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On subtractive varieties III: From ideals to congruences

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Abstract. As a sequel to [2] and [15] we investigate ideal properties focusing on subtractive varieties. Here we probe the relations between congruences and ideals in subtractive varieties, in order to give some means to recover the congruence structure from the ideal structure. To do so we consider mainly two operators from the ideal lattice to the congruence lattice of a given algebra and we classify subtractive varieties according to various properties of these operators. In the last section several examples are discussed in details.

0. Introduction

The aim of this paper is to investigate the relation ''congruences-ideals'' in subtractive algebras. In a sense we want to see how short the algebras fall from being 0-regular (in which case the relation is the best possible: a lattice isomorphism), and also give a sensible way to control the set

$$\mathrm{CON}(I) = \{\theta \in \mathrm{Con}(\mathbf{A}) \colon 0/\theta = I\}$$

of the congruences associated to an ideal I, in order to ascertain something of the structure of the family $\{\mathrm{CON}(I) \colon I \in \mathrm{I}(\mathbf{A})\}$.

A variety $\mathscr{V}$ has *normal ideals* if for all $\mathbf{A} \in \mathscr{V}$ every ideal of $\mathbf{A}$ is a congruence class; subtractive algebras do have normal ideals. In particular we consider the two natural mappings $(\ )^\delta$ and $(\ )^\varepsilon$ which associate to any ideal I the least (resp. the greatest) congruence in $\mathrm{CON}(I)$. If $\mathscr{V}$ is a variety with normal ideals we set $\mathscr{V}_\varepsilon = \{\mathbf{A}/(0)^\varepsilon_{\mathbf{A}} \colon \mathbf{A} \in \mathscr{V}\}$; then algebraic properties of $\mathscr{V}$ and closure properties of $\mathscr{V}_\varepsilon$ turn out to be connected with logical properties of the assertional logic $\mathsf{AL}_{\mathscr{V}}$ (in the sense of Blok and Pigozzi [6]). For instance if $\mathscr{V}$ has normal ideals, then

- $\mathscr{V}$ is subtractive iff $\mathsf{AL}_{\mathscr{V}}$ is protoalgebraic iff $\mathscr{V}_\varepsilon$ is closed under subdirect products iff $(\ )^\varepsilon$ is monotonic.
- $\mathscr{V}$ is ideal determined iff $\mathscr{V}_\varepsilon$ is a variety iff $(\ )^\varepsilon$ is a homomorphism iff $\mathsf{AL}_{\mathscr{V}}$ is strongly algebraizable.

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The last section of the paper is devoted to examples of varieties with (strongly) normal ideals; this serves both to illuminate the entire subject and also to emphasize that the concept of an algebra with (strongly) normal ideals applies to a fairly broad spectrum of structures.

We tried to make the present paper as self-contained as possible; the few results in [2], [15] to which we refer are easily proved if needed.

1. Preliminaries

All algebras (and varieties) considered will be of a signature with a fixed constant 0. If R is a binary relation on a set A and $a \in A$ we set $a/R = \{x : a\ R\ x\}$. If $\langle L, \leq \rangle$ is a poset, $a, b \in L$, then $[a, b] = \{x \in L : a \leq x \leq b\}$ (the *closed interval* from a to b).

If $\mathbf{A}$ is an algebra a *semicongruence* [2] of $\mathbf{A}$ is a reflexive subalgebra of $\mathbf{A} \times \mathbf{A}$; $\mathrm{Con}(\mathbf{A})$ is the congruence lattice of $\mathbf{A}$.

