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Affine complete varieties are congruence distributive

K. Kaarli[1] and R. McKenzie[2]

Abstract. An algebra is *affine complete* iff its polynomial operations are the same as all the operations over its universe that are compatible with all its congruences. A variety is affine complete iff all its algebras are. We prove that every affine complete variety is congruence distributive, and give a useful characterization of all arithmetical, affine complete varieties of countable type. We show that affine complete varieties with finite residual bound have enough injectives. We also construct an example of an affine complete variety without finite residual bound.

We prove several results concerning residually finite varieties whose finite algebras are congruence distributive, while leaving open the question whether every such variety must be congruence distributive.

1. Introduction

In K. Kaarli, A. Pixley [5] it was announced that all locally finite affine complete (AC) varieties are congruence distributive (CD). The proof of this result, due to R. McKenzie, was never published, although it is an easy application of tame congruence theory (see D. Hobby, R. McKenzie [2]). Since that time, both of us have tried to remove or weaken the assumption of local finiteness in this result. Recently we were able to prove, using different methods, that an AC variety is CD provided it has a finite residual bound. The first author observed that a homomorphically closed class is CD iff all its members have no skew congruences and then reached the result studying the action of unary polynomials. The second author attacked a more general problem: Is a residually finite variety necessarily CD if all of its finite members are CD? He was able to prove this under the assumption that the variety has a finite residual bound.

This paper is the result of combining our two approaches. The main result – every AC variety is CD – falls out rather easily. Later analysis allowed us to prove more. It turns out that all we need to prove congruence distributivity of an algebra **A** is that **A** and all of its homomorphic images are affine com-

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plete and residually finite. Our main result has several important consequences because some basic results about AC varieties were proved earlier under the assumption of congruence distributivity. Recall that by K. Kaarli, A. Pixley [5], all AC varieties are residually finite and every subdirectly irreducible (SI) member of an AC and CD variety has no subalgebras besides itself. Hence McKenzie's result that finite algebras in AC varieties are CD already implied that any SI algebra **A** of any AC variety $\mathcal{V}$ has no proper subalgebras. The latter condition is equivalent to the existence of unary terms $t_1, \ldots, t_n$ in the language of $\mathcal{V}$ such that $A = \{t_1(a), \ldots, t_n(a)\}$ for all $a \in A$ – in other words, the set $T = \{t_1, \ldots, t_n\}$ acts transitively on A. It is possible that some fixed finite set of unary terms acts transitively on all SI's of a variety $\mathcal{V}$. In this case, we say that $\mathcal{V}$ satisfies condition **(S)**.

We prove here that every AC variety of countable type satisfies condition **(S)**. This result sharpens Theorem 3.5 of [5] stating that every AC and CD variety of countable type has a finite residual bound. A well-known result of A. Pixley [7] implies that for an arithmetical variety, the condition **(S)** is sufficient for affine completeness. Hence, an arithmetical variety of countable type is AC if and only if it satisfies **(S)**.

It has been shown in K. Kaarli [4] that all finitely generated AC varieties are term equivalent to varieties of finite type. Obviously, a variety of finite type and finite residual bound is finitely generated. Thus we see that an AC variety is finitely generated iff it is term equivalent to a variety of finite type.

We prove that AC varieties with finite residual bound enjoy some nice categorical properties. They have enough injectives and consequently have the congruence extension and amalgamation properties.

We present an example of an arithmetical AC variety that has a finite residual bound but does not satisfy **(S)**. (Such a variety cannot be of countable type.) A related construction gives an arithmetical AC variety that has no finite residual bound.

The last section of this paper deals with the problem whether a residually finite variety all of whose finite members are CD must be CD. The problem remains open. We present several partial results. In particular, a counter-example cannot have a finite residual bound; and if there exists a counter-example, then there exists one of countable type, all of whose proper subvarieties are CD.

2. Congruence distributivity

The meet and join of two elements a, b of a lattice **L** will be denoted by ab and $a + b$ respectively.

Obviously, if we want to prove that a lattice $\mathbf{L}$ is distributive, it suffices to construct for every pair $\rho, \mu \in L$ with $\rho < \mu$ a lattice homomorphism from $\mathbf{L}$ into a distributive lattice $\mathbf{D}$ which separates ρ and μ. If $\mathbf{L} = \mathbf{Con\,A}$ for some algebra $\mathbf{A}$ then it enjoys the special property that every element is a meet of completely meet-irreducible elements ρ, i.e., congruences ρ such that the quotient algebra $\mathbf{A}/\rho$ is subdirectly irreducible. This property simplifies checking of congruence distributivity. Namely, it is enough to find the separating homomorphisms $\mathbf{L} \to \mathbf{D}$ only for pairs $\langle \rho, \mu \rangle$ where ρ is completely meet-irreducible and μ is its unique cover. Indeed, let $\alpha < \beta$ and choose $\rho \geq \alpha$ so that $\rho \ngeq \beta$ and ρ is completely meet-irreducible with unique cover μ. Then it is easy to see that if a homomorphism f separates ρ and μ, it must also separate α and β.

Often we shall be dealing with a lattice $\mathbf{L}$ that is not assumed to be complete. An element a in such a lattice will be termed *completely meet-irreducible* provided there is an element $b \in L$ (the cover of a) such that b is the smallest element of $\mathbf{L}$ strictly larger than a. Suppose now that ρ is a completely meet-irreducible element of a lattice $\mathbf{L}$, μ is its cover and a lattice homomorphism $f\colon \mathbf{L} \to \mathbf{D}$ separates ρ and μ. If $\mathbf{D}$ is a distributive lattice then there exists also a homomorphism $g\colon \mathbf{L} \to \mathbf{2}$ with the same separating property. Elementary calculations show that there is only one way to define this map: $g(\xi) = 0$ if and only if $\xi \leq \rho$. For sake of easy reference we state the above observations as a separate lemma.

LEMMA 2.1. *Let* $\mathbf{L}$ *be a lattice such that every element of* L *is a meet of completely meet-irreducible elements of* L. *The following conditions are equivalent*:

(1) $\mathbf{L}$ *is distributive*;

(2) *for every* $\rho, \mu \in L$ *where* ρ *is completely meet-irreducible and* μ *covers* ρ *there exists a lattice homomorphism* $\phi\colon \mathbf{L} \to \mathbf{2}$ *which separates* ρ *and* μ;

(3) *for every* $\rho, \beta, \gamma \in L$ *where* ρ *is completely meet-irreducible, if* $\beta\gamma \leq \rho$ *then either* $\beta \leq \rho$ *or* $\gamma \leq \rho$.

Recall that if an algebra $\mathbf{A}$ is included in $\mathbf{B} \times \mathbf{C}$ as a subdirect product and the congruence $\rho \in \mathrm{Con}\,\mathbf{A}$ cannot be represented as a restriction of $\sigma \times \tau$ for some congruences $\sigma \in \mathrm{Con}\,\mathbf{B}$, $\tau \in \mathrm{Con}\,\mathbf{C}$ then ρ is said to be a *skew congruence* of $\mathbf{A}$. Of course the property to be a skew congruence depends on the given subdirect decomposition. The following easy lemma (see [1]) gives a lattice-theoretic criterion for a congruence to be nonskew.

LEMMA 2.2. *Let* $\mathbf{A}$ *be a subdirect product of* $\mathbf{B}$ *and* $\mathbf{C}$ *and let* β *and* γ *be the kernels of projections* $\mathbf{A} \to \mathbf{B}$ *and* $\mathbf{A} \to \mathbf{C}$, *respectively*.

(1) *A congruence* ρ *of* $\mathbf{A}$ *is nonskew with respect to this subdirect decomposition if and only if* $(\beta + \rho)(\gamma + \rho) = \rho$.

(2) *If* $\mathbf{A}/\rho$ *is SI then* ρ *is nonskew with respect to this subdirect decomposition if and only if either* $\beta \leq \rho$ *or* $\gamma \leq \rho$.

A pair of congruences $\langle \beta, \gamma \rangle$ of an algebra **A** is associated with a subdirect decomposition of **A** as above iff $\beta\gamma = 0$ (the zero element of **Con A**). If $\beta\gamma = 0$, then the nonskew congruences with respect to this decomposition are those congruences ρ for which $\rho = (\beta + \rho)(\gamma + \rho)$. Note that if ρ is skew, then some completely meet-irreducible congruence above ρ is skew. Indeed, choose a completely meet-irreducible congruence $\rho' \geq \rho$ such that $\rho' \not\geq (\beta + \rho)(\gamma + \rho)$. Then $\rho' \geq \beta$ would imply $\rho' \geq \beta + \rho$, which is not the case. Likewise, $\rho' \not\geq \gamma$, and then $\rho' < (\beta + \rho')(\gamma + \rho')$ because ρ' is meet-irreducible.

If **A** is a subdirect product of algebras $\mathbf{A}_i$, $i \in I$, and $\mathscr{F}$ is a filter on I then the corresponding filter congruence of **A** will be denoted by $\theta_{\mathscr{F}}$. Recall that the elements $a, b \in A$ are congruent modulo $\theta_{\mathscr{F}}$ iff the set of those $i \in I$ for which a and b have the same projections in A_i is a member of $\mathscr{F}$.

We wish to consider the relationships between several lattice-theoretic properties of an algebra, and we now define them.

(1) **A** is CD; i.e., **Con A** is a distributive lattice.

(2) **A** has no skew congruence for subdirect decompositions into finitely many factors; i.e., whenever $\{\rho, \theta_1, \ldots, \theta_k\} \subseteq \text{Con } \mathbf{A}$ and $\theta_1 \cdots \theta_k = 0$ then $(\theta_1 + \rho)(\theta_2 + \rho) \cdots (\theta_k + \rho) = \rho$.

(3) **A** has no skew congruence for subdirect decompositions into two factors; i.e., whenever $\{\rho, \theta_1, \theta_2\} \subseteq \text{Con } \mathbf{A}$ and $\theta_1 \cdot \theta_2 = 0$ then $(\theta_1 + \rho)(\theta_2 + \rho) = \rho$.

(4) If $\mathbf{A}/\rho$ is SI and $\beta\gamma = 0$ (with $\rho, \beta, \gamma \in \text{Con } \mathbf{A}$) then $\rho \geq \beta$ or $\rho \geq \gamma$.

(J) If **A** is a subdirect product of algebras $\mathbf{A}_i$, $i \in I$, and ρ is a congruence of **A** such that $\mathbf{A}/\rho$ is SI, then there exists an ultrafilter $\mathscr{U}$ on I such that $\theta_{\mathscr{U}} \leq \rho$.

($\mathrm{J_{SI}}$) This is (J) for subdirect products with subdirectly irreducible factors.

If $\mathscr{K}$ is any class of algebras, and (P) is any property of algebras, then we say that $\mathscr{K}$ satisfies (P) iff all members of $\mathscr{K}$ satisfy (P). Our basic observation below will be that if $\mathscr{K}$ is a class of algebras closed under the formation of homomorphic images then all six properties above are equivalent for $\mathscr{K}$, in the sense that if $\mathscr{K}$ satisfies one, it satisfies all. It is important to note however, as we shall, that for a single algebra, there are actually three inequivalent properties among those defined above.

