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Non-embeddable simple relation algebras

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Abstract. This paper presents solutions or partial solutions for several problems in the theory of relation algebras. In a simple relation algebra $\mathfrak{A}$, an element x satisfying the condition (a) $0 \neq x; 1; \breve{x} + \breve{x}; 1; x \leq$ 1' must be an atom of $\mathfrak{A}$. It follows that x must also be an atom in every simple extension of $\mathfrak{A}$. Andréka, Jónsson and Németi [1, Problem 4] (see [12, Problem **P5**]) asked whether the converse holds: if x is an atom in every simple extension of a simple relation algebra, must it satisfy (a)? We show that the answer is "no".

The only known examples of simple relation algebras without simple proper extensions are the algebras of all binary relations on a finite set. Jónsson proposed finding all finite simple relation algebras without simple proper extensions [12, Problem **P6**]. We show how to construct many new examples of finite simple relation algebras that have no simple proper extensions, thus providing a partial answer for this second problem. These algebras are also integral and nonrepresentable.

Andréka, Jónsson, Németi [1, Problem 2] (see [12, Problem **P7**]) asked whether there is a countable simple relation algebra that cannot be embedded in a one-generated relation algebra. The answer is "yes". Givant [3, Problem 9] asked whether there is some k such that every finitely generated simple relation algebra can be embedded in a k-generated simple relation algebra. The answer is "no".

1. Introduction

Let U be an arbitrary set. $\mathfrak{Re}\,U$ is the representable relation algebra whose universe consists of all binary relations on U. If U is empty then $\mathfrak{Re}\,U$ is a trivial one-element algebra, so $\mathfrak{Re}\,U$ is not simple and has no atoms. On the other hand, if U is not empty, then $\mathfrak{Re}\,U$ is simple and atomic. The atoms of $\mathfrak{Re}\,U$ are the binary relations of the form $\{\langle a, b\rangle\}$, where $a, b \in U$. Every atom of $\mathfrak{Re}\,U$ satisfies the condition

$$0 \neq x; 1; \breve{x} + \breve{x}; 1; x \leq 1'. \tag{1}$$

One of the remarkable properties of relation algebras is that if x is an element of a simple relation algebra $\mathfrak{A}$ that satisfies (1), then x must be an atom of $\mathfrak{A}$. (See [7, Lemma 7.3] or [11, Theorem 41].) This has an interesting consequence for which we first make a definition.

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DEFINITION 1.1. Suppose x is an element of a simple relation algebra $\mathfrak{A}$. We say that x is a **persistent atom** of $\mathfrak{A}$ if x is an atom of $\mathfrak{A}$ and x is also an atom of every simple relation algebra that contains $\mathfrak{A}$ as a subalgebra.

In a simple relation algebra $\mathfrak{A}$, every element x that satisfies (1) is a persistent atom of $\mathfrak{A}$, for if $\mathfrak{B}$ is a simple relation algebra that contains $\mathfrak{A}$ as a subalgebra, then x is an element of $\mathfrak{B}$ that still satisfies (1), so x is also an atom of $\mathfrak{B}$. For example, all the atoms of $\mathfrak{Re}\,U$ are persistent. It follows that $\mathfrak{Re}\,U$ cannot be properly embedded in any simple relation algebra whenever $\mathfrak{Re}\,U$ is finite [7, Lemma 7.4]. These observations provoke some natural questions. Do all persistent atoms of simple relation algebras satisfy (1)? (See [1, Problem 4] or [12, Problem **P4**].) Are there any other finite simple relation algebras besides $\mathfrak{Re}\,U$ that cannot be properly embedded in any simple relation algebra? (See [12, Problem **P6**].)

We will show that the answer to the first question is "no" by exhibiting finite simple relation algebras in which the identity element 1' is a persistent atom that does not satisfy (1). See Theorems 6.1, 6.2, 7.1. A nontrivial relation algebra is said to be **integral** if $0 = x ; y$ implies either $x = 0$ or $y = 0$. Integrality has the following interesting characterization.

THEOREM 1.2. [8, *Theorem* 4.17] *Let* $\mathfrak{A}$ *be a relation algebra with at least two elements. The the following statements are equivalent.*

(1) 1' *is an atom of* $\mathfrak{A}$.

(2) *If* x *and* y *are elements of* $\mathfrak{A}$ *and* $0 = x ; y$, *then either* $x = 0$ *or* $y = 0$.

Every integral relation algebra is simple [8, Theorem 4.18(ii)]. All our examples are integral. A relation algebra is **Boolean** if it satisfies $x ; y = x \cdot y$, $\breve{x} = x$, and $1' = 1$ (and hence is essentially just a Boolean algebra). The identity element 1' of a relation algebra $\mathfrak{A}$ satisfies (1) only if $\mathfrak{A}$ is Boolean. A Boolean relation algebra is simple only if it has exactly two elements. There are 102 integral relation algebras having exactly sixteen elements (and four atoms). In all of them, $1' \neq 1$ and 1' fails to satisfy (1). A heuristic criterion was developed for selecting integral algebras in which the identity atom 1' might also be persistent, narrowing the list of 102 algebras down to five algebras. These algebras, dubbed $\mathfrak{C}_1$–$\mathfrak{C}_5$, are presented in Section 6. It turns out that 1' is indeed persistent in all five. However, in three of these algebras, namely $\mathfrak{C}_3$–$\mathfrak{C}_5$, all four atoms are persistent, so $\mathfrak{C}_3$–$\mathfrak{C}_5$ cannot be properly embedded in any simple relation algebra; see Theorem 6.3. None of $\mathfrak{C}_1$–$\mathfrak{C}_5$ is isomorphic to any $\mathfrak{Re}\,U$, because $\mathfrak{Re}\,U$ is not integral whenever U has more than one element, and $\mathfrak{Re}\,U$ has only two elements when U has exactly one element. The algebras $\mathfrak{C}_3$–$\mathfrak{C}_5$ therefore show that the answer to the second question is "yes".

The algebras $\mathfrak{C}_1$–$\mathfrak{C}_5$ share certain features that lead to the formulation of some general lemmas applicable to larger algebras. Lemma 4.1 below gives a simple criterion by which the persistence of one atom can be deduced from the persistence of another. Lemmas 5.4–5.6 arise from the key fact that the algebras $\mathfrak{C}_1$–$\mathfrak{C}_5$ fail to satisfy a certain identity, namely (L) in Section 7. Identity (L) holds in all representable relation algebras, so $\mathfrak{C}_1$–$\mathfrak{C}_5$ are not representable. The failure of (L) in $\mathfrak{C}_1$–$\mathfrak{C}_5$ is due to the presence of elements that satisfy conditions (35)–(40) in Section 5. These conditions form the hypotheses to Lemmas 5.4–5.6, and together with the hypotheses of Lemma 4.1, allow us to show that all the atoms of $\mathfrak{C}_3$–$\mathfrak{C}_5$ are persistent. For a finite integral relation algebra with a large number of atoms, the hypotheses of these lemmas determine only a small part of the structure of the algebra, suggesting the possibility of constructing, by varying the rest of the algebra, a huge number of finite integral relation algebras with no proper simple extensions. Section 7 presents such a construction. If B is a set of cardinality n, and Δ is any set of three-element subsets of B, then $\mathfrak{D}^B_\Delta$ is a complete atomic integral (hence simple) relation algebra with $n+3$ atoms that is not properly embeddable in any simple relation algebra. See Theorem 7.1. When B is a one-element set, this construction yields $\mathfrak{C}_3$ as a special case. Probably $\mathfrak{C}_4$ and $\mathfrak{C}_5$ are also special cases of general constructions, and there are probably many other ways to construct such algebras, including ways that do not depend on the failure of (L). However, every finite simple representable relation algebra can be embedded in $\mathfrak{Re}\,U$ for some set U, so every simple relation algebra without proper simple extensions is nonrepresentable.