We now recall the main notions from the theory of ideals [15]. If $\mathscr{K}$ is a class of similar algebras, a term $p(x_1, \ldots, x_m, y_1, \ldots, y_n)$ is a *$\mathscr{K}$-ideal term in $\vec{y}$* (and we write $p(\vec{x}, \vec{y}) \in \mathrm{IT}_{\mathscr{K}}(\vec{y})$) if the identity $p(\vec{x}, 0, \ldots, 0) \approx 0$ holds in $\mathscr{K}$. A nonempty subset I of $\mathbf{A} \in \mathscr{K}$ is a *$\mathscr{K}$-ideal* of $\mathbf{A}$ if for any $p(\vec{x}, \vec{y}) \in \mathrm{IT}_{\mathscr{K}}(\vec{y})$, for $\vec{a} \in A$ and $\vec{b} \in I$, $p(\vec{a}, \vec{b}) \in I$. Under inclusion, the set $\mathrm{I}_{\mathscr{K}}(\mathbf{A})$ of all $\mathscr{K}$-ideals of $\mathbf{A}$ is an algebraic lattice; if $H \subseteq A$, the ideal $\langle H \rangle_{\mathscr{K}}$ generated by H is easily seen to be the set $\{p(\vec{a}, \vec{b}) : p(\vec{x}, \vec{y}) \in \mathrm{IT}_{\mathscr{K}}(\vec{y}), \vec{a} \in A, \vec{b} \in H\}$; likewise one easily sees that if S is a semicongruence of $\mathbf{A}$, then $0/S \in \mathrm{I}(\mathbf{A})$. If $H = \{a_1, \ldots, a_n\}$ we will write $\langle a_1, \ldots, a_n \rangle_{\mathbf{A}}$ but, in (apparent) contrast with the previous notation, we will write $(a)^{\mathscr{K}}_{\mathbf{A}}$ whenever $H = \{a\}$; note that $(0)^{\mathscr{K}}_{\mathbf{A}} = \{0\}$.

When $\mathscr{K}$ is $\{\mathbf{A}\}$ (or, equivalently, the variety $\boldsymbol{V}\mathbf{A}$ generated by $\mathbf{A}$), then a $\mathscr{K}$-ideal of $\mathbf{A}$ will be called an *ideal* and we drop all the affixes and suffixes in sight. By $\mathrm{N}(\mathbf{A})$ we denote the set $\{0/\theta : \theta \in \mathrm{Con}(\mathbf{A})\}$ and trivially $\mathrm{N}(\mathbf{A}) \subseteq \mathrm{I}(\mathbf{A}) \subseteq \mathrm{I}_{\mathscr{K}}(\mathbf{A})$ whenever $\mathbf{A} \in \mathscr{K}$. $\mathrm{N}(\mathbf{A})$ inherits in a natural way the lattice structure of $\mathrm{Con}(\mathbf{A})$. One can easily check that for any $\mathbf{A} \in \mathscr{K}$ the following are equivalent:

(1) The mapping from $\mathrm{Con}(\mathbf{A})$ into $\mathrm{I}_{\mathscr{K}}(\mathbf{A})$ defined by $\theta \mapsto 0/\theta$ is a lattice homomorphism.
(2) $\mathrm{N}(\mathbf{A})$ is a sublattice of $\mathrm{I}_{\mathscr{K}}(\mathbf{A})$.

For $X \subseteq A$ we denote by $\sigma(X)$ the smallest congruence which makes all elements of X congruent, namely

$$\sigma(X) = \mathrm{Cg}_{\mathbf{A}}(X \times X).$$

Note that $X \in \mathrm{N}(\mathbf{A})$ iff $X = 0/\sigma(X)$, iff for every unary polynomial $g(x)$ and for any $a \in X$, $g(a) \in X$ iff $g(0) \in X$.

PROPOSITION 1.1. *For any* $X \subseteq A$

$$\mathrm{Cg}_{\mathbf{A}}(\{0\} \times X) = \mathrm{Cg}_{\mathbf{A}}(\{0\} \times \langle X \rangle) = \sigma(\langle X \rangle).$$

If moreover $0 \in X$, *then*

$$\sigma(X) = \mathrm{Cg}_{\mathbf{A}}(\{0\} \times X) = \sigma(\langle X \rangle).$$

Proof. It is enough to show that

$$\langle X \rangle \times \langle X \rangle \subseteq \mathrm{Cg}_{\mathbf{A}}(\{0\} \times X).$$

Let p, p' be ideal terms, $a = p(\vec{b}, \vec{x})$, $c = p'(\vec{d}, \vec{y})$ with $\vec{b}, \vec{d} \in A$ and $\vec{x}, \vec{y} \in X$. Then $(p(\vec{b}, \vec{x}), 0) \in \mathrm{Cg}_{\mathbf{A}}(\{0\} \times X)$ and similarly for $(p'(\vec{d}, \vec{y}), 0)$. Hence $(a, c) \in \mathrm{Cg}_{\mathbf{A}}(\{0\} \times X)$; if $0 \in X$, then $\mathrm{Cg}_{\mathbf{A}}(\{0\} \times X) = \mathrm{Cg}_{\mathbf{A}}(X \times X)$. □

Most often we are interested in the restriction of σ to $\mathrm{I}(\mathbf{A})$.