B. Jónsson [3] proved the very important result that (1) implies (J) for any algebra. Every SI algebra satisfies (2), (3) and (4) and hence these properties cannot imply (1). Also, (3) does not imply (2), as the following example shows.

Let **A** be an algebra whose congruence lattice is the lattice pictured in Figure 1.

(There exists such an algebra.) The completely meet-irreducible congruence ρ does not dominate any of α_1, α_2, α_3 yet $\alpha_1\alpha_2\alpha_3 = 0_A$. Nevertheless, it can be checked that whenever $\beta\gamma = 0_A$ in **Con A** and σ is any one of the four completely meet-irreducible elements of **Con A**, then $\sigma \geq \beta$ or $\sigma \geq \gamma$. Thus **A** satisfies (3) and (4) but not (2).

LEMMA 2.3. *For any algebra* **A** *we have*

(1) $\Rightarrow$ (2) $\Leftrightarrow$ (J) $\Leftrightarrow$ ($\mathrm{J_{SI}}$) $\Rightarrow$ (3) $\Leftrightarrow$ (4).

Proof. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Leftrightarrow$ (4) should be obvious from the preceding discussions, and (J) $\Rightarrow$ ($\mathrm{J_{SI}}$) is trivial. The proof that (2) implies (J) is due to B. Jónsson.

To prove that ($\mathrm{J_{SI}}$) $\Rightarrow$ (2), assume that (2) fails. Then **A** has, actually, a congruence ρ such that $\mathbf{A}/\rho$ is SI and for some congruences $\theta_1, \ldots, \theta_k$, where $\theta_1 \cdots \theta_k = 0$, the inclusion $\theta_i \leq \rho$ holds for no i. Each of the congruences θ_i is the meet of some completely meet-irreducible congruences, and in fact there must exist, for some set T presented as a union of subsets T_i, $i \in \{1, \ldots, k\}$, completely meet-irreducible congruences ρ_t, $t \in T$, such that

$$\theta_i = \bigwedge_{t \in T_i} \rho_t \qquad \text{for each } i \in \{1, \ldots, k\}.$$

Setting $\mathbf{A}_t = \mathbf{A}/\rho_t$, we have that **A** is a subdirect product of the SI algebras $\mathbf{A}_t$. If $\mathscr{U}$ is any ultrafilter on T, then some one of the sets T_i belongs to $\mathscr{U}$, implying that $\theta_i = \bigwedge_{t \in T_i} \rho_t \leq \theta_{\mathscr{U}}$ and so $\theta_{\mathscr{U}} \leq \rho$ must be false. Thus we have shown that when (2) fails then ($\mathrm{J_{SI}}$) fails also. $\square$

LEMMA 2.4. *Let* $\mathscr{K}$ *be a class of algebras closed with respect to homomorphic images. If* $\mathscr{K}$ *has any one of the properties* (1), (2), (3), (4), (J), ($\mathrm{J_{SI}}$) *then it has all of them.*

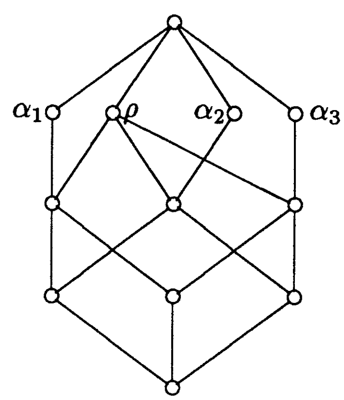

Figure 1

Proof. In view of Lemma 2.3, all that is required is a proof that if $\mathscr{K}$ satisfies (4) then it satisfies (1). Let $\mathbf{A} \in \mathscr{K}$ and assume that all quotient algebras of $\mathbf{A}$ satisfy (4). We show that $\mathbf{L} = \mathbf{Con\, A}$ satisfies condition (3) of Lemma 2.1, thus establishing that $\mathbf{Con\, A}$ is distributive. Thus, assume that $\rho, \beta, \gamma \in L$, ρ is completely meet-irreducible, and $\beta\gamma \leq \rho$.

Now in the algebra $\mathbf{B} = \mathbf{A}/\beta\gamma$, the congruence $\rho' = \rho/\beta\gamma$ is completely meet-irreducible – i.e., $\mathbf{B}/\rho'$ is SI – and $\beta'\gamma' = 0_B$ where $\beta' = \beta/\beta\gamma$ and $\gamma' = \gamma/\beta\gamma$. By condition (4) for $\mathbf{B}$, we have, say, $\beta' \leq \rho'$. This is equivalent to $\beta \leq \rho$, which shows that $\mathbf{L}$ satisfies condition (3) of Lemma 2.1 and hence is a distributive lattice. □

COROLLARY 2.1. *If $\mathscr{V}$ is a locally finite variety and the finite members of $\mathscr{V}$ have no skew congruences then $\mathscr{V}$ is CD.*

Now we prove a useful lemma showing how the existence of skew congruences can in certain situations be ruled out by assumptions weaker than congruence distributivity. We need to introduce the notion of Jónsson terms and the classical result of B. Jónsson concerning such terms.

DEFINITION 2.1. A finite sequence $t_0(x, y, z), \dots, t_n(x, y, z)$ of terms in three variables in the language of a variety $\mathscr{V}$ is called a sequence of *Jónsson terms* for $\mathscr{V}$ provided that the equations $t_0(x, x, z) \approx x \approx t_i(x, y, x) \approx t_n(z, x, x)$ (for all $i \leq n$) and the equations $t_{2j}(x, y, y) \approx t_{2j+1}(x, y, y)$ (for $0 \leq 2j < n$) and $t_{2j+1}(x, x, y) \approx t_{2j+2}(x, x, y)$ (for $0 \leq 2j \leq n-2$) are laws of $\mathscr{V}$. The first order sentence which asserts that all of these equations are valid for all x, y, z is called a *Jónsson sentence*. A sequence of Jónsson terms for an algebra $\mathbf{A}$ is just a sequence of terms in three variables such that the corresponding Jónsson sentence holds in $\mathbf{A}$ (or equivalently, the sequence is a Jónsson sequence for the variety generated by $\mathbf{A}$).

LEMMA 2.5. (B. Jónsson [3]) *A variety is congruence distributive iff it has a sequence of Jónsson terms.*

LEMMA 2.6. *Let* $\rho, \beta, \gamma \in \mathrm{Con}\, \mathbf{A}$ *where* $\mathbf{A}$ *is any algebra.*

(1) *If* $\mathbf{A}/\rho$ *generates a CD variety then* $(\rho + \beta)(\rho + \gamma) = \rho + \beta(\rho + \gamma)$.

(2) *If* $\mathbf{A}/(\rho\beta)$ *generates a CD variety then* $(\rho + \beta)(\rho + \gamma) = \rho + \beta\gamma$.

Proof. (1) Obviously $(\rho + \beta)(\rho + \gamma) \geq \rho + \beta(\rho + \gamma)$. Assume that $\langle a, b \rangle \in (\rho + \beta)(\rho + \gamma)$, in particular there are $c_0, c_1, \dots, c_m \in A$ such that $a = c_0$, $c_m = b$ and $\langle c_{i-1}, c_i \rangle \in \rho \cup \beta$ for every $i = 1, \dots, m$.

Let $t_0, t_1, \dots, t_n$ be Jónsson terms for $\mathbf{A}/\rho$ and define the elements $e_{ij} = t_i(a, c_j, b)$ where $i = 0, 1, \dots, n$ and $j = 0, 1, \dots, m$. Now consider the sequence

$$a, e_{00}, e_{01}, \ldots, e_{0m}, e_{1m}, e_{1,m-1}, \ldots, e_{10}, e_{20}, \ldots, e_{nm}, b. \tag{1}$$

Obviously $e_{00} = t_0(a, a, b)\rho a$, $e_{nm} = t_n(a, b, b)\rho b$ and all pairs of the form $\langle e_{i,j-1}, e_{ij}\rangle$ are in $\rho \cup \beta$. As for the pairs of form $\langle e_{i-1,m}, e_{im}\rangle$ with i even and $\langle e_{i-1,0}, e_{i0}\rangle$ with i odd, they all are in ρ by the definition of Jónsson terms. On the other hand, if $\theta = \mathrm{Cg}(a, b)$ then $\theta \le \rho + \gamma$ and $e_{ij} = t_i(a, c_j, b)\theta t_i(a, c_j, a)\rho a$ which implies $\langle e_{ij}, a\rangle \in \rho + \gamma$ for every i and j. This means that all members of the sequence (1) are contained in a single $(\rho + \gamma)$-block. Now the adjacent members of this sequence are congruent either modulo $\rho(\rho + \gamma) = \rho$ or modulo $\beta(\rho + \gamma)$. Hence $\langle a, b\rangle \in \rho + \beta(\rho + \gamma)$.

(2) In view of the first statement of the lemma it suffices to prove the inequality $\beta(\rho + \gamma) \le \beta\rho + \beta\gamma$. This can be done in a fairly similar way as the proof of the preceding statement. We take an arbitrary pair $\langle a, b\rangle \in \beta(\rho + \gamma)$ and pick the elements $c_0, c_1, \ldots, c_m \in A$ such that $a = c_0$, $c_m = b$ and $\langle c_{i-1}, c_i\rangle \in \rho \cup \gamma$ for every $i = 1, \ldots, m$. Then we take Jónsson terms $t_0, t_1, \ldots, t_n$ for $\mathbf{A}/(\rho\beta)$ and form again the sequence (1). Now $e_{00} = t_0(a, a, b)\rho\beta a$, $e_{nm} = t_n(a, b, b)\rho\beta b$ and all pairs of the form $\langle e_{i,j-1}, e_{ij}\rangle$ are in $\rho \cup \gamma$. As for the pairs of form $\langle e_{i-1,m}, e_{im}\rangle$ with i even and $\langle e_{i-1,0}, e_{i0}\rangle$ with i odd, they all are in $\rho\beta$ by the definition of Jónsson terms. On the other hand, if $\theta = \mathrm{Cg}(a, b)$ then $\theta \le \beta$ and $e_{ij} = t_i(a, c_j, b)\theta t_i(a, c_j, a)\rho\beta a$ which implies $\langle e_{ij}, a\rangle \in \beta$ for every i and j. This means that all members of the sequence (1) are contained in a single β-block. Now the adjacent members of this sequence are congruent either modulo $\beta\rho$ or modulo $\beta\gamma$. Hence $\langle a, b\rangle \in \beta\rho + \beta\gamma$. □

COROLLARY 2.2. *Let* $\mathbf{A}$ *be a subdirect product in* $\mathbf{B} \times \mathbf{C}$. *If* $\mathrm{Var}(\mathbf{A}/\rho, \mathbf{B})$ *is CD for every* $\rho \in \mathrm{Con}\,\mathbf{A}$ *with SI* $\mathbf{A}/\rho$ *then* $\mathbf{A}$ *has no skew congruences with respect to the given subdirect decomposition.*