An interesting byproduct of the construction of $\mathfrak{D}^B_\Delta$ is the solution of two more problems. If U is finite, then the relation algebra $\mathfrak{Re}\,U$ is generated by a single relation [13, Theorem 8.4(xiv)]. Indeed, any linear ordering of U generates $\mathfrak{Re}\,U$. It follows that every simple finitely representable relation algebra can be embedded in a one-generated simple representable relation algebra. A significant extension of this observation is that every finitely generated (and possibly infinite) representable simple relation algebra can be embedded in a one-generated representable simple relation algebra; see Tarski–Givant [13, Theorem 8.4(xv)]. Can results like these be extended to nonrepresentable relation algebras? Does there exist a countable (or finitely generated) simple relation algebra that is not embeddable in a one-generated relation algbera? (See [1, Problem 2] or [12, Problem **P7**].) We show that the answer is "yes". More generally, Givant [3, Problem 9] asked whether there is some k such that every finitely generated simple relation algebra can be embedded in a k-generated simple relation algebra. The answer is "no", because the construction of $\mathfrak{D}^B_\Delta$ happens to include algebras that require arbitrarily large numbers of generators. Indeed, given k, choose n so that $k+3 < \log_2(n+3)$, let B be an n-element set, and let Δ be either the empty set, or else the set of all three-element subsets of B. We prove in Section 7 that the resulting algebra $\mathfrak{D}^B_\Delta$ cannot be generated by fewer than $-3+\log_2(n+3)$ elements. Of course, $\mathfrak{D}^B_\Delta$ cannot be properly embedded in any simple relation algebra. With slightly more work we can also show that $\mathfrak{D}_{\Delta}^{B}$ cannot be embedded in any k-generated relation algebra, simple or not. See Theorem 7.2.

2. Relation algebras

DEFINITION 2.1. A **relation algebra** is an algebraic structure of the form $\langle A, +, ^{-}, ;, \breve{}, 1' \rangle$, where A is a nonempty set, $+$ and ; are binary operations on A, $^{-}$ and $\breve{}$ are unary operations on A, and 1' is a distinguished element of A, such that for all $x, y, z \in A$:

$$x + y = y + x \tag{2}$$

$$(x + y) + z = x + (y + z) \tag{3}$$

$$x = \overline{\bar{x} + y} + \overline{\bar{x} + \bar{y}} \tag{4}$$

$$x ; (y ; z) = (x ; y) ; z \tag{5}$$

$$(x + y) ; z = (x ; z) + (y ; z) \tag{6}$$

$$x ; 1' = x \tag{7}$$

$$\breve{\breve{x}} = x \tag{8}$$

$$(x + y)\breve{} = \breve{x} + \breve{y} \tag{9}$$

$$(x ; y)\breve{} = \breve{y} ; \breve{x} \tag{10}$$

$$\bar{y} = \bar{y} + (\breve{x} ; \overline{x ; y}) \tag{11}$$

Identities (2)–(4) state that the reduct $\langle A, +, ^{-} \rangle$ is a Boolean algebra [5] [6]. Define an additional binary operation $\cdot$ on A, two partial ordering relations $\leq$ and $\geq$ on A, and elements 0', 1, and 0 of $\mathfrak{A}$ as follows. For all $x, y \in A$,

$$x \cdot y = \overline{\bar{x} + \bar{y}},$$

$$x \leq y \quad \Leftrightarrow \quad x + y = y \quad \Leftrightarrow \quad x \cdot y = x,$$

$$x \geq y \quad \Leftrightarrow \quad x + y = x \quad \Leftrightarrow \quad x \cdot y = y,$$

$$0' = \overline{1'},$$

$$1 = 1' + 0',$$

$$0 = \bar{1}.$$

Parentheses may be omitted according to the convention that unary operations are always performed first, followed by ;, then $\cdot$, and finally $+$, with repeated binary operations associated to the left, *e.g.*, $x ; y ; z = (x ; y) ; z$. Here are other laws of relation algebras that we will use. Proofs can be found in [2], [8], or [10].

$$1' ; x = x, \tag{12}$$

$$x ; (y + z) = x ; y + x ; z, \tag{13}$$

$$(x \cdot y)\breve{} = \breve{x} \cdot \breve{y}, \tag{14}$$

$$\text{if } x \leq y \quad \text{and} \quad u \leq v \quad \text{then} \quad \breve{x} \leq \breve{y} \quad \text{and} \quad x ; u \leq y ; v, \tag{15}$$

$$x ; 0 = 0 = 0 ; x, \tag{16}$$

$$1 ; 1 = 1, \tag{17}$$

$$\breve{1}' = 1', \quad \breve{0}' = 0', \quad \breve{0} = 0, \quad \breve{1} = 1, \tag{18}$$

$$\bar{x} ; \breve{x} \leq 0', \quad \breve{x} ; \bar{x} \leq 0', \tag{19}$$

$$\text{if } x \leq 1' \quad \text{then} \quad \breve{x} = x, \tag{20}$$

$$\text{if } x, y \leq 1' \quad \text{then} \quad x ; y = x \cdot y \tag{21}$$

$$\text{if } x \leq 1' \quad \text{then} \quad x ; y \cdot z = x ; (y \cdot z) \quad \text{and} \quad z \cdot y ; x = (z \cdot y) ; x \tag{22}$$

$$x = (1' \cdot x ; \breve{x}) ; x = x ; (1' \cdot \breve{x} ; x) \tag{23}$$

$$x \cdot y ; z \leq (y \cdot x ; \breve{z}) ; z, \tag{24}$$

$$x \cdot y ; z \leq y ; (z \cdot \breve{y} ; x), \tag{25}$$

$$x ; 1 \cdot y ; z = (x ; 1 \cdot y) ; z, \quad y ; z \cdot 1 ; x = y ; (z \cdot 1 ; x), \tag{26}$$

$$z ; \breve{y} \cdot x = 0 \quad \Leftrightarrow \quad x ; y \cdot z = 0 \quad \Leftrightarrow \quad \breve{x} ; z \cdot y = 0. \tag{27}$$

The following lemma gives a well known alternate characterization of relation algebras, together with a variation on it that will be useful for us later.