PROPOSITION 1.2. *The map* $I \mapsto \sigma(I)$ *is a join homomorphism from* $\mathrm{I}(\mathbf{A})$ *into* $\mathrm{Con}(\mathbf{A})$: *for any* $I, J \in \mathrm{I}(\mathbf{A})$,

$$\sigma(I \vee J) = \sigma(I) \vee \sigma(J).$$

Proof. One inclusion is trivial. Next observe that

$$\sigma(I \vee J) \subseteq \mathrm{Cg}_{\mathbf{A}}(I \times J) \subseteq \sigma(I) \vee \sigma(J).$$

In fact $\sigma(I \vee J) = \sigma(\langle I \cup J \rangle) = \sigma(I \cup J)$. Let then $h \in I \cup J$; then $(0, h) \in I \times J$ or $(0, h) \in J \times I$, hence $\{0\} \times (I \cup J) \subseteq (I \times J) \cup (J \times I) \subseteq \mathrm{Cg}_{\mathbf{A}}(I \times J)$. Now pick $i \in I$, and $j \in J$: then $(0, i) \in \sigma(I)$, $(0, j) \in \sigma(J)$ which implies $(i, j) \in \sigma(I) \circ \sigma(J)$. □

Thus the interplay of the two mappings

$$0/ : \theta \in \mathrm{Con}(\mathbf{A}) \mapsto 0/\theta \in \mathrm{I}(\mathbf{A})$$

$$\sigma : I \in \mathrm{I}(\mathbf{A}) \mapsto \sigma(I) \in \mathrm{Con}(\mathbf{A})$$

is a key tool for our aim: $0/$ is of course a meet-homomorphism and σ is a join homomorphism.

Next we move toward subtractive varieties, but a weaker condition deserves some attention. A variety $\mathscr{V}$ (resp. an algebra $\mathbf{A}$) has *normal ideals* if $\mathrm{I}_{\mathscr{V}}(\mathbf{A}) = \mathrm{N}(\mathbf{A})$ for all $\mathbf{A}$ in $\mathscr{V}$ (resp. if $\mathrm{I}(\mathbf{A}) = \mathrm{N}(\mathbf{A})$). The following are equivalent (cfr. [1]):

(1) $\mathbf{A}$ has normal ideals;
(2) $I = 0/\sigma(I)$ for $I \in \mathrm{I}(\mathbf{A})$;
(3) $I/\sigma(J) = I \vee J$ for $I, J \in \mathrm{I}(\mathbf{A})$;
(4) σ is injective from $\mathrm{I}(\mathbf{A})$ into $\mathrm{Con}(\mathbf{A})$.

For $I \in \mathrm{I}(\mathbf{A})$ let us define

$$\mathrm{CON}(I) = \{\theta \in \mathrm{Con}(\mathbf{A}) \colon 0/\theta = I\}$$

$$I^{\delta} = \bigwedge \mathrm{CON}(I)$$

$$I^{\varepsilon} = \bigvee \mathrm{CON}(I).$$

Observe that for $\theta \in \mathrm{CON}(I)$, $I = I/\theta$. Moreover $0/I^{\delta} = I$, hence $I^{\delta} \in \mathrm{CON}(I)$. Also if $a\, I^{\varepsilon}\, 0$, then for some $\theta_1, \dots, \theta_n \in \mathrm{CON}(I)$, $b_1, \dots, b_n \in A$ we have

$$0 = b_1\, \theta_1\, b_2\, \theta_2\, b_3\, \theta_3 \cdots \theta_n\, b_n = a.$$

Hence $b_2 \in I$, $b_3 \in I/\theta_2 = I \dots$ and finally $a \in I$; hence also $0/I^{\varepsilon} \in \mathrm{CON}(I)$. We have thus proved the following:

PROPOSITION 1.3. *If* $\mathbf{A}$ *has normal ideals, then* $\mathrm{CON}(I) = [I^{\delta}, I^{\varepsilon}]$ *is an interval in* $\mathrm{Con}(\mathbf{A})$.