Proof. Let β and γ be the kernels of the projections $\mathbf{A} \to \mathbf{B}$ and $\mathbf{A} \to \mathbf{C}$, respectively. Then whenever $\mathbf{A}/\rho$ is SI we have that $\mathbf{A}/(\rho\beta)$ generates a CD variety and by Lemma 2.6 $(\rho + \beta)(\rho + \gamma) = \rho + \beta\gamma = \rho$, since $\beta\gamma = 0_A$. Now if δ is any congruence of $\mathbf{A}$ then we have $\delta = \bigwedge_{i \in I} \rho_i$ where all $\mathbf{A}/\rho_i$ are SI. Thus

$$(\delta + \beta)(\delta + \gamma) \le (\rho_i + \beta)(\rho_i + \gamma) = \rho_i \qquad \text{for all } i$$

and consequently $(\delta + \beta)(\delta + \gamma) = \delta$. □

3. The compatible function lifting property

By a *compatible function* on the algebra $\mathbf{A}$ we mean a finitary operation over the universe of the algebra which respects all the congruences of the algebra. If $\mathbf{A}$ is any member of an AC variety and $\rho \in \mathrm{Con}\,\mathbf{A}$ then any compatible function g on $\mathbf{A}/\rho$ has a compatible lift on $\mathbf{A}$, i.e., a compatible function f on $\mathbf{A}$ that induces g on $\mathbf{A}/\rho$. This is obvious from the fact that g is a polynomial function. We generalize this property as follows. Let A be any set and $\mathbf{L}$ be a sublattice of $\mathbf{Eqv}\,A$. Thus $\mathbf{L}$ is a lattice of equivalence relations on A. Throughout this paper, sublattices of $\mathbf{Eqv}\,A$ are assumed to contain the least and largest members of $\mathbf{Eqv}\,A$. We attach to the pair $\langle A, \mathbf{L}\rangle$ the clone F of all $\mathbf{L}$-compatible function on A. Given any $\rho \in L$, we have a pair $\langle A/\rho, \mathbf{L}_\rho\rangle$ where $\mathbf{L}_\rho$ is the lattice of all equivalences of A/ρ that have the form σ/ρ where $\sigma \in L$, $\rho \leq \sigma$. Letting F_ρ denote the clone of all $\mathbf{L}_\rho$-compatible functions for the pair $\langle A/\rho, \mathbf{L}_\rho\rangle$, there is a canonical mapping $F \to F_\rho$ that takes any function f to the function it induces modulo ρ. In what follows, the members of F_ρ will be referred to as the compatible functions on A/ρ.

DEFINITION 3.1. The pair $\langle A, \mathbf{L}\rangle$ is said to satisfy the *compatible function lifting property* **(CFL)** iff all the canonical mappings $F \to F_\rho$, $\rho \in L$, are surjective.

It turns out that this property together with residual finiteness implies the distributivity of the lattice $\mathbf{L}$. In proving this, we first treat the finite case and use for this a modification of the second author's original argument for proving that locally finite AC varieties are CD. The proof should be understandable for readers not conversant with tame congruence theory.

We need the notion of induced algebra. Let $\mathbf{A}$ be any algebra and U be a subset of its universe. Let P_U denote the set of all polynomial operations of $\mathbf{A}$ under which U is closed, and let $P_U|_U$ denote the set of all restrictions to U of the operations in P_U. Then the algebra $\langle U, P_U|_U\rangle$ is the *algebra induced on U by* $\mathbf{A}$. We shall need the following elementary lemma that follows by applying the well-known lemma of P. P. Pálfy and P. Pudlák (see [2]) to the algebra induced by $\mathbf{A} = \langle A, e\rangle$ on the subset $U = e(A)$.

LEMMA 3.1. *Let A be a set and* $\mathbf{L}$ *a sublattice of* $\mathbf{Eqv}\,A$. *If e is a unary idempotent* $\mathbf{L}$*-compatible function on A and $U = e(A)$ then the restriction map is a surjective lattice homomorphism from* $\mathbf{L}$ *to* $\mathbf{L}|_U$, *the sublattice of the lattice* $\mathbf{Eqv}\,U$ *with universe $L|_U = \{\rho|_U \mid \rho \in L\}$.*

Our proof is based on the next lemma.

LEMMA 3.2. *Let A be a set and* $\mathbf{L}$ *a sublattice of* $\mathbf{Eqv}\,A$ *such that the pair $\langle A, \mathbf{L}\rangle$ satisfies* **(CFL)**. *If e is a unary idempotent* $\mathbf{L}$*-compatible function on A and $U = e(A)$ then the pair $\langle U, \mathbf{L}|_U\rangle$ also satisfies* **(CFL)**.

Proof. Let $\rho \in L$ and let g be an m-ary compatible function for the pair $\langle U/\rho|_U, (\mathbf{L}|_U)_{\rho|_U}\rangle$. Consider the function $f: (A/\rho)^m \to A/\rho$ defined via $f(a_1/\rho, \ldots, a_m/\rho) = g(e(a_1)/\rho, \ldots, e(a_m)/\rho)$. Clearly f is a compatible function on A/ρ. Hence, by **(CFL)** there exists a function $h \in F$ such that $h(a_1, \ldots, a_m)/\rho = f(a_1/\rho, \ldots, a_m/\rho)$ for all $a_1, \ldots, a_m \in A$. In particular, for $u_1, \ldots, u_m \in U$ we have $h(u_1, \ldots, u_m)/\rho = g(u_1/\rho, \ldots, u_m/\rho) = u/\rho$ for some $u \in U$. Thus, $eh(u_1, \ldots, u_m)/\rho = e(u)/\rho = u/\rho$ and the function $eh: U^m \to U$ is a compatible lift for g. □

THEOREM 3.1. *Let A be a finite set and* $\mathbf{L}$ *a sublattice of* **Eqv** *A. If the pair $\langle A, \mathbf{L}\rangle$ satisfies* **(CFL)** *then the lattice* $\mathbf{L}$ *is distributive.*

Proof. Our proof is by induction on the size of A. The claim is obvious if $|A| = 1$. Assume that our claim is valid for all sets of size less than $|A|$. Denote by Φ the smallest congruence on $\mathbf{L}$ with $\mathbf{L}/\Phi$ distributive. We prove that $\langle \rho, \mu\rangle \notin \Phi$ for all ρ and μ in L such that ρ is meet-irreducible and ρ is covered by μ. Then Lemma 2.1 applies.

Let F be the clone of all $\mathbf{L}$-compatible functions on A. First consider the case with $|A/\rho| \geq 3$. Let $a, b \in A$ be such that $\langle a, b\rangle \in \mu - \rho$. Choose an arbitrary idempotent surjective function $g: A/\rho \to \{a/\rho, b/\rho\}$. Obviously $g \in F_\rho$, hence by the property **(CFL)** it is induced by some function $f \in F$; i.e., $g(x/\rho) = f(x)/\rho$ for all $x \in A$. Since A is finite, an appropriate power of f is idempotent. Denote this idempotent function by e and let $U = e(A)$. Obviously $U/\rho = \{a/\rho, b/\rho\}$, hence $U \neq A$. Denote the restriction mapping $L \to L|_U$ by π_ρ. By Lemma 3.1, π_ρ maps $\mathbf{L}$ homomorphically onto $\mathbf{L}|_U$ and separates ρ and μ. Applying Lemma 3.2 and the induction hypothesis, we have that the lattice $\mathbf{L}|_U$ is distributive. Since $\Phi \leq \operatorname{Ker} \pi_\rho$, we have $\langle \rho, \mu\rangle \notin \Phi$.

The assumption $|A/\rho| \geq 3$ was needed in order to construct a unary idempotent nonidentical function $f \in F$ separating ρ and μ. Hence we must handle separately the case with $|A/\rho| = 2$ and where the only unary idempotent function $f \in F$ separating ρ and μ $(= A \times A)$ is the identity function. We shall show that this is possible only if A is a two-element set, which obviously yields that $\mathbf{L}$ is distributive. The argument to show this comes from tame congruence theory.

Let $f(x, y) \in F$ be a binary function on A which induces a semilattice operation on A/ρ. Any function on a two-element set is compatible, so by condition **(CFL)** there exists such a function $f(x, y)$.

Let X and Y be the ρ-equivalence classes and assume that $f(X, Y) \subseteq Y$. Putting $g(x) = f(x, x)$, we have that $g \in F$ and an appropriate power g^n is idempotent. Note that $g(X) \subseteq X$ and $g(Y) \subseteq Y$, hence g^n separates ρ from μ and by our assumption is the identity function. Put $h(x, y) = g^{n-1}(f(x, y))$. Now h also induces the semilattice operation and $h(x, x) = x$.

Now put $k(x, y) = h(x, h(x, \ldots, h(x, y)) \cdots))$ where h occurs $m!$ times with $m = |A|$. Then we have that $k(x, k(x, y)) = k(x, y)$ for all $x, y \in A$. Also, k induces the semilattice operation on A/ρ and $k(x, x) = x$. Moreover, for all b in X, z in A we have $k(b, z) = z$ – this is because the map $z \mapsto k(b, z)$ is idempotent and maps X into X and Y into Y.

Now put $p(x, y) = k(k(\cdots k((x, y), y) \cdots), y)$ so that $p(p(x, y), y) = p(x, y)$. Then p induces the semilattice operation on A/ρ and $p(b, x) = x = p(x, b)$ for all $b \in X$ and $x \in A$. From these equations it follows that $b_1 = p(b_1, b_2) = b_2$ for all $b_1, b_2 \in X$, hence $|X| = 1$. The equality $|Y| = 1$ can be proved by similar arguments. □

COROLLARY 3.1. *Let A be a finite set and* **L** *a sublattice of* **Eqv** *A. If the pair $\langle A, \mathbf{L}\rangle$ satisfies* **(CFL)** *then the following are true*:

(1) *the clone F of all* **L**-*compatible functions contains a near unanimity function*;
(2) $\mathbf{L} = \mathbf{Con\,A}$ *where* $\mathbf{A} = \langle A, F\rangle$;
(3) **A** *generates an affine complete variety*.

Proof. Since **L** is distributive, by R. Quackenbush, B. Wolk [8] the congruence lattice of the algebra **A** coincides with **L**. Hence **A** has all its quotient algebras affine complete and by Theorem 3.2 from [4] F contains a near unanimity function. Then, in particular, the variety $\mathscr{V}$ generated by **A** is CD. Moreover, since F contains all constants, the **A** has no proper subalgebras and so by Jónsson's lemma every SI of $\mathscr{V}$ is a homomorphic image of **A**. If $\mathbf{A}/\rho_1, \ldots, \mathbf{A}/\rho_n$ are all SI homomorphic images of some $\mathbf{B} \in \mathscr{V}$ then **B** belongs to a CD variety generated by $\mathbf{A}/\rho$ where $\rho = \rho_1 \cdots \rho_n$. Now $\mathbf{A}/\rho$ and all its quotients are affine complete and **B** contains a canonical isomorphic copy of $\mathbf{A}/\rho$ (since **A** has no proper subalgebras). Hence by Theorem 4.1 from [4] the algebra **B** is affine complete. This proves that the variety $\mathscr{V}$ is affine complete. □

Now we prove the infinite version of Theorem 3.1. We call a pair $\langle A, \mathbf{L}\rangle$ where **L** is a lattice of equivalence relations on A *residually finite* if every $\rho \in L$ is the intersection of equivalence relations $\sigma \in L$ of finite index. The latter means that A/σ is a finite set.