LEMMA 2.2. *The following statements are equivalent.*
(1) $\mathfrak{A}$ *is a relation algebra*
(2) $\mathfrak{A}$ *satisfies* (2)–(4), (5), (7), *and* (27).
(3) $\mathfrak{A}$ *satisfies* (2)–(4), (7), (27), *and*

$$v ; x \cdot w ; y = 0 \quad \Leftrightarrow \quad \breve{v} ; w \cdot x ; \breve{y} = 0. \tag{28}$$

Proof. The equivalence of the first two statements is due to Chin–Tarski [2, Theorem 2.2]. The first statement implies the third, for if $\mathfrak{A}$ is a relation algebra then (2)–(27) hold, and from (5) and (27) we get

$$v\,;x\cdot w\,;y=0 \quad\Leftrightarrow\quad \breve{v}\,;(w\,;y)\cdot x=0 \quad\Leftrightarrow\quad (\breve{v}\,;w)\,;y\cdot x=0 \quad\Leftrightarrow\quad x\,;\breve{y}\cdot\breve{v}\,;w=0,$$

so (28) follows by the commutativity of $\cdot$, a consequence of (2)–(4). To complete the proof, we assume $\mathfrak{A}$ satisfies (2)–(4), (7), (27), and (28), and derive (5). From (2)–(4) we get all the laws of Boolean algebras. From (7) and (27) we have

$$x\cdot y=0 \quad\Leftrightarrow\quad x\,;1\text{'}\cdot y=0 \quad\Leftrightarrow\quad \breve{x}\,;y\cdot 1\text{'}=0 \quad\Leftrightarrow\quad \breve{\breve{x}}\,;1\text{'}\cdot y=0 \quad\Leftrightarrow\quad \breve{\breve{x}}\cdot y=0.$$

Taking y to be first $\bar{x}$ and then $\bar{\breve{\breve{x}}}$, we conclude by various Boolean algebraic laws that $x=\breve{\breve{x}}$, so (8) holds. From (8), (27), and (28) we get

$$x\,;(y\,;z)\cdot w=0 \quad\Leftrightarrow$$

$$\breve{x}\,;w\cdot y\,;z=0 \quad\Leftrightarrow$$

$$\breve{x}\,;w\cdot y\,;\breve{\breve{z}}=0 \quad\Leftrightarrow$$

$$x\,;y\cdot w\,;\breve{z}=0 \quad\Leftrightarrow$$

$$(x\,;y)\,;z\cdot w=0.$$

Applying this with $w=\overline{x\,;(y\,;z)}$ shows that $(x\,;y)\,;z\le x\,;(y\,;z)$, and the opposite inclusion follows by taking $w=\overline{(x\,;y)\,;z}$. Thus (5) holds. □

3. Simplicity

An algebra is simple if it contains at least two elements and has exactly two congruence relations, namely the identity relation and the universal relation on its underlying domain [4, Def. 0.2.34]. For relation algebras there is a particularly useful characterization of simplicity.

THEOREM 3.1. [8, Theorem 4.10] *Let* $\mathfrak{A}$ *be a relation algebra in which* $0\neq 1$. *Then the following conditions are equivalent*:

(1) $\mathfrak{A}$ *is simple*,
(2) *for every* $x\in A$, *if* $x\neq 0$ *then* $1=1\,;x\,;1$,
(3) *for all* $x, y\in A$, *if* $0=x\,;1\,;y$ *then* $x=0$ *or* $y=0$.

4. Persistent atoms

LEMMA 4.1. *Assume* $\mathfrak{A}$ *is a simple relation algebra and* x *is a persistent atom of* $\mathfrak{A}$. *Suppose* $\mathfrak{A}$ *has elements* y, z *such that*

$$0 \neq z \leq x \,;\, y, \tag{29}$$

$$\breve{z} \,;\, z \cdot \breve{y} \,;\, y \leq 1\text{'}. \tag{30}$$

Then z *is also a persistent atom of* $\mathfrak{A}$.

Proof. Suppose, to the contrary, that z is not a persistent atom of $\mathfrak{A}$. Then there is a simple relation algebra $\mathfrak{B}$, containing $\mathfrak{A}$ as a subalgebra, with elements $z_0, z_1 \in B$ such that

$$z_0 + z_1 = z, \quad z_0 \cdot z_1 = 0, \quad z_0 \neq 0, \quad z_1 \neq 0. \tag{31}$$

From (29) and (31) we get $0 \neq z_0 \leq z \leq x \,;\, y$, hence $0 \neq z_0 \,;\, \breve{y} \cdot x$ by (27). Since x is a persistent atom of $\mathfrak{A}$, it is an atom of $\mathfrak{B}$, so $0 \neq x \leq z_0 \,;\, \breve{y}$. Similarly, $0 \neq x \leq z_1 \,;\, \breve{y}$. Thus we have

$$0 \neq x \leq z_0 \,;\, \breve{y} \cdot z_1 \,;\, \breve{y}. \tag{32}$$

From (31) we get $z_1 \leq \overline{z_0}$, hence $\breve{z}_0 \,;\, z_1 \leq \breve{z}_0 \,;\, \overline{z_0} \leq 0\text{'}$ by (15) and (19), so $\breve{z}_0 \,;\, z_1 \cdot \breve{y} \,;\, y \leq \breve{z} \,;\, z \cdot 0\text{'} \cdot \breve{y} \,;\, y$. From this we get $0 = \breve{z}_0 \,;\, z_1 \cdot \breve{y} \,;\, y$ by (30), hence $0 = z_0 \,;\, \breve{y} \cdot z_1 \,;\, \breve{y}$ by (28) and (8), contradicting (32). □

5. Coherence

DEFINITION 5.1. Let x be an element of a relation algebra $\mathfrak{A}$. We say that x is a **coherent** element in $\mathfrak{A}$ if $e \,;\, x = x \,;\, e$ whenever $1\text{'} \geq e \in A$.

For example, all subidentity elements are coherent, that is, if $x \leq 1\text{'}$ then x is coherent. It is easy to show that the only coherent elements in $\mathfrak{Re}\,U$ are subidentity elements. On the other hand, in an integral relation algebra, where 1' is an atom, every element is coherent by various parts of Lemma 5.2. The converse holds when the algebra is simple by Lemma 5.3.

LEMMA 5.2. *Suppose* x *and* y *are elements of a relation algebra* $\mathfrak{A}$.

(1) *If* x *is coherent and* $y \leq x$ *then* y *is coherent*.

(2) *If* x *is coherent then* $\breve{x}$ *is coherent.*
(3) *If* x *and* y *are both coherent then* $x + y$ *and* $x ; y$ *are coherent.*
(4) *If* $x \leq 1'$ *and* x *is an atom, then* $x ; 1 ; x$ *is coherent.*

Proof. Assume x is coherent and $y \leq x$. For any $e \leq 1'$, we have, by (22),

$$e ; y = e ; (y \cdot x) = y \cdot e ; x = y \cdot x ; e = (y \cdot x) ; e = y ; e,$$

so y is coherent, and $\breve{e} = e$ by (20), so, by (10),

$$e ; \breve{x} = \breve{e} ; \breve{x} = (x ; e)\breve{} = (e ; x)\breve{} = \breve{x} ; \breve{e} = \breve{x} ; e.$$

This shows that $\breve{x}$ is coherent. Assume x and y are both coherent. For any $e \leq 1'$, we have $e ; x = x ; e$ and $e ; y = y ; e$, so, by (6), (13), and (5),

$$e ; (x + y) = e ; x + e ; y = x ; e + y ; e = (x + y) ; e,$$

and

$$e ; (x ; y) = (e ; x) ; y = (x ; e) ; y = x ; (e ; y) = x ; (y ; e) = (x ; y) ; e.$$

Thus $x + y$ and $x ; y$ are coherent. Finally, suppose $x \leq 1'$ and x is an atom. Let $e \leq 1'$. Since x is an atom, either $e \cdot x = 0$ or else $e \cdot x = x$. If $e \cdot x = 0$ then, by (5), (21), and (16), $e ; (x ; 1 ; x) = (e ; x) ; 1 ; x = (e \cdot x) ; 1 ; x = 0 ; 1 ; x = 0$, and, similarly, $0 = x ; 1 ; x ; e$. If $e \cdot x = x$, then $e ; (x ; 1 ; x) = (e \cdot x) ; 1 ; x = x ; 1 ; x = x ; 1 ; x ; e$. Therefore, $x ; 1 ; x$ is coherent.