Also observe that

PROPOSITION 1.4. *If* $\mathbf{A}$ *has normal ideals and* $\theta \in \mathrm{Con}(\mathbf{A})$ *then*

$$0/\theta = 0/\sigma(0/\theta).$$

Proof. In fact one inclusion holds trivially. Let $a \in 0/\sigma(0/\theta)$; then there are $a_0 = 0, a_1, \dots, a_n = a$, $(0, u_i) \in \{0\} \times 0/\theta$ $(i = 0, \dots, n)$ and unary polynomials $f_0, \dots, f_n$ such that

$$\{a_i, a_{i+1}\} = \{f_i(0), f_i(u_i)\}.$$

Thus $f_i(0)\ \theta\ f_i(u_i)$, and $a_i\ \theta\ a_{i+1}$; hence $0\ \theta\ a$. □

Observe that, for $\theta \in \mathrm{Con}(\mathbf{A})$, $(0/\theta)^\delta$ is the least congruence ψ such that $0/\psi = 0/\theta$; thus we have

$$\sigma(0/\theta) = (0/\theta)^\delta.$$

In fact $0/\theta = 0/\sigma(0/\theta)$, thus $(0/\theta)^\delta \subseteq \sigma(0/\theta)$; on the other hand $0/(0/\theta)^\delta = 0/\theta$, thus $\sigma(0/\theta) = \sigma(0/(0/\theta)^\delta) \subseteq (0/\theta)^\delta$.

Let us define, for $\theta \in \mathrm{Con}(\mathbf{A})$

$$\theta_0 = (0/\theta)^\delta = \sigma(0/\theta)$$

$$\theta_1 = (0/\theta)^\varepsilon$$

$$\hat{\theta} = \mathrm{CON}(0/\theta)$$

hence $\hat{\theta} = [\theta_0, \theta_1]$. Note that $(\theta_0)_0 = (\theta_1)_0 = \theta_0$, $(\theta_1)_1 = (\theta_0)_1 = \theta_1$ and that $\theta_0 = \varphi_0$ iff $0/\theta = 0/\varphi$ iff $0/\theta_0 = 0/\varphi_0$ iff $\theta_1 = \varphi_1$ iff $0/\theta_1 = 0/\varphi_1$ for all $\theta, \varphi \in \mathrm{Con}(\mathbf{A})$.

Next observe that $\{\hat{\theta} : \theta \in \mathrm{Con}(\mathbf{A})\}$ is a partition of $\mathrm{Con}(\mathbf{A})$, corresponding to the equivalence relation defined by

$$\theta \sim \varphi \quad \text{iff} \quad 0/\theta = 0/\varphi.$$

Also remark that for $\theta, \varphi \in \mathrm{Con}(\mathbf{A})$

$$(\theta \wedge \varphi)_0 \leq \theta_0 \wedge \varphi_0 \tag{*}$$

since

$$\{\alpha \wedge \beta : 0/\alpha = 0/\theta, 0/\beta = 0/\varphi\} \subseteq \{\gamma : 0/\gamma = 0/(\varphi \wedge \theta)\}$$

and consequently

$$\text{if} \quad \theta \leq \varphi \quad \text{then} \quad \theta_0 \leq \varphi_0.$$

To get something more we now move to subtractive algebras. A variety $\mathscr{V}$ is *subtractive* if for some binary term $s(x, y)$

$$s(x, 0) \approx x \qquad s(x, x) \approx 0$$

hold in $\mathscr{V}$ (we also say that $\mathscr{V}$ is s-subtractive); an algebra $\mathbf{A}$ is subtractive if $\boldsymbol{V}\mathbf{A}$ is. In a subtractive algebra there is an important derived ternary term $u(x, y, z) = s(x, s(s(x, y), z))$ (see [15]) satisfying

$$u(x, 0, 0) \approx 0 \qquad u(x, x, 0) \approx x$$

$$u(x, y, s(x, y)) \approx x.$$

Several characterizations of subtractive varieties can be found in [1] and [15]. Here we are mainly interested in recalling that the following are equivalent:

(1) $\mathscr{V}$ is subtractive;
(2) for any $\mathbf{A} \in \mathscr{V}$ the congruences of $\mathbf{A}$ permute at 0;
(3) for $\mathbf{A} \in \mathscr{V}$, the mapping $0/$ is a lattice homomorphism from $\mathrm{Con}(\mathbf{A})$ into $\mathrm{I}(\mathbf{A})$.

Moreover any subtractive variety has normal ideals, thus $0/$ is in fact a complete epimorphism.

PROPOSITION 1.5. *Let* $\mathbf{A}$ *be subtractive*; *then*
(1) $\sim$ *is a congruence of* $\mathrm{Con}(\mathbf{A})$ *and* $\mathrm{I}(\mathbf{A})$ *is isomorphic with* $\mathrm{Con}(\mathbf{A})/\sim$.
(2) *For* $\theta, \varphi \in \mathrm{Con}(\mathbf{A})$

$$\theta_1 \vee \varphi_1 \leq (\theta \vee \varphi)_1.$$

Hence if $\theta \leq \varphi$, *then* $\theta_1 \leq \varphi_1$.