LEMMA 3.3. *Let A be a set*, **L** *be a sublattice of* **Eqv** *A, F be the clone of all* **L**-*compatible functions and $\mathbf{A} = \langle A, F\rangle$. Assume that the pair $\langle A, \mathbf{L}\rangle$ is residually finite and satisfies* **(CFL)**. *Let $\beta, \rho, \mu \in L$ be such that $\beta \nleq \rho$, $\beta\rho = 0$, ρ is completely meet-irreducible in* **L** *and μ is its cover in* **L**. *Then, given any two-element set U contained in one μ/ρ-block of $\mathbf{A}/\rho$, there exists a unary function $f \in F$ which induces a bijective function on U and a constant function on A/β.*

Proof. Let $a_1, a_2 \in A$ be such that $\langle a_1, a_2 \rangle \in \beta - \rho$. Since $\beta\rho = 0$, we have the canonical subdirect embedding $A \leq A/\rho \times A/\beta$. Hence we may write $a_i = \langle b_i, c \rangle$, for some $b_i \in A/\rho$, $c \in A/\beta$, $i = 1, 2$.

Since ρ is completely meet-irreducible and $\langle A, \mathbf{L} \rangle$ is residually finite, ρ is of finite index. Clearly, the induced pair $\langle A/\rho, \mathbf{L}_\rho \rangle$ satisfies **(CFL)**, hence by Corollary 3.1 the congruence lattice of the algebra $\mathbf{A}/\rho$ is exactly $\mathbf{L}_\rho$. In particular, μ/ρ is the monolithic congruence of $\mathbf{A}/\rho$. Let e be an idempotent unary member of F_ρ such that $e(\{b_1, b_2\}) = U$. Denote $e(b_i) = u_i$, $i = 1, 2$. By condition **(CFL)**, the function e is induced by some $f \in F$. Then $f(a_i) = \langle u_i, d \rangle$ for some $d \in A/\beta$, $i = 1, 2$.

Now consider the function $h = e \times g$ on $A/\rho \times A/\beta$ where g is a constant function: $g(A/\beta) = \{d\}$. Clearly $h(A) \subseteq A$. We prove that the restriction of h to A belongs to F by showing that this function (which we also denote by h) preserves all $\sigma \in L$ of finite index. For any such σ, the equivalence relation $\rho\sigma \in L$ is of finite index, too. Hence there exists a near unanimity term for $\mathbf{A}/\rho\sigma$ and this algebra generates a CD variety. Hence by Lemma 2.6, the equivalence σ cannot be skew with respect to the decomposition $A \leq A/\rho \times A/\beta$. However, the function h obviously preserves all non-skew equivalences $\tau \in L$. □

THEOREM 3.2. *Let A be a set and $\mathbf{L}$ a sublattice of* $\mathbf{Eqv}\,A$. *If the pair $\langle A, \mathbf{L} \rangle$ is residually finite and satisfies* **(CFL)**, *then the lattice $\mathbf{L}$ is distributive*.

Proof. Let F be the clone of all $\mathbf{L}$-compatible functions on A. Thus all $\rho \in L$ are congruences of the algebra $\mathbf{A} = \langle A, F \rangle$.

We are going to prove that the lattice $\mathbf{L}$ satisfies condition (3) from Lemma 2.1. (It is true that every member of L is the meet of completely meet-irreducible elements, due to the fact that every member is the meet of elements of finite index.) Thus, let $\rho, \beta, \gamma \in L$ be such that ρ is completely meet-irreducible and $\beta\gamma \leq \rho$.

Let μ be the cover of ρ in $\mathbf{L}$ and $U = \{u_1, u_2\}$ be a two-element set contained in one μ/ρ-block of $\mathbf{A}/\rho$. Assume that $\beta \not\leq \rho$, i.e., there exists a pair $\langle a_1, a_2 \rangle \in \beta - \rho$. Then by Lemma 3.3 there exists a function $f \in F_{\rho\beta}$ which induces a bijection on U and a constant map on A/β. Similarly, if $\gamma \not\leq \rho$, then there exists a function $g \in F_{\rho\gamma}$ which induces a bijection on U and a constant map on A/γ. By condition **(CFL)**, the functions f and g are induced by some functions $f_1, g_1 \in F$, respectively. Now the composed function $f_1 g_1$ still induces a bijection on U but a constant map on $A/\beta\gamma$. Hence $f_1 g_1$ induces a constant on A/ρ as well, since $\beta\gamma \leq \rho$. Since $U \subseteq A/\rho$, this is a contradiction. □

4. Affine complete varieties

An immediate corollary of Theorem 3.2 is that every algebra all of whose homomorphic images are residually finite and affine complete, is congruence distributive. For varieties, this yields the main result of our paper. We summarize in the next theorem three important properties of affine complete varieties, partly known earlier.

THEOREM 4.1. *Every affine complete variety is congruence distributive and residually finite, and its subdirectly irreducible members have no proper sub-algebras.*

Proof. The residual finiteness of AC varieties was proved in K. Kaarli and A. Pixley [5]. Hence it follows from Theorem 3.2 that AC varieties are CD. In [5] it was proved that an SI member of an AC and CD variety has no subalgebras other than itself. Since an SI in an AC variety is finite, it generates a locally finite AC variety which was known to be CD due to the unpublished result of the second author. Thus it was known, and mentioned in several papers, that the SI members of any AC variety have no proper subalgebras. □

That an algebra **A** has no proper subalgebras is equivalent to the condition that the set of unary term functions acts transitively on A, i.e., for every $a, b \in A$ there exists a unary term t such that $t(a) = b$. If **A** is finite, this condition takes the form $\mathbf{A} \models \Phi(T)$ where $T = \{t_1, \ldots, t_n\}$ is some finite set of unary terms and $\Phi(T)$ is the sentence

$$(\forall x, y)\{t_1(x) = y \vee \cdots \vee t_n(x) = y\}.$$

Now we introduce a condition which plays a significant role in the theory of affine complete varieties.

DEFINITION 4.1. We say that a variety $\mathscr{V}$ satisfies condition **(S)** if there exists a finite set T of unary terms such that all subdirectly irreducible members of $\mathscr{V}$ satisfy the sentence $\Phi(T)$.

Obviously, condition **(S)** implies that all SI members of the variety have at most n elements and no proper subalgebras. In the arithmetical case, this condition is sufficient for affine completeness.

THEOREM 4.2. *Every arithmetical variety with property* **(S)** *is affine complete.*

Proof. Assume that the unary terms $t_1, \ldots, t_n$ witness **(S)** for $\mathscr{V}$ (so that $\mathscr{V}$ is residually $\leq n$). Let f be an m-ary compatible function for the algebra $\mathbf{A}$ in $\mathscr{V}$. We can assume that $\mathbf{A}$ is subdirect in $\Pi\{\mathbf{A}_i : i \in I\}$ where the $\mathbf{A}_i$ are SI. Let $c \in A$ and put $C = \{t_1(c), \ldots, t_n(c)\}$.

Because $\mathscr{V}$ is arithmetical and C is finite, there is an m-ary polynomial g of $\mathbf{A}$ which agrees with f on C. Because of condition **(S)**, the set C projects onto every A_i. Then, since both g and f are compatible for $\mathbf{A}$, it easily follows that $g(x_1, \ldots, x_m) = f(x_1, \ldots, x_m)$ for all $x_1, \ldots, x_m \in A$. □

Now it is natural to ask whether the condition **(S)** is satisfied in all AC varieties. As we shall see later, this is not the case. However, in some important classes of AC varieties, the condition does hold.

THEOREM 4.3. *Suppose that $\mathscr{V}$ is an affine complete variety whose class of subdirectly irreducible members is first order definable. Then $\mathscr{V}$ has property* **(S)**.

Proof. Assume that Ψ is a first order sentence in the language of $\mathscr{V}$ so that an algebra in $\mathscr{V}$ is SI iff it satisfies Ψ. Let T_i, $i \in I$, be the collection of all finite sets of unary terms in the language of $\mathscr{V}$ and consider the set of first order sentences

$$\mathrm{Id}(\mathscr{V}) \cup \{\Psi\} \cup \{\neg\Phi(T_i) \mid i \in I\}.$$

This set is inconsistent. Indeed, if it were not then it would have a model $\mathbf{A}$, which would be an SI algebra in $\mathscr{V}$ that fails to satisfy any of the sentences $\Phi(T_i)$. This contradicts Theorem 4.1.

Hence by the logical compactness theorem there is a finite set $\{i_1, \ldots, i_k\}$ such that the set

$$\mathrm{Id}(\mathscr{V}) \cup \{\Psi, \neg\Phi(T_{i_1}), \ldots, \neg\Phi(T_{i_k})\}$$

is inconsistent. This means that any SI algebra $\mathbf{A} \in \mathscr{V}$ satisfies at least one of the sentences $\Phi(T_{i_j})$ where $1 \leq j \leq k$. Obviously then, every SI in $\mathscr{V}$ satisfies the sentence $\Phi(T_{i_1} \cup \cdots \cup T_{i_k})$. □

COROLLARY 4.1. *A discriminator variety is affine complete if and only if it has property* **(S)**.

Proof. Recall that discriminator varieties are those having a term $t(x, y, z)$ so that an algebra in $\mathscr{V}$ is subdirectly irreducible iff it satisfies the first order sentence

$$(\forall x, y, z)\{t(x, x, z) = z \& (x \neq y \rightarrow t(x, y, z) = x)\}.$$

Now the necessity is immediate from Theorem 4.3. Since discriminator varieties are known to be arithmetical, the sufficiency follows from Theorem 4.2. □

THEOREM 4.4. *Every affine complete variety of countable type has property* **(S)**.

Proof. Let $\mathscr{V}$ be an AC variety of countable type and t_n, $n \in \omega$, be a list of all the unary terms in the language of $\mathscr{V}$. First assume, to obtain a contradiction, that for every n there is an SI algebra $\mathbf{A}_n \in \mathscr{V}$ and an element $a_n \in \mathbf{A}_n$ for which the set $\{t_0(a_n), \ldots, t_n(a_n)\}$ does not intersect any of the nontrivial equivalence classes of the monolithic congruence μ_n of $\mathbf{A}_n$.

We choose b_n, c_n in $\mathbf{A}_n$ so that $b_n \neq c_n$ and $\langle b_n, c_n \rangle \in \mu_n$. Let

$$\mathbf{P} = \prod\{\mathbf{A}_n \times \mathbf{A}_n : n \in \omega\}$$

and a be the element $\langle a_0, a_0, a_1, a_1, \ldots \rangle$ of $\mathbf{P}$. A general element $x \in P$ will be denoted as $\langle x(0), x(1), x(2), \ldots \rangle$ so that for example $a(2n) = a(2n+1) = a_n$ for every $n \in \omega$. We take $\mathbf{D}$ to be the subalgebra of $\mathbf{P}$ generated by a, which consists of all the elements $t_n^{\mathbf{P}}(a)$ for $n \in \omega$. Then we take $\mathbf{D}'$ to be the set of all elements of $\mathbf{P}$ which for some $x \in \mathbf{D}$ differ only at finitely many places from x. For every n, let f_n be the identity function on A_n and take g_n to be the function that differs from f_n only at c_n and has $g_n(c_n) = b_n$. Both f_n and g_n are compatible functions of $\mathbf{A}_n$. Let

$$F = f_0 \times g_0 \times f_1 \times g_1 \times \cdots \times f_n \times g_n \cdots,$$

a function mapping D' into P. Since for every $x \in D'$ there is an n such that for all sufficiently large k,

$$x(2k) = x(2k+1) = t_n(a_k) \neq c_k$$

it follows that $F(x)$ eventually agrees with x and so $F(x) \in D'$.