LEMMA 5.3. *Let* $\mathfrak{A}$ *be a relation algebra. The following statements are equivalent:*
(1) 1 *is coherent in* $\mathfrak{A}$.
(2) *Every element in* $\mathfrak{A}$ *is coherent.*
(3) $\mathfrak{A}$ *satisfies the identity*

$$(1' \cdot x) ; 1 = 1 ; (1' \cdot x). \tag{33}$$

(4) $\mathfrak{A}$ *satisfies the identity*

$$y ; 1 = 1 ; y, \tag{34}$$

(5) $\mathfrak{A}$ *is a subdirect product of integral relation algebras.*

In particular, a relation algebra is integral if and only if it is simple and 1 *is coherent*.

Proof. The first statement is a special case of the second, but the first also implies the second by Lemma 5.2. The equivalence of the first and third statements is immediate from the definition of coherence. The fourth statement obviously implies the third. For the converse, assume (33) holds for all x. Then

$$\begin{aligned} y ; 1 &= y ; (1' \cdot \breve{y} ; y) ; 1 && (23)\\ &= y ; ((1' \cdot \breve{y} ; y) ; 1) && (5)\\ &= y ; (1 ; (1' \cdot \breve{y} ; y)) && (33) \text{ with } x = \breve{y} ; y\\ &\leq 1 ; (1 ; (1 ; y)) && (15)\\ &= 1 ; y && (5), (17) \end{aligned}$$

and, similarly, $1 ; y \leq y ; 1$, so (34) holds. In an integral relation algebra, the identity (33) holds since 1' is an atom, hence $(1' \cdot x) ; 1$ and $1 ; (1' \cdot x)$ are either both 0 or both 1. All subdirect products of integral relation algebras also satisfy (33). Thus the fifth statement implies the third. We complete the proof by showing the third and fourth statements imply the fifth. Suppose $\mathfrak{A}$ is a relation algebra satisfying (33) or (34). Every relation algebra is a subdirect product of simple relation algebras [7, Corollary 4.6]. Suppose $\mathfrak{A}$ is a subdirect product of simple relation algebras $\mathfrak{B}_i$, $i \in I$. Each $\mathfrak{B}_i$ is a homomorphic image of $\mathfrak{A}$, so $\mathfrak{B}_i$ also satisfies (33) or (34). We prove that 1' is an atom in $\mathfrak{B}_i$. Assume $0 \neq e \leq 1'$. We have $e ; 1 = 1 ; e$ by (33) or (34), so $e ; 1 ; (\bar{e} \cdot 1') = 0$ since, by (5), (21), and (16),

$$e ; 1 ; (\bar{e} \cdot 1') = 1 ; e ; (\bar{e} \cdot 1') = 1 ;\ (e ; (\bar{e} \cdot 1')) = 1 ; (e \cdot \bar{e} \cdot 1') = 1 ; (0 \cdot 1') = 0.$$

$\mathfrak{B}_i$ is simple, so $e = 0$ or $\bar{e} \cdot 1' = 0$ by Lemma 3.1. But $e \neq 0$ by assumption, so $\bar{e} \cdot 1' = 0$, hence $e = 1'$. Thus 1' is an atom and $\mathfrak{B}_i$ is integral by Theorem 1.2. □

The identity (33) in the previous theorem can be replaced by the identity $(1' \cdot x) ; y = y ; (1' \cdot x)$, for if (33) holds then

$$\begin{aligned} (1' \cdot x) ; y &= (1' \cdot x) ; (y \cdot 1)\\ &= y \cdot (1' \cdot x) ; 1 && (22)\\ &= y \cdot 1 ; (1' \cdot x) && (33)\\ &= (y \cdot 1) ; (1' \cdot x) && (22)\\ &= y ; (1' \cdot x), \end{aligned}$$

but (34) cannot be replaced by $x ; y = y ; x$ because there are noncommutative integral relation algebras.

Next we have three technical lemmas. They are designed for application to the five examples presented in the next section. In these algebras we can find elements v, w, x, y, z such that

$$v = \breve{w} ; w \cdot 0' \tag{35}$$

$$v ; v \cdot v = 0 \tag{36}$$

$$v < x ; \breve{y} \tag{37}$$

$$w ; x \cdot w ; y \leq w ; z \tag{38}$$

$$x ; \breve{z} \leq 0' \tag{39}$$

$$y ; \breve{z} \leq 0' \tag{40}$$

The hypotheses for all three lemmas are that $\mathfrak{A}$ *is a relation algebra with elements* v, w, x, y, z *satisfying conditions* (35)–(40).

LEMMA 5.4. *If* $w \geq w_0 \in A$ *then* $(w_0)\breve{} ; (w \cdot \overline{w_0}) = 0$.

LEMMA 5.5. v *is a coherent element of* $\mathfrak{A}$.

LEMMA 5.6. *If* $\mathfrak{A}$ *is integral*, *then* w *is an atom of* $\mathfrak{A}$.

Proof. First note that v is symmetric, since, by (14), (10), (18), and (8),

$$\breve{v} = (\breve{w} ; w \cdot 0')\breve{} = (\breve{w} ; w)\breve{} \cdot \breve{0'} = \breve{w} ; \breve{\breve{w}} \cdot 0' = \breve{w} ; w \cdot 0' = v.$$

Suppose $w_0 \leq w$. Let $w_1 = \overline{w_0} \cdot w$. Then

$$w_0 \cdot w_1 = 0 \quad \text{and} \quad w_0 + w_1 = w. \tag{41}$$

We have

$$(w_0)\breve{} ; w_1 \leq v \quad \text{and} \quad (w_1)\breve{} ; w_0 \leq v \tag{42}$$

since

$$
\begin{aligned}
(w_0)\breve{}\,;w_1 &= (w_0)\breve{}\,;(\overline{w_0}\cdot w) && \text{def. of } w_1\\
&\le (w_0)\breve{}\,;\overline{w_0}\cdot \breve{w}\,;w && (15)\\
&\le 0'\cdot(1'+v) && (19), (35)\\
&\le v && 0'\cdot 1' = 0
\end{aligned}
$$

and the other equation is proved similarly. By (42) and hypothesis (37) we get $(w_0)\breve{}\,;w_1 \le x\,;\breve{y}$, so $(w_0)\breve{}\,;w_1 = (w_0)\breve{}\,;w_1\cdot x\,;\breve{y}$. It follows by (28) that $(w_0)\breve{}\,;w_1 = 0$ iff $w_0\,;x\cdot w_1\,;y = 0$. We will prove the latter equality. By (41), (15), and hypothesis (38) and we have

$$
w_0\,;x\cdot w_1\,;y \le w\,;x\cdot w\,;y \le w\,;z, \tag{43}
$$

hence

$$
\begin{aligned}
w_0\,;x\cdot w_1\,;y &= w_0\,;x\cdot w_1\,;y\cdot w\,;z && (43)\\
&= w_0\,;x\cdot w_1\,;y\cdot(w_0+w_1)\,;z && (41)\\
&= w_0\,;x\cdot w_1\,;y\cdot w_0\,;z + w_0\,;x\cdot w_1\,;y\cdot w_1\,;z && (6)\\
&\le w_0\,;x\cdot w_1\,;1\cdot w_0\,;z + w_0\,;1\cdot w_1\,;y\cdot w_1\,;z. && (15)
\end{aligned}
$$