Proof. While (1) is trivial, for (2) observe that

$$0/(\theta_1 \vee \varphi_1) = 0/\theta_1 \vee 0/\varphi_1 = 0/\theta \vee 0/\varphi = 0/(\theta \vee \varphi). \qquad \square$$

Thus the structure of $\mathrm{Con}(\mathbf{A})/\sim$, which coincides with $\{\hat{\theta} : \theta \in \mathrm{Con}(\mathbf{A})\}$, can be described as follows

$$\hat{\theta} \leq \hat{\varphi} \quad \text{iff} \quad \theta \leq \varphi$$

$$\hat{\theta} \vee \hat{\varphi} = (\theta \vee \varphi)^{\wedge} = [(\theta \vee \varphi)_0, (\theta \vee \varphi)_1]$$

$$\hat{\theta} \wedge \hat{\varphi} = (\theta \wedge \varphi)^{\wedge} = [(\theta \wedge \varphi)_0, (\theta \wedge \varphi)_1].$$

Hence when **A** is subtractive, I(**A**) is isomorphic with this lattice of intervals of Con(**A**) via the lattice isomorphism

$$\mathrm{CON}: I \to \mathrm{CON}(I)$$

where for any $\theta \in \mathrm{CON}(I)$

$$\mathrm{CON}(I) = [I^{\delta}, I^{\varepsilon}] = [\theta_0, \theta_1].$$

In general this representation is not very perspicuous (even in this kind of ''explosion'' of a lattice into a lattice of intervals may deserve further inquiry), because we would like to be able to choose ''uniformly'', or at least homomorphically, a representative in $\hat{\theta}$ for the ideal $0/\theta$.

Given a mapping F from I(**A**) into Con(**A**) we set for $\theta \in \mathrm{Con}(\mathbf{A})$, $\theta_F = F(0/\theta)$ and

$$\mathrm{Con}_F(\mathbf{A}) = \{\theta_F : \theta \in \mathrm{Con}(\mathbf{A})\}.$$

For instance

$$\mathrm{Con}_\delta(\mathbf{A}) = \{(0/\theta)^\delta : \theta \in \mathrm{Con}(\mathbf{A})\} = \mathrm{Con}_\sigma(\mathbf{A}) = \{\sigma(0/\theta) : \theta \in \mathrm{Con}(\mathbf{A})\}.$$

We say that F is a *normalizer* for **A** if

(1) F is a lattice homomorphism from I(**A**) into Con(**A**);
(2) $0/F(I) = I$ for $I \in \mathrm{I}(\mathbf{A})$.

A *complete normalizer* will be a normalizer which is a complete lattice homomorphism. Thus if **A** is subtractive, and hence $0/$ is a homomorphism from Con(**A**) onto I(**A**), then a complete normalizer F is a section of $0/$ in the category of complete lattices. Observe that any normalizer F for **A** is in fact an embedding of I(**A**) into Con(**A**):

$$F(I) = F(J) \quad \text{implies} \quad I = J$$

for $I, J \in \mathrm{I}(\mathbf{A})$; so I(**A**) is isomorphic with $\mathrm{Con}_F(\mathbf{A})$.

PROPOSITION 1.6. *If* **A** *is subtractive and* F *is a mapping from* I(**A**) *into* Con(**A**) *then the following are equivalent*:
(1) F *is a* (*complete*) *normalizer for* **A**;

(2) *the mapping sending* $\theta \mapsto \theta_F$ *is a* (*complete*) *lattic endomorphism of* Con(**A**) *and* $0/\theta_F = 0/\theta$.

Proof. (1) implies (2), simply because the mapping is the composition of the hoomorphisms 0/ and F.

Assume (2) and let $I = 0/\theta$, $J = 0/\varphi$; then

$$\begin{aligned} F(I \vee J) &= F(0/\theta \vee 0/\varphi) = F(0/\theta \vee \varphi) \\ &= (\theta \vee \varphi)_F = \theta_F \vee \varphi_F = F(0/\theta) \vee F(0/\varphi) \\ &= F(I) \vee F(J) \end{aligned}$$

thus F is a homomorphism. Moreover, if $I = 0/\theta$, then

$$0/F(I) = 0/\theta_F = 0/\theta = I.$$

For the complete case the proof is similar. □

We will say that **A** has *strongly normal ideals* if there is a normalizer for **A**; a variety $\mathscr{V}$ has *strongly normal ideals* if any algebra in $\mathscr{V}$ has strongly normal ideals. Pointed sets show at once that, for a variety, having strongly normal ideals is not a Mal'cev condition.