The function F is a compatible function of $\mathbf{D}'$. Indeed, let $x, y \in D'$ and choose m large enough so that $F(x)$ agrees with x and $F(y)$ agrees with y at $2m$ and at all later places. Identifying $\mathbf{D}'$ with a subdirect product of $\mathbf{A}_0 \times \mathbf{A}_0 \times \cdots \times \mathbf{A}_{m-1} \times \mathbf{A}_{m-1}$ and a certain subalgebra $\mathbf{D}_m$ of $\mathbf{A}_m \times \mathbf{A}_m \times \cdots$, we have that

$$x = \langle x_0, \ldots, x_{2m-1}, u\rangle, \qquad F(x) = \langle x_0, g_0(x_1), \ldots, g_{m-1}(x_{2m-1}), u\rangle,$$

$$y = \langle y_0, \ldots, y_{2m-1}, v\rangle, \qquad F(y) = \langle y_0, g_0(y_1), \ldots, g_{m-1}(y_{2m-1}), v\rangle.$$

This subdirect decomposition of $\mathbf{D}'$ into $2m+1$ factors has no skew congruences (since the variety $\mathscr{V}$ is CD). From this, together with the fact that each of the coordinate functions $f_0, g_0, f_1, g_1, \ldots, f_{m-1}, g_{m-1}$ of F is compatible, it should be clear that F is a compatible function of $\mathbf{D}'$. However, if $G(x) = t(x, d^1, \ldots, d^r)$ is any polynomial function of $\mathbf{D}'$ where t is a term and $\{d^1, \ldots, d^r\} \subseteq D'$, and if we choose m large enough so that for $k \geq m$ we have

$$\langle d^1(2k), \ldots, d^r(2k)\rangle = \langle d^1(2k+1), \ldots, d^r(2k+1)\rangle$$

then the coordinate functions of $G(x)$ at the $2k$ and $2k+1$ coordinates will be equal when $k \geq m$. Since $F(x)$ does not have this property, it follows that $G(x)$ cannot be identical with $F(x)$ on D'. This contradicts the fact that the algebra $\mathbf{D}'$ is affine complete.

The contradiction establishes that there exists an integer M such that whenever a belongs to an SI algebra $\mathbf{A}$ in $\mathscr{V}$ then among $t_0(a), \ldots, t_M(a)$ there is an element which lies in a nontrivial class of the monolith of $\mathbf{A}$. All that remains is to show that there is an integer N so that whenever a lies in a nontrivial monolith class for an SI algebra $\mathbf{A}$ in $\mathscr{V}$ then $A = \{t_0(a), \ldots, t_N(a)\}$ for then the terms $t_i(t_j(x))$ $(i \leq N,\ j \leq M)$ will satisfy the requirement of the theorem.

Assume to the contrary that for all n there is an SI algebra $\mathbf{A}_n$ in $\mathscr{V}$ and distinct elements a_n, b_n congruent modulo the monolith of $\mathbf{A}_n$ and an element $c_n \in A_n - \{t_0(a_n), \ldots, t_n(a_n)\}$. We form $\mathbf{P}$, $\mathbf{D}$ and $\mathbf{D}'$ just as before. Here we take f_n and g_n to be the compatible functions of $\mathbf{A}_n$ such that $f_n(x) = a_n$ is constant and $g_n(x) = a_n$ for all x except $x = c_n$ where $g_n(c_n) = b_n$. Exactly the same argument as before reaches precisely the same contradiction. □

COROLLARY 4.2. *Every affine complete variety of countable type has a finite residual bound.*

COROLLARY 4.3. *For an affine complete variety* $\mathscr{V}$ *the following are equivalent*:

(1) $\mathscr{V}$ *is finitely generated*;
(2) $\mathscr{V}$ *is locally finite*;
(3) $\mathscr{V}$ *is equivalent to a variety of finite type.*

Proof. In [4] it is proved that every locally finite affine complete variety has a near unanimity term. It is known that a clone on a finite set is finitely generated whenever it contains a near unanimity term. This yields the implication (1) $\Rightarrow$ (3). Now, if $\mathcal{V}$ is of finite type, then by Corollary 4.2 the sizes of SI's of $\mathcal{V}$ have a finite bound. However there are only finitely many different algebras of a given finite type on a given set. Hence there are only finitely many SI's in $\mathcal{V}$ implying that $\mathcal{V}$ is finitely generated. This proves the implication (3) $\Rightarrow$ (1).

Obviously (1) implies (2). Since the SI's of $\mathcal{V}$ have no proper subalgebras, they all are homomorphic images of $\mathbf{F}_1$, the free algebra of $\mathcal{V}$ in one generator. If $\mathcal{V}$ is locally finite then $\mathbf{F}_1$ is finite and again there is only finitely many different SI's. Hence (2) implies (1). □

In view of Theorem 4.2 we also have the following corollary.

COROLLARY 4.4. *The following are equivalent for an arithmetical variety* $\mathcal{V}$ *of countable type*:

(1) $\mathcal{V}$ *is affine complete*;
(2) $\mathcal{V}$ *has property* **(S)**;
(3) $\mathcal{V}$ *has a finite set* T *of unary terms such that* $\mathcal{V}$ *is generated by the algebras* $\mathbf{A} \in \mathcal{V}$ *on which the set of functions* $t^{\mathbf{A}}$, $t \in T$, *acts transitively*.

Now we prove that all AC varieties with finite residual bound have some important categorical properties. Actually, these properties follow already from congruence distributivity and the absence of proper subalgebras in SI's.

Recall that an algebra $\mathbf{E}$ in a variety $\mathcal{V}$ is said to be *injective in* $\mathcal{V}$ if for every $\mathbf{A} \in \mathcal{V}$, every homomorphism $\alpha: \mathbf{A} \to \mathbf{E}$ can be extended to every $\mathbf{B} \in \mathcal{V}$ that contains $\mathbf{A}$ as a subalgebra. A variety $\mathcal{V}$ *has enough injectives* iff every $\mathbf{A} \in \mathcal{V}$ is a subalgebra of some injective member of $\mathcal{V}$.

LEMMA 4.1. *Let* $\mathcal{V}$ *be a congruence distributive variety whose subdirectly irreducible algebras have no proper subalgebras. Assume that* $\mathbf{A}$, $\mathbf{B}$ *and* $\mathbf{S}$ *are finite algebras in* $\mathcal{V}$, *that* $\mathbf{A} \leq \mathbf{B}$, *that* $\mathbf{S}$ *is subdirectly irreducible, and* $\alpha: \mathbf{A} \to \mathbf{S}$ *is any homomorphism. Then there exists a homomorphism* $\beta: \mathbf{B} \to \mathbf{S}$ *that extends* α.

Proof. Let $\mathbf{B}$ be subdirect in $\mathbf{S}_1 \times \cdots \times \mathbf{S}_n$ where all the $\mathbf{S}_i$ are SI. Then $\mathbf{A}$ is also subdirect in $\mathbf{S}_1 \times \cdots \times \mathbf{S}_n$. Since $\mathbf{S}$ has no proper subalgebra, α is surjective. Let ρ be the kernel of α and θ_i be the kernel of the projection π_i of $\mathbf{B}$ to $\mathbf{S}_i$. Note that π_i is surjective and $\pi_i|_A$ is likewise surjective, because $\mathbf{S}_i$ has no proper subalgebras. Because of congruence distributivity and the fact that ρ is meet-irreducible, there is an i with $\theta_i|_A \leq \rho$. This gives a homomorphism $\gamma: \mathbf{S}_i \to \mathbf{S}$ for which $\gamma\pi_i(a) = \alpha(a)$ for all $a \in A$. Thus $\gamma\pi_i: \mathbf{B} \to \mathbf{S}$ extends α. □

THEOREM 4.5. *Suppose that $\mathscr{V}$ is a congruence distributive variety with a finite residual bound, and no subdirectly irreducible algebra in $\mathscr{V}$ possesses a proper subalgebra. Then $\mathscr{V}$ has enough injectives; hence $\mathscr{V}$ has the congruence extension property and the amalgamation property.*

Proof. It is known that the congruence extension property and the amalgamation property are true in any variety with enough injectives (see [6]). Obviously, every algebra in $\mathscr{V}$ can be embedded in a product of SI's in $\mathscr{V}$, and it is well-known (and easily verified) that the class of injective algebras in $\mathscr{V}$ is closed under direct products. Hence all parts of this theorem will follow if we can prove that every SI algebra in $\mathscr{V}$ is injective in $\mathscr{V}$.

To prove this, let $\mathbf{A} \leq \mathbf{B} \in \mathscr{V}$ and let $\alpha: \mathbf{A} \to \mathbf{S}$ where $\mathbf{S}$ is an SI algebra in $\mathscr{V}$. We can assume that $\mathbf{B}$ is subdirect in $\Pi_{i \in I} \mathbf{B}_i$ where each $\mathbf{B}_i$ is SI. Then $\mathbf{A}$ is also subdirect in $\Pi_{i \in I} \mathbf{B}_i$. We denote by ρ the kernel of α and remark that α is surjective, so that ρ is a completely meet-irreducible congruence of $\mathbf{A}$.

By Lemma 2.4, we can choose an ultrafilter $\mathscr{U}$ on I such that $\theta_{\mathscr{U}}|_A \leq \rho$. Let $\pi: \mathbf{B} \to \Pi_{i \in I} \mathbf{B}_i / \theta_{\mathscr{U}}$ be the natural map, whose kernel is $\theta_{\mathscr{U}}|_B$. Since $\theta_{\mathscr{U}}|_A \leq \rho$, there is a homomorphism $\gamma: \mathbf{A}_1 \to \mathbf{S}$, where $\mathbf{A}_1 = \pi(\mathbf{A})$, such that $\gamma\pi(a) = \alpha(a)$ for all $a \in A$. Now the algebra $\mathbf{B}_1 = \pi(\mathbf{B}) \leq \Pi_{i \in I} \mathbf{B}_i / \theta_{\mathscr{U}}$ is finite, since the $\mathbf{B}_i$ are of bounded finite size. By Lemma 4.1, there is a homomorphism $\delta: \mathbf{B}_1 \to \mathbf{S}$ such that $\delta(\pi(a)) = \gamma(\pi(a)) = \alpha(a)$ for all $a \in A$. Thus $\delta\pi|_B$ extends α to $\mathbf{B}$, as desired. □

We present now a characterization of affine complete varieties of countable type. The result is not entirely satisfactory and we hope that it can be improved.