We can show that the last two terms are both 0, by two nearly identical proofs. The first term is 0 since

$$
\begin{aligned}
&w_0\,;x\cdot w_1\,;1\cdot w_0\,;z\\
&\quad\le w_1\,;1\cdot(w_0\cdot w_0\,;x\,;\breve{z})\,;z && (24)\\
&\quad\le w_1\,;1\cdot(w_0\cdot w_0\,;0')\,;z && \text{(5), hypothesis (39), (15)}\\
&\quad= (w_0\cdot w_1\,;1\cdot w_0\,;0')\,;z && (26)\\
&\quad\le (w_0\cdot w_1\,;(1\cdot(w_1)\breve{}\,;w_0)\cdot w_0\,;0')\,;z && (25), (15)\\
&\quad\le (w_0\cdot w_1\,;v\cdot w_0\,;0')\,;z && (42), (15)\\
&\quad\le (w_1\,;v\cdot w_0\,;(0'\cdot(w_0)\breve{}\,;w_0))\,;z && (25), (15)\\
&\quad\le (w_1\,;v\cdot w_0\,;(0'\cdot\breve{w}\,;w))\,;z && (41), (15)\\
&\quad= (w_1\,;v\cdot w_0\,;v)\,;z && (35)\\
&\quad\le w_1\,;(v\cdot(w_1)\breve{}\,;w_0\,;v)\,;z && (25), (5), (15)
\end{aligned}
$$

$$
\begin{aligned}
&\le w_1 ; (v \cdot v ; v) ; z && (42), (15)\\
&= w_1 ; 0 ; z && \text{hypothesis (36)}\\
&= 0 && (16)
\end{aligned}
$$

and the second term is 0 since

$$
\begin{aligned}
&w_0 ; 1 \cdot w_1 ; y \cdot w_1 ; z\\
&\quad\le w_0 ; 1 \cdot (w_1 \cdot w_1 ; y ; \breve{z}) ; z && (24)\\
&\quad\le w_0 ; 1 \cdot (w_1 \cdot w_1 ; 0') ; z && (5),\ \text{hypothesis (40)}, (15)\\
&\quad= (w_1 \cdot w_0 ; 1 \cdot w_1 ; 0') ; \mathrm{z} && (26)\\
&\quad\le (w_1 \cdot w_0 ; (1 \cdot (w_0)\breve{} ; w_1) \cdot w_1 ; 0') ; z && (25),(15)\\
&\quad\le (w_1 \cdot w_0 ; v \cdot w_1 ; 0') ; z && (42), (15)\\
&\quad\le (w_0 ; v \cdot w_1 ; (0' \cdot (w_1)\breve{} ; w_1)) ; z && (25), (15)\\
&\quad\le (w_0 ; v \cdot w_1 ; (0' \cdot \breve{w} ; w)) ; z && (41), (15)\\
&\quad= (w_0 ; v \cdot w_1 ; v) ; z && (35)\\
&\quad\le w_0 ; (v \cdot (w_0)\breve{} ; w_1 ; v) ; z && (25), (5), (15)\\
&\quad\le w_0 ; (v \cdot v ; v) ; z && (42), (15)\\
&\quad= w_0 ; 0 ; z && \text{hypothesis (36)}\\
&\quad= 0 && (16)
\end{aligned}
$$

This completes the proof of $(w_0)\breve{} ; w_1 = 0$, and shows that Lemma 5.4 holds. For Lemma 5.5, we show that v is coherent in $\mathfrak{A}$. Assume $e \le 1'$. Let

$$w_0 = w ; e \quad \text{and} \quad w_1 = w ; (\bar{e} \cdot 1'). \tag{44}$$

Then, by (13) and (7),

$$w_0 + w_1 = w ; e + w ; (\bar{e} \cdot 1') = w ; (e + \bar{e} \cdot 1') = w ; 1' = w,$$

and

$$w_0 \cdot w_1 = w ; e \cdot w ; (\bar{e} \cdot 1')$$

$$\begin{aligned}
&= (w\ ;\ e \cdot w)\ ;\ (\bar{e} \cdot 1') && (22)\\
&= ((w \cdot w)\ ;\ e)\ ;\ (\bar{e} \cdot 1') && (22)\\
&= w\ ;\ (e\ ;\ (\bar{e} \cdot 1')) && (5)\\
&= w\ ;\ (e \cdot \bar{e} \cdot 1') && (21)\\
&= w\ ;\ 0\\
&= 0. && (16)
\end{aligned}$$

Hence

$$w_1 = w \cdot \overline{w_0}.$$

It follows by Lemma 5.4 that $0 = (w_0)\breve{}\ ;\ w_1$, so

$$\begin{aligned}
e\ ;\ v\ ;\ (\bar{e} \cdot 1') &\le e\ ;\ (\breve{w}\ ;\ w)\ ;\ (\bar{e} \cdot 1') && (35), (15)\\
&= e\ ;\ \breve{w}\ ;\ (w\ ;\ (\bar{e} \cdot 1')) && (5)\\
&= \breve{e}\ ;\ \breve{w}\ ;\ (w\ ;\ (\bar{e} \cdot 1')) && (20)\\
&= (w\ ;\ e)\breve{}\ ;\ (w\ ;\ (\bar{e} \cdot 1')) && (10)\\
&= (w_0)\breve{}\ ;\ w_1 && (44)\\
&= 0.
\end{aligned}$$

This implies $e\ ;\ v = e\ ;\ v\ ;\ e$ since

$$e\ ;\ v = e\ ;\ v\ ;\ 1' = e\ ;\ v\ ;\ (e + \bar{e} \cdot 1') = e\ ;\ v\ ;\ e + e\ ;\ v\ ;\ (\bar{e} \cdot 1') = e\ ;\ v\ ;\ e + 0 = e\ ;\ v\ ;\ e.$$

From $e\ ;\ v = e\ ;\ v\ ;\ e$ we also get $v\ ;\ e = e\ ;\ v\ ;\ e$ as follows, using the symmetry of v, (20), (10), and (5).

$$v\ ;\ e = \breve{v}\ ;\ \breve{e} = (e\ ;\ v)\breve{} = (e\ ;\ v\ ;\ e)\breve{} = \breve{e}\ ;\ (\breve{v}\ ;\ \breve{e}) = e\ ;\ v\ ;\ e.$$

Thus $e\ ;\ v = v\ ;\ e$, which finishes the proof of Lemma 5.5. For Lemma 5.6, we prove the contrapositive. Suppose w is not an atom of $\mathfrak{A}$. There there is some $w_0 \in A$ such that $0 \ne w_0 \le w$ and $0 \ne w \cdot \bar{w}_0$. By Lemma 5.4 we have $0 = (w_0)\breve{}\ ;\ (w \cdot \bar{w}_0)$, so $\mathfrak{A}$ is not integral. □

6. Five examples

In this section we present five finite integral nonrepresentable relation algebras with four atoms. Four of these algebras, $\mathfrak{C}_1$–$\mathfrak{C}_4$, are symmetric (satisfy $\breve{x} = x$), while the fifth, $\mathfrak{C}_5$, is not symmetric. In all of these algebras, 1' and at least one atom included in 0' are persistent atoms. In three of them, *all* atoms are persistent. $\mathfrak{C}_1$–$\mathfrak{C}_4$ have the same Boolean part, namely, a 4-atom Boolean algebra whose atoms are 1', *a*, *b*, *c*. The atom 1' is chosen as the distinguished element. Conversion is trivial, in the sence that $\mathfrak{C}_1$–$\mathfrak{C}_4$ satisfy $\breve{x} = x$. The tables below define relative multiplication on the atoms. Plus signs are omitted to save space. For example, "*abc*" replaces "$a + b + c$". Relative products involving the remaining 12 elements can be computed from these tables, using the fact that relative multiplication is a normal operator, that is, it distributes over Boolean join and $0 ; x = 0 = x ; 0$ for every *x*. From the tables it is clear that the atom 1' does happen to be a (two-sided) identity for relative multiplication. Note that relative multiplication is commutative in these algebras, so that the tables are symmetric about the main diagonal. By the way, it follows immediately from (10) that relative multiplication is commutative in any symmetric relation algebra.