If **A** is a subtractive algebra and F is a normalizer for **A**, then we have the lattice isomorphisms

$$\mathrm{I}(\mathbf{A}) \cong \mathrm{Con}(\mathbf{A})/\sim \; \cong \mathrm{Con}_F(\mathbf{A})$$

whence I(**A**) is both a sublattice and a quotient of Con(**A**). As a matter of fact we have a *homomorphic choice function* on Con(**A**)/~, namely a morphism

$$C\colon \mathrm{Con}(\mathbf{A})/\sim \; \to \mathrm{Con}(\mathbf{A})$$

such that $C(\hat{\theta}) \in \hat{\theta}$ for $\theta \in \mathrm{Con}(\mathbf{A})$.

In fact, define $\tilde{F}(\hat{\theta}) = F(0/\theta)$; this is a well defined mapping $\tilde{F}$ from Con(**A**)/~ into Con(**A**). Moreover $\tilde{F}(\hat{\theta}) \in \hat{\theta}$ and $\tilde{F}$ is a homomorphism: $\hat{F}$ preserves meets trivially and also

$$\begin{aligned} \tilde{F}(\hat{\theta} \vee \hat{\varphi}) = \tilde{F}((\theta \vee \varphi)^\wedge) &= F(0/\theta \vee \varphi) \\ &= F(0/\theta \vee 0/\varphi) = F(0/\theta) \vee F(0/\varphi) \\ &= \tilde{F}(\hat{\theta}) \vee \tilde{F}(\hat{\varphi}). \end{aligned}$$

Conversely, if C is a homomorphic choice function on $\mathrm{Con}(\mathbf{A})/\sim$, then let us define

$$\tilde{C}(I) = C(\mathrm{CON}(I)).$$

Then $\tilde{C}$ is easily shown to be a homomorphism from $\mathrm{I}(\mathbf{A})$ into $\mathrm{Con}(\mathbf{A})$ and moreover

$$0/\tilde{C}(I) = I$$

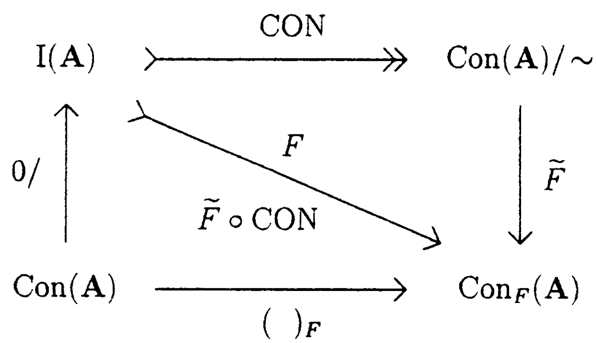
(in fact $C(\mathrm{CON}(I)) \in \mathrm{CON}(I)$). Notice also that, if F is a normalizer, then

$$\tilde{\tilde{F}} = \tilde{F} \circ \mathrm{CON} = F$$

and if C is a homomorphic choice function on $\mathrm{Con}(\mathbf{A})/\sim$, then

$$\tilde{\tilde{C}} = C.$$

Thus if $\mathbf{A}$ is a subtractive algebra and F is a normalizer then the following diagram commutes in the category of lattices:



Thus we conclude:

PROPOSITION 1.7. *If* $\mathbf{A}$ *is subtractive the following are equivalent*:
(1) $\mathbf{A}$ *has strongly normal ideals*;
(2) *there is a homomorphic choice function on* $\mathrm{Con}(\mathbf{A})/\sim$.

While the mappings $(\ )^{\delta}$ and $(\ )^{\varepsilon}$ satisfy

$$0/I^{\delta} = 0/I^{\varepsilon} = I$$

they are not in general normalizers. In the next sections we will study them, giving conditions under which they are normalizers, and consequences thereof. The behavior of the commutator of ideals ([2], [15]) will be investigated in a different paper.

2. The least congruence

We begin investigating the operator on Con(**A**)

$$(\ )_0 \colon \theta \mapsto \theta_0.$$

Let us call $\mathrm{Con}_0(\mathbf{A})$ its codomain $\{\theta_0 \colon \theta \in \mathrm{Con}(\mathbf{A})\}$. If we want $(\ )^{\delta}$ to be a normalizer for a subtractive algebra **A**, then surely $(\ )_0$ is to be an endomorphism of Con(**A**), being the composition of $(\ )^{\delta}$ with $0/$.