DEFINITION 4.2. A variety $\mathscr{V}$ will be said to have the *polynomial restriction property provided that whenever* $\mathbf{A} \leq \mathbf{B} \in \mathscr{V}$, f is a polynomial operation of $\mathbf{B}$, and $\mathbf{A}$ is closed under f, then $f|_A$ is a polynomial operation of $\mathbf{A}$. If $\mathbf{A}$ is any algebra, $X \subseteq A^n$ and $f: X \to A$ then f is said to be *compatible for* $\mathbf{A}$ if it respects congruences of $\mathbf{A}$ restricted to X.

THEOREM 4.6. *Let $\mathscr{V}$ be a variety of countable type. Then $\mathscr{V}$ is affine complete iff $\mathscr{V}$ is a congruence distributive, has the polynomial restriction property, and has condition* (S) *relative to some terms $t_1, \ldots, t_n$ such that the following is true: whenever $\mathbf{A} \in \mathscr{V}$, $a \in A$, $X = \{t_1(a), \ldots, t_n(a)\}^m$ for some $m < \omega$, and $f: X \to A$ is compatible for $\mathbf{A}$, then there exists an algebra $\mathbf{B} \in \mathscr{V}$ with $\mathbf{A} \leq \mathbf{B}$ and a polynomial g of $\mathbf{B}$ such that $g(x) = f(x)$ for all $x \in X$.*

Proof. Suppose first that $\mathscr{V}$ is affine complete. By Theorems 4.1, 4.4 and 4.5, $\mathscr{V}$ is congruence distributive, has **(S)**, and has the congruence extension property. That $\mathscr{V}$ has the polynomial restriction property is then easy to show: If $\mathbf{A} \leq \mathbf{B} \in \mathscr{V}$ and f is a polynomial of $\mathbf{B}$ under which A is closed, we have that $f|_A$ is compatible for $\mathbf{A}$ since all congruences of $\mathbf{A}$ are restrictions of congruences of $\mathbf{B}$. Thus $f|_A$ is a polynomial of $\mathbf{A}$.

Let $t_1(x), \ldots, t_n(x)$ be terms witnessing that $\mathscr{V}$ has **(S)**. Suppose that $\mathbf{A} \in \mathscr{V}$, $a \in A$, $X \subseteq A^m$ and $f: X \to A$ are as in the statement of this theorem. We can assume that $\mathbf{A}$ is subdirect in $\mathbf{B} = \Pi_{t \in T} \mathbf{B}_t$, each $\mathbf{B}_t$ a SI algebra in $\mathscr{V}$. Since f is compatible for $\mathbf{A}$ and $\{t_1(a), \ldots, t_n(a)\}$ projects onto each $\mathbf{B}_t$, there are m-ary, fully-defined, operations g_t on B_t, compatible for $\mathbf{B}_t$ such that where $g = \Pi_t g_t$, g agrees with f on X. If μ is any congruence of $\mathbf{B}$ such that $\mathbf{B}/\mu$ is SI, and $\theta_{\mathscr{U}}$ is an ultrafilter congruence on $\mathbf{B}$ with $\theta_{\mathscr{U}} \leq \mu$, then g respects $\theta_{\mathscr{U}}$ (since it is defined coordinatewise) and every member of $\mathbf{B}$ is $\theta_{\mathscr{U}}$-equivalent to some member of $\{t_1(a), \ldots, t_n(a)\}$. Moreover, $g|_X$ respects $\theta_{\mathscr{U}}|_A$ and $\mu|_A$. Putting these facts together gives the proof that g respects μ. Then g must respect all congruences of $\mathbf{B}$. The compatible operation g is thus a polynomial. That completes our proof of the necessity of the given conditions in order that $\mathscr{V}$ be affine complete.

To begin the proof of sufficiency, assume that the conditions hold. Let $\mathbf{A} \in \mathscr{V}$ and let f be any m-ary operation over A, compatible for $\mathbf{A}$. Let $a \in A$ and $X \subseteq A^m$ be defined as above. Then we have an algebra $\mathbf{B} \in \mathscr{V}$, $\mathbf{A} \leq \mathbf{B}$, and a polynomial g of $\mathbf{B}$ such that g agrees with f on X. Obviously, we can assume that $\mathbf{B} = \Pi_{t \in T} \mathbf{B}_t$ as above with all $\mathbf{B}_t$ SI. Now A, even $\{t_1(a), \ldots, t_n(a)\}$, projects onto every B_t, and so f is the restriction to A of a product $\Pi_t f_t$ where each f_t is a compatible m-ary operation of $\mathbf{B}_t$. Likewise, $g = \Pi_t g_t$. We have $f_t = g_t$ for all t as determined by the fact that f and g agree on X. Thus $f = g|_A$. Now the polynomial restriction property supplies the desired conclusion, that f is a polynomial of $\mathbf{A}$. □

PROBLEM 1. Find a good internal characterization of affine complete varieties.

5. Two counter-examples

We present here two examples showing that Theorem 4.4 and Corollary 4.2 cannot be extended to varieties of arbitrary type.

EXAMPLE 1. There exists an arithmetical, affine complete variety of uncountable type which is residually ≤ 4 and fails to satisfy **(S)**.

As is well-known, there is an uncountable family $\mathscr{S}$ of infinite subsets of ω with the property that every two members of $\mathscr{S}$ have finite intersection. For example we may construct such family as follows. For every real number r pick a rational sequence $\langle a_i \mid i \in \omega \rangle$ converging to r and put $S_r = \{\{a_0, \ldots, a_i\} \mid i \in \omega\}$. Then the set of all S_r forms an uncountable family of subsets in the countable set of all finite rational sequences and two distinct such sets can have only finitely many members in common. We can assume that $\mathscr{S}$ is maximal, that is that every infinite subset of ω has infinite intersection with some member of $\mathscr{S}$. Also, if $U_0, \ldots, U_n$ are finitely many members of $\mathscr{S}$ then $\omega - (U_0 \cup \cdots \cup U_n)$ is infinite, because it contains all but finitely many of the elements in any set $U \in \mathscr{S}$ different from $U_0, \ldots, U_n$.

We construct a certain variety which will be generated by SI algebras $\mathbf{S}_n$, $n \in \omega \cup \{-1\}$. Operations of $\mathscr{V}$:

ternary operations $p(x, y, z)$, $t_n(x, y, z)$ for $n < \omega$

constants: $0, 1, c_n, d_n, e_U$ for $n < \omega$ and $U \in \mathscr{S}$.

The universe of $\mathbf{S}_{-1}$ is $S_{-1} = \{0, 1\}$ and its operation $p(x, y, z\}$ is the ternary discriminator, i.e., $p(x, y, z)$ is z if $x = y$ and x if $x \neq y$. We put $t_n(x, y, z) = x$ and $c_n = d_n = e_U = 1$ in $\mathbf{S}_{-1}$ and 0 and 1 are interpreted as themselves. The universe of $\mathbf{S}_n$, $n \geq 0$, is $S_n = S_0 = \{0, 1\} \times \{0, 1\}$. The reduct of $\mathbf{S}_n$ to the language $\{p, 0, 1\}$ is the algebra $\mathbf{S}_{-1} \times \mathbf{S}_{-1}$. For $k < n$ we define $t_k(x, y, z) = x$ and $c_k = d_k = \langle 1, 1 \rangle$ in $\mathbf{S}_n$ and for $k \geq n$ we put t_k equal to the discriminator operation on S_n and put $c_k = \langle 0, 1 \rangle$ and $d_k = \langle 1, 0 \rangle$. For $U \in \mathscr{S}$ we give the constant e_U in $\mathbf{S}_n$ the value $e_U = \langle 1, 1 \rangle$ if $n \notin U$ and $e_U = \langle 0, 1 \rangle$ if $n \in U$. This defines $\mathbf{S}_n$.

Note that all the algebras $\mathbf{S}_n$, $n \in \omega \cup \{-1\}$, are simple (actually quasiprimal) and have no proper subalgebras.

Let $\mathscr{V} = \mathrm{Var}\{\mathbf{S}_n : n \in \omega \cup \{-1\}\}$. Since $p(x, y, y) = p(x, y, x) = p(y, y, x) = x$ holds identically in all generating algebras of $\mathscr{V}$, this variety is arithmetical. Notice that $\mathscr{V}$ does not satisfy **(S)**: Let $\mathscr{F}$ be any finite subset of the operation symbols of $\mathscr{V}$. Then choose $n \in \omega$ to be larger than all k such that c_k or d_k or t_k occurs in $\mathscr{F}$ and to lie outside of every set $U \in \mathscr{S}$ such that e_U belongs to $\mathscr{F}$. The algebra $\mathbf{S}_n|_{\mathscr{F}}$, or $\mathbf{S}_n$ reduced to $\mathscr{F}$, is then term-equivalent to $\mathbf{S}_{-1} \times \mathbf{S}_{-1}$ and has no unary term t satisfying $t(\langle 1, 1 \rangle) = \langle 0, 1 \rangle$.

Our first task is to find all the SI algebras in $\mathscr{V}$. Let $\mathbf{S}$ be any SI in $\mathscr{V}$. Since $\mathscr{V}$ is CD, we can write $\mathbf{S}$ as isomorphic to $\mathbf{D}/\theta$ where $\mathbf{D} \leq \mathbf{E} = \Pi_{i \in I} \mathbf{B}_i/\theta_{\mathscr{U}}$, each $\mathbf{B}_i$ is one of our generating SI's, and $\mathscr{U}$ is an ultrafilter on I. For $n \in \omega \cup \{-1\}$ let I_n be the set of all $i \in I$ such that $\mathbf{B}_i = \mathbf{S}_n$. If the ultrafilter $\mathscr{U}$ contains none of the sets I_n, then $\mathbf{E}$ is isomorphic to an algebra which, except for the interpretations of the constants e_U (which may be rather wild), is $\mathbf{S}_{-1} \times \mathbf{S}_{-1}$. The class $HS(\mathbf{E})$, which includes $\mathbf{S}$, has up to isomorphism at most two SI members, each of which is isomorphic to an algebra which is $\mathbf{S}_{-1}$ except for the interpretations of the constants e_U. On the other hand, if $I_n \in \mathscr{U}$, then $\mathbf{E}$, $\mathbf{D}$, $\mathbf{S}$ are all isomorphic to $\mathbf{S}_n$. Let $\mathscr{V}_{-1}$ denote the class of all algebras in $\mathscr{V}$ whose reduct to the language

$$\{p\} \cup \{t_n : n \in \omega\} \cup \{c_n, d_n : n \in \omega\}$$

is equal to the reduct of $\mathbf{S}_{-1}$. This is a set of SI algebras containing at most exp(exp(2)) (or "two raised to the power 2^{ω}") algebras. Up to isomorphism, the SI algebras in $\mathscr{V}$ are the $\mathbf{S}_n$, $n \geq 0$, and the members of $\mathscr{V}_{-1}$.