$\mathfrak{C}_1$	1'	a	b	c
1'	1'	a	b	c
a	a	1'c	bc	abc
b	b	bc	1'ac	abc
c	c	abc	abc	1'ab

$\mathfrak{C}_2$	1'	a	b	c
1'	1'	a	b	c
a	a	1'c	bc	abc
b	b	bc	1'abc	abc
c	c	abc	abc	1'ab

$\mathfrak{C}_3$	1'	a	b	c
1'	1'	a	b	c
a	a	1'c	bc	abc
b	b	bc	1'ab	ac
c	c	abc	ac	1'ab

$\mathfrak{C}_4$	1'	a	b	c
1'	1'	a	b	c
a	a	1'c	bc	abc
b	b	bc	1'a	ac
c	c	abc	ac	1'ab

The atoms of the Boolean part of $\mathfrak{C}_5$ are 1', *a*, *b*, and $\breve{b}$. Again, 1' is chosen as distinguished element. Conversion in $\mathfrak{C}_5$ is not trivial. It leaves 1' and *a* unchanged, i.e., $\breve{1}' = 1'$ and $\breve{a} = a$, but it interchanges *b* and $\breve{b}$. The definition of conversion is reflected in the choice of notation: we use "$\breve{b}$" in place of "*c*" as a reminder that *c* is the converse of *b*. The restriction of relative multiplication to the atoms of $\mathfrak{C}_5$ is given in the table below.

$\mathfrak{C}_5$	1'	a	b	$\breve{b}$
1'	1'	a	b	$\breve{b}$
a	a	1'$b\breve{b}$	$ab\breve{b}$	$ab\breve{b}$
b	b	$ab\breve{b}$	$a\breve{b}$	1'a
$\breve{b}$	$\breve{b}$	$ab\breve{b}$	1'a	ab

In the next section we prove that $\mathfrak{C}_3$ is a relation algebra. Similar proofs show that $\mathfrak{C}_1$, $\mathfrak{C}_2$, $\mathfrak{C}_4$, and $\mathfrak{C}_5$ are also relation algebras.

THEOREM 6.1. 1' *and* a *are persistent atoms of* $\mathfrak{C}_1$, $\mathfrak{C}_2$, $\mathfrak{C}_3$, *and* $\mathfrak{C}_4$.

Proof. Suppose that $\mathfrak{A}$ is a simple relation algebra that has one of $\mathfrak{C}_1$, $\mathfrak{C}_2$, $\mathfrak{C}_3$, or $\mathfrak{C}_4$ as a subalgebra. Let $w = a$, $x = a$, $y = b$, $z = c$, and $v = \breve{w} ; w \cdot 0'$. Then

$$v = \breve{w} ; w \cdot 0' = \breve{a} ; a \cdot 0' = a ; a \cdot 0' = (1' + c) \cdot 0' = c$$

and conditions (36)–(40) hold, since

$$v ; v \cdot v = c ; c \cdot c = (1' + a + b) \cdot c = 0,$$

$$v = c \leq b + c = a ; b = a ; \breve{b} = x ; \breve{y},$$

$$w ; x \cdot w ; y = a ; a \cdot a ; b = (1' + c) \cdot (b + c) = c \leq 0' = a ; c = w ; z.$$

$$x ; \breve{z} = a ; \breve{c} = a ; c = 0',$$

$$y ; \breve{z} = b ; \breve{c} = b ; c \leq 0'.$$

By Lemma 4.4, c is coherent in $\mathfrak{A}$. $\mathfrak{A}$ is assumed to be simple, so by Lemma 5.2, $c + c ; c$ is coherent in $\mathfrak{A}$. But $c + c ; c = c + \bar{c} = 1$, so 1 is coherent in $\mathfrak{A}$. By Lemma 5.3, $\mathfrak{A}$ is integral. Hence 1' must be an atom of $\mathfrak{A}$, and it follows from Lemma 5.6 that a is an atom of $\mathfrak{A}$. This shows 1' and a are persistent atoms of $\mathfrak{C}_1$–$\mathfrak{C}_4$.

THEOREM 6.2. 1', b, and $\breve{b}$ *are persistent atoms of* $\mathfrak{C}_5$.

Proof. Suppose that $\mathfrak{A}$ is a simple relation algebra and $\mathfrak{C}_5 \subseteq \mathfrak{A}$. Let $w = b$, $x = b$, $y = b$, $z = a$ and $v = \breve{b} ; b \cdot 0' = (1' + a) \cdot 0' = a$. Then (35)–(40) hold, so from Lemma 5.5, Lemma 5.2, and $a + a ; a = a + 1' + b + \breve{b} = 1$ we conclude that 1 is coherent in $\mathfrak{A}$. $\mathfrak{A}$ must be integral by Lemma 5.3, and b is an atom of $\mathfrak{A}$ by Lemma 5.6. There happens to be an automorphism of $\mathfrak{C}_5$ that interchanges b and $\breve{b}$, so we can interchange b and $\breve{b}$ in the previous argument. Hence $\breve{b}$ is also an atom of $\mathfrak{A}$. Thus 1', b, and $\breve{b}$ are persistent atoms of $\mathfrak{C}_5$. □

We know that $\mathfrak{C}_1$ and $\mathfrak{C}_2$ have arbitrarily large proper integral (and hence simple) extensions. On the other hand, $\mathfrak{C}_3$, $\mathfrak{C}_4$, and $\mathfrak{C}_5$ have no such extensions.

THEOREM 6.3. $\mathfrak{C}_3$, $\mathfrak{C}_4$, *and* $\mathfrak{C}_5$ *have no proper simple extensions*.

Proof. Since the algebras involved are finite, it suffices to show that all the atoms of $\mathfrak{C}_3$, $\mathfrak{C}_4$, and $\mathfrak{C}_5$ are persistent. We have shown in Theorem 6.1 that 1' and

a are persistent in $\mathfrak{C}_3$ and $\mathfrak{C}_4$. Next we apply Lemma 4.1 to $\mathfrak{C}_3$ and $\mathfrak{C}_4$. Let $z = c$ and $x = y = a$. Then (29) holds since $c \leq a \,;\, a$, (30) holds since $c \,;\, c \cdot a \,;\, a \leq 1'$, and a is persistent, so c is persistent. Apply Lemma 4.1 again, with $z = b$, $x = c$, and $y = a$. Since $b \leq c \,;\, a$, $b \,;\, b \cdot a \,;\, a \leq 1'$, and c is persistent, we conclude that b is also persistent. Apply Lemma 4.1 to $\mathfrak{C}_5$ with $z = a$ and $x = y = b$. Then (29) holds since $a \leq b \,;\, b$, (30) holds since $\breve{a} \,;\, a \cdot \breve{b} \,;\, b \leq 1'$, and b is persistent by Theorem 6.2, so a is persistent in $\mathfrak{C}_5$. □