PROPOSITION 2.1. *If* **A** *is any algebra then* $(\ )_0$ *preserves meets in* Con(**A**) *iff* $\mathrm{Con}_0(\mathbf{A})$ *is closed under meets.*

If **A** *is subtractive then* $(\ )_0$ *preserves joins in* Con(**A**) *iff* $\mathrm{Con}_0(\mathbf{A})$ *is closed under joins.*

Proof. Certainly if $(\ )_0$ preserves meets, then the conclusion holds. Conversely if $\theta_0 \wedge \varphi_0 = \psi_0$ for some $\psi \in \mathrm{Con}(\mathbf{A})$, then by $(*)$ in Section 1

$$\begin{aligned}\theta_0 \wedge \varphi_0 \le \psi_0 = \psi_{00} = (\theta_0 \wedge \varphi_0)_0 \\ \le (\theta \wedge \varphi)_0 \le \theta_0 \wedge \varphi_0\end{aligned}$$

Again if $(\ )_0$ preserves joins then $\mathrm{Con}_0(\mathbf{A})$ is closed under joins. Conversely let $\theta_0 \vee \varphi_0 = \psi_0$ for some $\psi \in \mathrm{Con}(\mathbf{A})$. Then $\psi_0 = \psi_{00} = (\theta_0 \vee \varphi_0) \ge (\theta \vee \varphi)_0$; in fact

$$0/(\theta \vee \varphi) = 0/\theta \vee 0/\varphi = 0/\theta_0 \vee 0/\varphi_0 = 0/(\theta_0 \vee \varphi_0) = 0/(\theta_0 \vee \varphi_0)_0.$$

On the other side trivially $(\theta \vee \varphi)_0 \ge \theta_0 \vee \varphi_0$. □

Thus we conclude that in a subtractive algebra **A** the following are equivalent:

(a) $(\ )_0$ is a lattice homomorphism;
(b) $\mathrm{Con}_0(\mathbf{A})$ is a sublattice of Con(**A**).

Next we show that $(\,)^\delta$ is a join homomorphism on any subtractive algebra.

PROPOSITION 2.2. *If* **A** *is subtractive and* $I, J \in \mathrm{I}(\mathbf{A})$ *then*
(1) *if* $I \subseteq J$ *then* $I^\delta \subseteq J^\delta$;
(2) $(I \vee J)^\delta = I^\delta \vee J^\delta$.

Proof. Note that $I^\delta = \theta_0$ whenever $\theta \in \mathrm{CON}(I)$; but $\theta_0 = \sigma(I)$ and (1) follows. One inclusion in (2) follows from (1); moreover

$$0/I^\delta \vee J^\delta = 0/I^\delta \vee 0/J^\delta = I \vee J,$$

hence $(I \vee J)^\delta \subseteq I^\delta \vee J^\delta$. □

PROPOSITION 2.3. *If* **A** *is subtractive than the following are equivalent*:
(1) $(\,)^\delta$ *preserves meets*;
(2) $(\,)_0$ *preserves meets*.

Proof. Assume (1) and compute

$$\begin{aligned}\theta_0 \wedge \varphi_0 &= (0/\theta)^\delta \wedge (0/\varphi)^\delta = (0/\theta \cap 0/\varphi)^\delta \\ &= (0/\theta \wedge \varphi)^\delta = (\theta \wedge \varphi)_0.\end{aligned}$$

Assume (2) and compute

$$\begin{aligned}I^\delta \wedge J^\delta &= (I^\delta)_0 \wedge (J^\delta)_0 = (I^\delta \wedge J^\delta)_0 = (0/(I^\delta \wedge J^\delta))^\delta \\ &= (0/I^\delta \cap 0/J^\delta)^\delta = (I \wedge J)^\delta.\end{aligned}$$

□

We summarize our findings in the following:

THEOREM 2.4. *For a subtractive algebra* **A** *the following are equivalent*:
(1) $(\,)^\delta$ *is a* (*complete*) *normalizer for* **A**;
(2) $(\,)^\delta$ *preserves* (*arbitrary*) *meets in* $\mathrm{I}(\mathbf{A})$;
(3) $(\,)_0$ *is a* (*complete*) *lattice endomorphism of* $\mathrm{Con}(\mathbf{A})$;
(4) $\mathrm{Con}_0(\mathbf{A})$ *is a* (*complete*) *sublattice of* $\mathrm{Con}(\mathbf{A})$;
(5) $\mathrm{Con}_0(\mathbf{A})$ *is closed under* (*arbitrary*) *meets in* $\mathrm{Con}(\mathbf{A})$.