We have shown that our variety is arithmetical and residually ≤ 4, and every SI algebra in $\mathscr{V}$ has no proper subalgebras. We now show that $\mathscr{V}$ is affine complete. Let f be a compatible unary function on the algebra $\mathbf{A} \in \mathscr{V}$. (The proof now to be given can be easily extended to show that compatible n-ary operations are polynomial.) We assume that $\mathbf{A}$ is subdirect in $\Pi_{i \in I} \mathbf{B}_i$ where every $\mathbf{B}_i$ belongs to $\mathscr{V}_{-1}$ or is equal to some $\mathbf{S}_n$, $n \in \omega$. Again, let I_n be the set of $i \in I$ where $\mathbf{B}_i = \mathbf{S}_n$, and let I_{-1} be the set of i where $\mathbf{B}_i \in \mathscr{V}_{-1}$. Let f_i denote the compatible function which f induces on $\mathbf{B}_i$ through the i-th coordinate projection.

First, suppose that there are infinitely many $n \in \omega$ for which some f_i, $i \in I_n$, is not compatible for the algebra $\langle B_i, p\rangle$, i.e., does not preserve both kernels of the projections of $B_i = S_n$ onto S_{-1}. Then, because of the maximality of $\mathscr{S}$, the set of those n must have an infinite intersection with some $U \in \mathscr{S}$. That is, there exists a set $U \in \mathscr{S}$ and an infinite subset $V \subseteq U$ and for every $n \in V$ an $i_n \in I_n$ such that f_{i_n} fails to be compatible for $\langle S_n, p\rangle$. Consider the elements 0, 1, e_U and $k_U = p(0, e_U, 1)$ in $\mathbf{A}$. At every i_n, $n \in V$, these project to the four elements $\langle 0, 0\rangle$, $\langle 1, 1\rangle$, $\langle 0, 1\rangle$, $\langle 1, 0\rangle$ of B_{i_n}. Since f is compatible and $\mathscr{V}$ is arithmetical, there is a polynomial $g(x) = t(x, \bar{a})$ of $\mathbf{A}$, t a term, such that g agrees with f on the set $\{0, 1, e_U, k_U\} \subseteq A$. Pick $n \in V$ larger than all the k such that t_k occurs in t. Then there is a term t' involving only p and constants such that at $i = i_n$, t' and g induce the same function. But this implies that f and t' induce the same function at i_n, contrary to the assumption that f_{i_n} cannot be compatible for $\langle S_n, p\rangle$.

Thus we can assume that there is $m \in \omega$ such that where

$$I' = I_0 \cup \cdots \cup I_m \quad \text{and}$$

$$I'' = I_{-1} \cup I_{m+1} \cup I_{m+2} \cup \cdots,$$

we have that for all $i \in I''$, f_i is compatible for the algebra $\langle B_i, p\rangle$.

Put

$$q''(x) = p(p(0, x, f(0)), p(0, x, f(1)), f(1)),$$

so that we have $q''(0) = f(0)$ and $q''(1) = f(1)$. Since both $f(x)$ and $q''(x)$ induce functions on every B_i, $i \in I''$, that are products of functions on S_{-1} at the two coordinates, and since 0 and 1 project onto S_{-1} through all these combined projections, it follows that for all $x \in A$, $q''(x)|_{I''} = f(x)|_{I''}$.

Let $q'(x)$ be a polynomial of $\mathbf{A}$ which agrees with $f(x)$ on $\{0, 1, c_m, d_m\}$. Note that 0, 1, c_m and d_m project onto B_i for all $i \in I'$. This implies, of course, that $q'(x)|_{I'} = f(x)|_{I'}$ for all $x \in A$. We claim that where

$$q(x) = p(p(c_m, d_m, q''(x)), p(c_m, d_m, q'(x)), q'(x)),$$

we have $q(x) = f(x)$ for all $x \in A$. Notice that when $y \in A$, $p(c_m, d_m, y)$ agrees with c_m at all $i \in I'$ and with y at all $i \in I''$. Then the claim readily follows. □

EXAMPLE 2. There exists an arithmetical, affine complete variety which has no finite residual bound.

This example is a modification of the previous one. Put $S_{-1} = \{0, 1\}$ and for $n < \omega$ put $S_n = \{0, 1\}^{n+2}$. We take P to be the set $\Pi_{n \in \omega} S_n$. For the operations of $\mathscr{V}$ we take:

ternary operations $p(x, y, z)$, $t_n(x, y, z)$ for $n < \omega$

constants: 0, 1, c_y for $y \in P$.

The operation $p(x, y, z)$ is interpreted as the discriminator in $\mathbf{S}_{-1}$ and is interpreted in $\mathbf{S}_n$ as the operation such that $\langle S_n, p \rangle = \langle \{0, 1\}, p \rangle^{n+2}$. The operation $t_n(x, y, z)$ is interpreted as x in $\mathbf{S}_{-1}$ and in $\mathbf{S}_k$ for all $k > n$ and interpreted as the discriminator on the universe in each $\mathbf{S}_k$ with $k \leq n$. The constants 0 and 1 are interpreted as themselves in $\mathbf{S}_{-1}$ and as the tuples $\langle 0, 0, \ldots, 0 \rangle$ and $\langle 1, 1, \ldots, 1 \rangle$ in $\mathbf{S}_n$. Each constant c_y, $y \in P$, is interpreted the same as 1 in $\mathbf{S}_{-1}$ and in each $\mathbf{S}_n$ gets the value $y(n)$.

We define $\mathscr{V} = \mathrm{Var}\{\mathbf{S}_k : k \in \{-1, 0, 1, 2, \ldots\}\}$. The same proof used in the previous example shows that every SI algebra in $\mathscr{V}$ is isomorphic either to some $\mathbf{S}_n$, $n < \omega$, or is isomorphic to an algebra whose universe and operations $p, t_n, 0, 1$ $(n < \omega)$ are the same as those of $\mathbf{S}_{-1}$. The collection of all two-element SI algebras in $\mathscr{V}$ sharing those components with $\mathbf{S}_{-1}$ is denoted by $\mathscr{V}_{-1}$.

The proof that $\mathscr{V}$ is affine complete is almost the same as in the last example. Let f be a compatible unary function on the algebra $\mathbf{A} \in \mathscr{V}$. We assume that $\mathbf{A}$ is subdirect in $\Pi_{i \in I} \mathbf{B}_i$ where every $\mathbf{B}_i$ belongs to $\mathscr{V}_{-1}$ or is equal to some $\mathbf{S}_n$. We use I_n to denote the set of $i \in I$ where $\mathbf{B}_i = \mathbf{S}_n$, and I_{-1} is $I - \bigcup_{n \geq 0} I_n$. Let f_i denote the compatible function which f induces on $\mathbf{B}_i$ through the i-th coordinate projection.

The proof now is essentially different from that in the last example only in the argument that there cannot exist infinitely many $n \in \omega$ for which some f_i, $i \in I_n$, is not compatible for the algebra $\langle B_i, p \rangle$. Suppose that this fails. Choose an infinite set $V \subseteq \omega$ and for every $n \in V$ we choose $i_n \in I_n$ and $a(n), b(n) \in B_{i_n}$ $(= S_n)$ so that $f_{i_n}(a(n))$ and $f_{i_n}(b(n))$ fail to agree at some one among their $n+2$ coordinates where $a(n)$ and $b(n)$ agree. Choose $y, z \in P$ so that for every $n \in V$, $y(n) = a(n)$ and $z(n) = b(n)$. Since f is compatible and $\mathscr{V}$ is arithmetical, there is a polynomial $g(x) = t(x, \bar{c})$ of $\mathbf{A}$, t a term, such that $g(c_y) = f(c_y)$ and $g(c_z) = f(c_z)$. Pick $n \in V$ larger than all the k such that t_k occurs in t and let t' be the term involving only p and constants such that the polynomials $g(x)$ and $t'(x, \bar{c})$ induce the same function at $i = i_n$. This implies that $f_{i_n}(a(n)) = t'(a(n), \bar{c}(i_n))$ and $f_{i_n}(b(n)) = t'(b(n), \bar{c}(i_n))$. Of course, this is impossible because in the algebra $\mathbf{B}_{i_n}$ the polynomial $t'(x, \bar{c}(i_n))$ is a product of functions acting coordinatewise in the $n+2$ different coordinates.

The remainder of the proof that f is a polynomial on $\mathbf{A}$ can be found by examining the proof in the prior example and making a small modification. □

6. Almost congruence distributive varieties

DEFINITION 6.1. A variety is said to be *finitely congruence distributive* (*FCD*) if all its finite members are CD. A variety will be called *almost congruence distributive* (*ACD*) if it is residually finite and finitely congruence distributive, but the variety itself is not CD.

Note that actually every finite member of an ACD variety generates a CD subvariety. Indeed, this subvariety must be locally finite, hence its free algebra on three generators is finite.

It is not known whether there exists any ACD variety. This section reports the few results we obtained while attempting to prove or disprove the existence of ACD varieties. Our first observation is that every ACD variety contains a minimal ACD subvariety. In fact this is a special case of the following general fact.

LEMMA 6.1. *Every non-CD variety contains a minimal non-CD subvariety.*

Proof. Take a non-CD variety $\mathscr{V}$. Consider the partially ordered set of non-CD subvarieties of $\mathscr{V}$. We want to use Zorn's lemma; hence we need to show that the intersection of every chain of this ordered set is non-CD as well.

Let $\langle \mathscr{W}_i \mid i \in I \rangle$ be a chain of non-CD subvarieties of $\mathscr{V}$ and let $\mathscr{W}$ be the intersection of all the varieties $\mathscr{W}_i$, $i \in I$.

Suppose on the contrary that $\mathscr{W}$ is CD. Then there exists a sequence of Jónsson terms $t_0, t_1, \ldots, t_n$ for $\mathscr{W}$ and corresponding Jónsson sentence ε such that $\mathscr{W} \models \varepsilon$. (Definition 2.1.) Hence then $\mathrm{Id}(\mathscr{W}) \cup \{\neg\varepsilon\}$ is inconsistent. So, by the first order compactness theorem of logic, some finite subset $\{\sigma_1, \ldots, \sigma_m, \neg\varepsilon\}$ is inconsistent as well. But, by construction $\{\sigma_1, \ldots, \sigma_m\} \subseteq \mathrm{Id}(\mathscr{W}_i)$ for some $i \in I$ and $\mathscr{W}_i \not\models \varepsilon$, a contradiction. Hence, $\mathscr{W}$ is not CD. □

LEMMA 6.2. *An almost congruence distributive variety cannot be a join of two congruence distributive subvarieties.*

Proof. Let an ACD variety $\mathscr{V}$ be the join of CD subvarieties $\mathscr{V}_1$ and $\mathscr{V}_2$. First assume that whenever $\mathbf{A}$ is a subdirect product of two algebras $\mathbf{B} \in \mathscr{V}_1$ and $\mathbf{C} \in \mathscr{V}_2$ then $\mathbf{A}$ has no skew congruences with respect to this decomposition. Under this assumption, it follows that every algebra in $\mathscr{V}$ has a representation as a subdirect product of an algebra in $\mathscr{V}_1$ and an algebra in $\mathscr{V}_2$. Now let $\mathbf{A} \in \mathscr{V}$ and suppose that $\mathbf{A}$ is a subdirect product in $\mathbf{B} \times \mathbf{C}$ where $\mathbf{B}$ and $\mathbf{C}$ are arbitrary algebras from $\mathscr{V}$. Let ρ be any congruence of $\mathbf{A}$. Each of $\mathbf{B}$ and $\mathbf{C}$ has a subdirect representation, $\mathbf{B} \leq \mathbf{B}_1 \times \mathbf{B}_2$, $\mathbf{C} \leq \mathbf{C}_1 \times \mathbf{C}_2$ where $\mathbf{B}_i, \mathbf{C}_i \in \mathscr{V}_i$, $i = 1, 2$. Then obviously $\mathbf{A}$ is subdirect in $\mathbf{A}_1 \times \mathbf{A}_2$ where $\mathbf{A}_i$ is the projection of $\mathbf{A}$ in $\mathbf{B}_i \times \mathbf{C}_i$, $i = 1, 2$. Hence, by our assumption, $\rho = (\rho_1 \times \rho_2)|_A$ for appropriate congruences $\rho_i \in \mathrm{Con}\, \mathbf{A}_i$, $i = 1, 2$. However, $\mathbf{A}_i$ are members of CD varieties. Hence $\rho_i = (\sigma_i \times \tau_i)|_{A_i}$ for appropriate congruences $\sigma_i \in \mathrm{Con}\, \mathbf{B}_i$, $\tau_i \in \mathrm{Con}\, \mathbf{C}_i$, $i = 1, 2$. Now it is easy to see that

$$\rho = ((\sigma_1 \times \sigma_2)|_B \times (\tau_1 \times \tau_2)|_C)|_A.$$

So we have seen that $\mathscr{V}$ has no skew congruences and thus must be CD provided that subdirect products of an algebra from $\mathscr{V}_1$ with an algebra from $\mathscr{V}_2$ always have no skew congruences.