7. More examples

Let B be a set disjoint from $\{1', a, c\}$. Let $B^{[3]}$ be the set of 3-element subsets of B:

$$B^{[3]} = \{\{b, b', b''\} : b, b', b'' \in B, b \neq b' \neq b'' \neq b\}.$$

Choose any $\Delta \subseteq B^{[3]}$. For all $b, b' \in B$ let

$$f_\Delta(b, b') = \{b, b'\} \cup \{b'' : \{b, b', b''\} \in \Delta\}.$$

Note that, for every such choice of B, Δ, and any $b, b' \in B$, we have $f_\Delta(b, b) = \{b\}$ and $\{b, b'\} \subseteq f_\Delta(b, b') = f_\Delta(b', b) \subseteq B$. If B has fewer than three elements then $\{b, b'\} = f_\Delta(b, b')$ for all $b, b' \in B$. Let $\{1', a, c\} \cup B$ be the set of atoms of an atomic Boolean algebra $\mathfrak{A}_0$. Let the atom 1' be the distinguished element. Define conversion on $\mathfrak{A}_0$ by setting $\breve{x} = x$ for every x. Define the binary operation ; on $\mathfrak{A}_0$ by first defining it on the atoms of $\mathfrak{A}_0$ in the table below, and extending it to arbitrary elements x and y according to the formula

$$x \,;\, y = \sum_{\substack{p, q \in \mathrm{At}\,\mathfrak{A}_0 \\ x \geq p,\, y \geq q}} p \,;\, q. \tag{45}$$

The values for ; given in the table apply whenever b and b' are any two distinct elements of B.

$\mathfrak{D}^B_\Delta$	1'	a	c	b	b'
1'	1'	a	c	b	b'
a	a	$1' + c$	0'	$0' \cdot \bar{b}$	$0' \cdot \bar{a}$
c	c	0'	$\bar{c}$	$0' \cdot \bar{b}$	$0' \cdot \overline{b'}$
b	b	$0' \cdot \bar{a}$	$0' \cdot \bar{b}$	$\bar{c}$	$a + c + f_\Delta(b, b')$
b'	b'	$0' \cdot \bar{a}$	$0' \cdot \overline{b'}$	$a + c + f_\Delta(b', b)$	$\bar{c}$

An alternative way to define ; is to specify the set of triples $\langle x, y, z \rangle$ such that $x, y, z \in \{1', a, c\} \cup B$ and $x ; y \cdot z = 0$. To do so, first let

$$
\begin{aligned}
F = {} & \{\langle 1', x, y \rangle : x, y \in \{1', a, c\} \cup B, x \neq y\} \\
& \cup \{\langle a, a, b \rangle : b \in B\} \\
& \cup \{\langle c, c, c \rangle\} \\
& \cup \{\langle b, b, c \rangle : b \in B\} \\
& \cup \{\langle b, b', b'' \rangle : \{b, b', b''\} \in B^{[3]} \sim \Delta\}.
\end{aligned}
$$

Then

$$
\begin{aligned}
& 0 = x ; y \cdot z \\
\Leftrightarrow \quad & \emptyset \neq F \cap \{\langle x, y, z \rangle, \langle x, z, y \rangle, \langle y, x, z \rangle, \langle y, z, x \rangle, \langle z, x, y \rangle, \langle z, y, x \rangle\}.
\end{aligned}
\tag{46}
$$

Let $\mathfrak{D}^B_\Delta$ be the Boolean algebra with operators obtained by supplementing the Boolean algebra $\mathfrak{A}_0$ with the binary operation ;, trivial conversion, and the distinguished element 1'. It is easy to see from the table and (45) that 1' is an identity element for ;. It is obvious from (46) that (27) holds. Rather more tedious is the task of proving (46) from the table, or of deriving the table from (46). Once this is done, it is easy to see that ; is associative, since one need only check that (28) holds whenever v, w, x, y are atoms of $\mathfrak{D}^B_\Delta$. Indeed, pairwise comparison of the entries in the table shows that if $v ; x \cdot w ; y = 0$ then one of v, w, x, y must be 1', in which case (28) holds by (7), (12), and (27).

If B has only one element, then $B^{[3]} = \emptyset$ and the table for $\mathfrak{D}^B_\Delta$ reduces to the table for $\mathfrak{C}_3$. We indicate this by writing $\mathfrak{D}^1_\emptyset = \mathfrak{C}_3$.

$\mathfrak{D}^1_\emptyset$	1'	a	c	b
1'	1'	a	c	b
a	a	$1' + c$	0'	$0' \cdot \bar{a}$
c	c	0'	$\bar{c}$	$0' \cdot \bar{b}$
b	b	$0' \cdot \bar{a}$	$0' \cdot \bar{b}$	$\bar{c}$

=

$\mathfrak{C}_3$	1'	a	c	b
1'	1'	a	c	b
a	a	1'c	abc	bc
c	c	abc	1'ab	ac
b	b	bc	ac	1'ab

If B is a two-element set, then $B^{[3]} = \emptyset$ and the only algebra we get is $\mathfrak{D}^2_\emptyset$. When B has exactly three elements, there are two choices for Δ, namely $\emptyset$ and $\{B\}$, giving two (nonisomorphic) algebras, $\mathfrak{D}^3_\emptyset$ and $\mathfrak{D}^3_{\{B\}}$. If B is a finite set of cardinality k, then $\mathfrak{D}^B_\Delta$ is a relation algebra with $3 + k$ atoms. There are $2^{\frac{1}{6}k(k-1)(k-2)}$ such algebras, since the number of 3-element subsets of B is $\frac{1}{6}k(k-1)(k-2)$, but the number of isomorphism types is smaller, since $\mathfrak{D}^B_\Delta$ and $\mathfrak{D}^B_{\Delta'}$ are isomorphic whenever there is a permutation of B that carries Δ onto Δ'.

THEOREM 7.1. *If* B *is a finite and* $\Delta \subseteq B^{[3]}$, *then* $\mathfrak{D}^B_\Delta$ *is a finite symmetric integral nonrepresentable relation algebra with no proper simple extensions.*

Proof. Simply repeat those parts of the proofs of Theorems 6.1 and 6.3 which show that $\mathfrak{C}_3$ has no proper simple extensions. (The parts of those proofs that concern b apply to every element of B.) It follows that $\mathfrak{D}^B_\Delta$ is not representable, but it is also easy to give a direct proof. Consider the equation

$$t ; u \cdot v ; w \cdot x ; y \leq t ; [\breve{t} ; v \cdot u ; \breve{w} \cdot (\breve{t} ; x \cdot u ; \breve{y}) ; (\breve{x} ; v \cdot y ; \breve{w})] ; w. \tag{L}$$

