, y, z \mid x^3 = y^3 = z^3 = [x, y]^3 = [x, z] = [y, z] = e\rangle.$$

Then G is of exponent 3, and $Z(G)/G'$ is generated by z, hence is cyclic. By Theorem 3.13, G is absolutely closed. Since $3G = \{e\}$,

$$G/(3G)G' \cong G/G' \cong (Z/3Z)^3,$$

so $G/(3G)G'$ is not cyclic. This shows that the condition in Corollary 3.18 is not necessary in general.

REFERENCES

[1] HIGGINS, P. M., *Epimorphisms and amalgams*, Colloq. Math. *56* no. *1* (1988), 1–17. MR:89m:20083

[2] ISBELL, J. R., *Epimorphisms and dominions*. In Proc. of the Conference on Categorical Algebra, La Jolla 1965, (Lange and Springer, New York 1966). MR:35#105a (The statement of the Zigzag Lemma for *rings* in this paper is incorrect. The correct version is stated in [3].)

[3] ISBELL, J. R., *Epimorphisms and dominions IV*, J. London Math. Soc. (2), *1* (1969), 265–273. MR:41#1774

[4] MACHENRY, T., *The tensor product and the 2nd nilpotent product of groups*, Math. Z. *73* (1960), 134–145. MR:22#11027a

[5] MAGIDIN, A., *Dominions in varieties of nilpotent groups*, Comm. Alg. to appear.

[6] MAIER, B. J., *Amalgame nilpotenter Gruppen der Klasse zwei II*, Publ. Math. Debrecen *33* (1986), 43–52. MR:87k:20050

[7] SARACINO, D., *Amalgamation bases for nil-2 groups*, Algebra univers. *16* (1983), 47–62. MR:84i:20035

Oficina 112
Instituto de Matemáticas
Universidad Nacional Autónoma de México
04510 Mexico City
Mexico
e-mail: magidin@matem.unam.mx