So now suppose that $\mathbf{A}$ is a subdirect product of two algebras $\mathbf{B} \in \mathscr{V}_1$ and $\mathbf{C} \in \mathscr{V}_2$. We note that it follows from Corollary 2.2 that if either $\mathscr{V}_1$ or $\mathscr{V}_2$ is finitely generated then no completely meet-irreducible congruence of $\mathbf{A}$ is skew with respect to this decomposition, and hence no congruence of $\mathbf{A}$ is skew with respect to the decomposition. Thus if $\mathscr{V}_1$ or $\mathscr{V}_2$ is finitely generated, then $\mathscr{V}$ must be CD. Finally, assume that ρ is a completely meet-irreducible congruence of $\mathbf{A}$. Then $\mathrm{Var}(\mathbf{A}/\rho)$ is finitely generated; so it follows by what we just proved that $\mathrm{Var}(\mathbf{A}/\rho) \vee \mathrm{Var}(\mathbf{B})$ is CD. But then it also follows from Corollary 2.2 that **A** has no skew congruences for this decomposition. □

LEMMA 6.3. *Let $\mathscr{V}$ be a variety, N be a natural number, and let $\mathscr{V}_{\leq N}$ denote the class of all algebras in $\mathscr{V}$ whose cardinality does not exceed N. Suppose that every member of $\mathscr{V}_{\leq N}$ generates a congruence distributive variety. Then $\mathscr{V}_{\leq N}$ is contained in the union of finitely many congruence distributive subvarieties $\mathscr{V}_0, \ldots, \mathscr{V}_n$ of $\mathscr{V}$.*

Proof. Let $\{j_i \mid i \in I\}$ be all of the Jónsson sentences in the language of $\mathscr{V}$. Then the set of first order sentences

$$\mathrm{Id}(\mathscr{V}) \cup \{\neg j_i \mid i \in I\} \cup \{\exists x_1 \cdots x_N \forall y \ (y = x_1 \vee \cdots \vee y = x_N)\} \tag{2}$$

is inconsistent. Indeed, if it were not then there should exist $\mathbf{A} \in \mathscr{V}_{\leq N}$ with all Jónsson sentences failing in it. Hence the set (2) must contain a finite inconsistent subset S. This means exactly that there are finitely many sentences $j_{i_0}, \ldots, j_{i_n}$ such that every member of $\mathscr{V}_{\leq N}$ satisfies one of them. Hence $\mathscr{V}_{\leq N}$ is contained in the union of the varieties $\mathscr{V}_k = \mathrm{Mod}(\mathrm{Id}(\mathscr{V}) \cup \{j_{i_k}\})$, $0 \leq k \leq n$. □

COROLLARY 6.1. *An ACD variety cannot have a finite residual bound.*

Proof. This is a consequence of the last two lemmas. □

DEFINITION 6.2. Let $L' \subseteq L$ where L is the set of operation symbols of a variety $\mathscr{V}$. Then $\mathscr{V}|_{L'}$ denotes the variety generated by all the reducts of algebras in $\mathscr{V}$ obtained by removing the operations denoted by symbols in $L - L'$. It is the variety of type L' axiomatized by all the equations constructed from L' that hold in $\mathscr{V}$.

THEOREM 6.1. *Every ACD variety $\mathscr{V}$ has a reduct $\mathscr{V}|_{L'}$ which is an ACD variety of countable type.*

Let $\mathscr{V}$ be an ACD variety. By Lemma 6.2 and Lemma 6.3, there are Jónsson sentences $j_2, j_3, \ldots, j_n, \ldots$ such that where $\mathscr{V}_n = \mathscr{V} \cap \mathrm{Mod}(j_n)$, we have $\mathscr{V}_{\leq n} \subseteq \mathscr{V}_n$ and $\mathscr{V}_n$ is CD. For each n, there is a finite conjunction ε_n of equations valid in $\mathscr{V}$ such that every model of ε_n which has at most n elements satisfies j_n. Let L_0 be the set of all the operation symbols of $\mathscr{V}$ that occur in any of the sentences ε_n or j_n, $2 \leq n < \omega$. Then L_0 is a countable set of operation symbols.

Now if L' is any countable subset of the operation symbols of $\mathscr{V}$, then there is another countable subset $L'' \supseteq L'$ such that every universal sentence defined in L'

and valid in $\mathscr{V}$ is implied by some finite subset of the valid equations of $\mathscr{V}|_{L''}$. Thus there must exist a sequence of countable sets of operation symbols, $L_0 \subseteq L_1 \subseteq \cdots \subseteq L_n \subseteq \cdots$ such that L_0 is the set constructed above and for every n, every universal sentence defined in L_n and valid in $\mathscr{V}$ is implied by some finite subset of the valid equations of $\mathscr{V}|_{L_{n+1}}$. Setting $L' = \bigcup_n L_n$, and $\mathscr{W} = \mathscr{V}|_{L'}$, we have that the valid universal sentences of the variety $\mathscr{W}$ are precisely the valid universal sentences of $\mathscr{V}$ in which only operation symbols from L' occur.

An equivalent condition to this last is that $\mathscr{W}$ is equal to the class of subalgebras of reducts to L' of algebras in $\mathscr{V}$. This implies that every SI algebra in $\mathscr{W}$ is a subalgebra of a reduct to L' of some SI algebra in $\mathscr{V}$. Indeed, let $\mathbf{A} \in \mathscr{W}$ be SI and let $\mathbf{A} \leq \mathbf{A}'$ and $\mathbf{A}'$ be a reduct of $\mathbf{B} \in \mathscr{V}$ and let $\mathbf{B}$ be a subdirect product in $\Pi_{t \in T} \mathbf{B}_t$ where every algebra $\mathbf{B}_t$ is SI. The projection homomorphisms $\mathbf{B} \to \mathbf{B}_t$ map $\mathbf{A}$ homomorphically to subalgebras of the L'-reducts of the algebras $\mathbf{B}_t$. One of these maps must be injective, due to the subdirect irreducibility of $\mathbf{A}$.

What was proved above implies that $\mathscr{W}$ is residually finite, just as $\mathscr{V}$ is. Obviously, no Jónsson sentence j can be valid in $\mathscr{W}$, since as a conjunction of finitely many equations built from L', it would have to be valid in $\mathscr{V}$ as well. Thus $\mathscr{W}$ is not CD. Since $\mathscr{W}$ counts ε_M (a conjunction of equations) among its valid sentences, then all members of $\mathscr{W}_{\leq M}$ satisfy j_m. So $\mathscr{W}$ is FCD and, finally, ACD. □

THEOREM 6.2. *Suppose that there exists an ACD variety. Then there exists an ACD variety $\mathscr{W}$ with the following properties*:

(1) *$\mathscr{W}$ is of countable type*;

(2) *every proper subvariety of $\mathscr{W}$ is CD*;

(3) *$\mathscr{W}$ has CD subvarieties $\mathscr{W}_2 \subseteq \mathscr{W}_3 \subseteq \cdots \subseteq \mathscr{W}_n \subseteq$ such that $\mathscr{W}_{\leq n} \subseteq \mathscr{W}_n$; every proper subvariety of $\mathscr{W}$ is contained in some $\mathscr{W}_n$; and $\mathscr{W}_n$ is defined for some Jónsson sentence j_n as all the algebras in $\mathscr{W}$ which satisfy j_n.*

Proof. By Lemma 6.1 and Theorem 6.1, there is an ACD variety $\mathscr{W}$ which is of countable type and minimal. Property (3) is a consequence of (1) and (2). Indeed, let $\{j_n : n < \omega\}$ be a list of all the Jónsson sentences in the language of $\mathscr{W}$. For any Jónsson sentence j, put $\mathscr{W}(j) = \mathscr{W} \cap \mathrm{Mod}(j)$. By Lemmas 6.2 and 6.3, we can find inductively, for each n a k_n such that

$$\mathscr{W}_{\leq n} \cup \mathscr{W}(j_0) \cup \mathscr{W}(j_1) \cup \cdots \cup \mathscr{W}(j_n) \cup \mathscr{W}(j_{k_{n-1}}) \subseteq \mathscr{W}(j_{k_n}).$$

The varieties $\mathscr{W}_n = \mathscr{W}(j_{k_n})$ fulfill (3). □

PROBLEM 2. Does there exist any ACD variety?

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REFERENCES

[1] BURRIS, S. and SANKAPPANAVAR, H. P., *A Course in Universal Algebra*. Graduate Texts in Mathematics, Springer-Verlag, New York, 1981.

[2] HOBBY, D. and MCKENZIE, R., *The Structure of Finite Algebras*. AMS Contemporary Mathematics, 1986.

[3] JÓNSSON, B., *Algebras whose congruence lattices are distributive*, Math. Scand. *21* (1967), 110–121.

[4] KAARLI, K., *Locally finite affine complete varieties*, J. Austral. Math. Soc. (Series A) *62* (1997), 141–159.

[5] KAARLI, K. and PIXLEY, A., *Affine complete varieties*, Algebra univers. *24* (1987), 74–90.

[6] KISS, E. W., MÁRKI, L., PRÖHLE, P. and THOLEN, W., *Categorical algebraical properties. A compendium of amalgamation, congruence extension, epimorphisms, residual smallness and injectivity*, Stud. Sci. Math. Hungar. *18* (1983), 79–141.

[7] PIXLEY, A., *Characterizations of arithmetical varieties*, Algebra univers. *9* (1979), 87–98.

[8] QUACKENBUSH, R. W. and WOLK, B., *Strong representation of congruence lattices*, Algebra univers. *1* (1971), 165–166.

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