This equation was involved in the initial discovery of nonrepresentable relation algebras by Lyndon [9]; see Chin-Tarski [2, p. 354]. It is not difficult to show that (L) holds in every representable relation algebra. However, (L) fails in $\mathfrak{D}^B_\Delta$ when $t = a$, $u = a$, $v = a$, $w = b$, $x = a$, and $y = c$, since

$$t ; u \cdot v ; w \cdot x ; y = a ; a \cdot a ; b \cdot a ; c = (1' + c) \cdot (0' \cdot \bar{a}) \cdot (0') = c$$

but

$$\begin{aligned}
&t ; [\breve{t} ; v \cdot u ; \breve{w} \cdot (\breve{t} ; x \cdot u ; \breve{y}) ; (\breve{x} ; v \cdot y ; \breve{w})] ; w \\
&\quad = a ; [\breve{a} ; a \cdot a ; \breve{b} \cdot (\breve{a} ; a \cdot a ; \breve{c}) ; (\breve{a} ; a \cdot c ; \breve{b})] ; b \\
&\quad = a ; [a ; a \cdot a ; b \cdot (a ; a \cdot a ; c) ; (a ; a \cdot c ; b)] ; b \\
&\quad = a ; [(1' + c) \cdot (0' \cdot \bar{a}) \cdot ((1' + c) \cdot 0') ; ((1' + c) \cdot (0' \cdot \bar{b}))] ; b \\
&\quad = a ; [c \cdot c ; c] ; b \\
&\quad = a ; [c \cdot \bar{c}] ; b \\
&\quad = a ; 0 ; b \\
&\quad = 0.
\end{aligned}$$

The proof of Theorem 7.1 shows that (L) fails in $\mathfrak{C}_3$ when $t = a$, $u = a$, $v = a$, $w = b$, $x = a$, and $y = c$. (L) fails in $\mathfrak{C}_1$, $\mathfrak{C}_2$, and $\mathfrak{C}_4$ under the same assignment of variables. (L) also fails in $\mathfrak{C}_5$ when $t = b$, $u = a$, $v = b$, $w = b$, $x = b$, and $y = \breve{b}$.

THEOREM 7.2. *Assume that* B *is finite,* $|B| = n$, *and* $\mathfrak{D}$ *is either* $\mathfrak{D}^B_{B[3]}$ *or* $\mathfrak{D}^B_\emptyset$.

(i) *If* $\{1', a, c\} \subseteq C \subseteq D$ *and* C *is closed under the Boolean operations then* C *is a subuniverse of* $\mathfrak{D}$.

(ii) $\mathfrak{D}$ *cannot be generated by fewer than* $\log_2(3 + n) - 3$ *elements.*

(iii) *If* $m \leq \log_2(3+n) - 3$ *then* $\mathfrak{D}$ *cannot be embedded in any* n*-generated relation algebra.*

Proof. (i): By assumption, C is a subuniverse of the Boolean part of $\mathfrak{D}$ and C contains 1'. Let $\mathfrak{C}_0$ be the Boolean subalgebra of the Boolean part of $\mathfrak{D}$ whose universe is C. Since $\mathfrak{D}$ is symmetric, C is closed under conversion, so we need only show that the relative product of any two atoms of $\mathfrak{C}_0$ is the join of elements of C. Note that 1', a, and c are atoms of $\mathfrak{C}_0$, and that almost every entry in the two possible tables for $\mathfrak{D}$, shown below, is obtained solely from 1', a, and c by Boolean operations. The only exceptions are relative products involving c and another atom $x \in \mathrm{At}\,\mathfrak{C}_0 \sim \{1', a, c\}$. There are just two cases. Either there are distinct b, $b' \in B$ such that $x \geq b + b'$, in which case $c\,;x \geq c\,;(b+b') = 0'$ so $c\,;x = 0' \in C$, or else $x \in B$ and $c\,;x = 0' \cdot \bar{x} \in C$.

$\mathfrak{D}^{B}_{B[3]}$	1'	a	c	b	b'
1'	1'	a	c	b	b'
a	a	$1' + c$	0'	$0' \cdot \bar{b}$	$0' \cdot \bar{a}$
c	c	0'	$\bar{c}$	$0' \cdot \bar{b}$	$0' \cdot \overline{b'}$
b	b	$0' \cdot \bar{a}$	$0' \cdot \bar{b}$	$\bar{c}$	0'
b'	b'	$0' \cdot \bar{a}$	$0' \cdot \overline{b'}$	0'	$\bar{c}$

$\mathfrak{D}^{B}_{\emptyset}$	1'	a	c	b	b'
1'	1'	a	c	b	b'
a	a	$1' + c$	0'	$0' \cdot \bar{b}$	$0' \cdot \bar{a}$
c	c	0'	$\bar{c}$	$0' \cdot \bar{b}$	$0' \cdot \overline{b'}$
b	b	$0' \cdot \bar{a}$	$0' \cdot \bar{b}$	$\bar{c}$	$a + c$
b'	b'	$0' \cdot \bar{a}$	$0' \cdot \overline{b'}$	$a + c$	$\bar{c}$

(ii): The subuniverse of $\mathfrak{D}$ generated by $X \subseteq D$ is contained in the closure of $\{1', a, c\} \cup X$ under the Boolean operations, and hence contains no more than $2^{2^{3+|X|}}$ elements. But $\mathfrak{D}$ has 2^{3+n} elements, so if X generates $\mathfrak{D}$ then X must have at least $\log_2(3+n) - 3$ elements.

(iii): Suppose $\mathfrak{D} \subseteq \mathfrak{A} \in \mathsf{RA}$. Every relation algebra is semisimple, i.e., it is a subdirect product of simple relation algebras [7, Corollary 4.6]. It follows that there is a homomorphism h from $\mathfrak{A}$ onto a simple relation algebra $\mathfrak{B}$. Since $\mathfrak{D}$ is simple, the restriction h' of h to D is an isomorphism from $\mathfrak{D}$ to a subalgebra of $\mathfrak{B}$. By Theorem 7.1, h' is actually an isomorphism *onto* $\mathfrak{B}$, so composing its inverse with h produces a homomorphism h'' from $\mathfrak{A}$ onto $\mathfrak{D}$:

$$h''(x) = (h')^{-1}(h(x)).$$

In general, homomorphic images of k-generated algebras are k-generated. It follows that $\mathfrak{A}$ cannot be generated by fewer than $\log(3+n)-3$ elements. □

8. Conclusions

Let $\pi(w)$ be the formula $0 \neq w ; 1 ; \breve{w} + \breve{w} ; 1 ; w \leq 1'$. It has been known for decades that $\pi(w)$ defines atoms in simple relation algebras. It was natural to wonder whether there are other such formulas. Our work shows that there are. Let $\varphi(w)$ be the formula

$$(\breve{w} ; w \cdot 0') ; (\breve{w} ; w \cdot 0') = \overline{\breve{w} ; w} + 1' \wedge$$

$$\exists x, y, z[(\breve{w} ; w \cdot 0' \leq x ; \breve{y}) \wedge (w ; x \cdot w ; y \leq w ; z) \wedge (x ; \breve{z} + y ; \breve{z} \leq 0')].$$

The proofs of Theorems 6.1 and 6.2 show that $\varphi(w)$ defines atoms in simple relation algebras. Another such formula, $\psi(w)$, obtained from the use of Lemma 4.1 in the proof of Theorem 6.3, is

$$\exists v[\varphi(v) \wedge \exists x[(0 \neq w \leq v ; x) \wedge (\breve{w} ; w \cdot \breve{x} ; x \leq 1')]].$$

One interesting contrast between these formulas is that, with trivial exceptions, atoms that satisfy π can occur only in nonintegral simple relation algebras, while atoms that satisfy φ or ψ can occur only in integral relation algebras. Atoms that satisfy π, φ, or ψ in simple relation algebras are persistent, but it seems unlikely that these formulas exhaust the ways in which persistent atoms can arise. Some of the problems solved here have simple formulations, requiring only "yes" or "no" as an answer. On the other hand, Jónsson posed the problem of finding *all* finite simple relation algebras in which every atom is persistent [12, Problem **P6**]. Our construction of large numbers of such algebras suggests that this goal is probably unattainable.

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