Proof. (1) and (3) are clearly equivalent by Proposition 1.6 and surely (3) implies (4) and (4) implies (5). By Proposition 2.3, (5) implies (2) and by Proposition 2.2, (2) implies (1). In the complete case one has of course to prove that Propositions 2.1, 2.2 and 2.3 hold for complete homomorphisms and arbitrary meets; but this is just routine. □

Now we investigate necessary and sufficient conditions for a class of subtractive algebras to be strongly normal with respect to $(\ )^\delta$. First we recall that, if $\theta, \varphi \in \mathrm{Con}(\mathbf{A})$ with $\theta \leq \varphi$ and $I \in \mathrm{I}(\mathbf{A})$

$$I/\theta = \{a/\theta : a \in I\} \in \mathrm{I}(\mathbf{A}/\theta)$$

$$\varphi/\theta = \{(a/\theta, b/\theta) : (a, b) \in \varphi\} \in \mathrm{Con}(\mathbf{A}/\theta).$$

PROPOSITION 2.5. *Let* $\mathbf{A}$ *be any algebra with a constant* 0 *in its type; then*

(1) *If* $\theta, \varphi \in \mathrm{Con}(\mathbf{A})$ *and* $\varphi \geq \theta$ *then* $(0/\theta)/(\varphi/\theta) = (0/\varphi)/\theta$.

(2) *If* $\theta \in \mathrm{Con}(\mathbf{A})$ *then the mapping* $I \mapsto I/\theta$ *is a complete lattice isomorphism between* $\mathrm{I}(\mathbf{A}/\theta)$ *and* $\{I \in \mathrm{I}(\mathbf{A}) : 0/\theta \subseteq I\}$.

(3) *If* $\mathbf{A}$ *is also subtractive,* $I \in \mathrm{I}(\mathbf{A})$ *and* $\theta \in \mathrm{Con}(\mathbf{A})$ *and* $\theta \leq I^\delta$, *then* $(I/\theta)^\delta = (I^\delta)/\theta$.

Proof. (1) and (2) can be proved with easy calculations. For (3) note that, by (1)

$$(0/\theta)/(I^\delta/\theta) = (0/I^\delta)/\theta = I/\theta$$

hence $(I^\delta/\theta)^\delta \subseteq I^\delta/\theta$.

Conversely let $\alpha \in \mathrm{Con}(\mathbf{A}/\theta)$ with $0/\alpha = I/\theta$; then there exists a $\varphi \in \mathrm{Con}(\mathbf{A})$ with $\varphi \geq \theta$ such that $\alpha = \varphi/\theta$ hence

$$(0/\theta)(\varphi/\theta) = (0/\varphi)/\theta = I/\theta.$$

Since $\theta \leq \varphi$ we have $0/\theta \leq 0/\varphi$ and hence by (2) and the above equation $0/\varphi = I$. But then $\varphi \geq I^\delta$, implying $\varphi/\theta = \alpha \geq I^\delta/\theta$; so $I^\delta/\theta \leq (I/\theta)^\delta$ and in fact equality holds. □

In parallel with the notion of subdirect irreducibility an algebra is *ideal irreducible* if $\mathrm{I}(\mathbf{A})$ contains a minimal nonzero ideal, which of course must be principal, generated by a *monolithic element*; it is *finitely ideal irreducible* if the meet of two nonzero ideal is not (0). In general subdirect irreducibility and ideal irreducibility are independent concepts, even in subtractive varieties (see [2] for examples of this fact). However:

THEOREM 2.6. *Let* $\mathscr{K}$ *be a class of subtractive algebras closed under homomorphic images; then the following are equivalent.*

(1) $(\ )^\delta$ *is a complete normalizer for* $\mathscr{K}$.

(2) *For any* $\mathbf{A} \in \mathscr{K}$ *and any family* $(I_\lambda)_{\lambda \in \Lambda}$ *of ideals of* $\mathbf{A}$

$$\bigwedge_{\lambda\in\Lambda} I_\lambda = (0) \quad \textit{implies} \quad \bigwedge_{\lambda\in\Lambda} I_\lambda^\delta = 0_{\mathbf{A}}.$$

Moreover each of (1) *and* (2) *implies*:

(3) *Any subdirectly irreducible algebra in* $\mathscr{K}$ *is also ideal irreducible*.

Proof. Since $(0)^\delta = 0_{\mathbf{A}}$, via Theorem 2.4 above we get that (1) implies (2).

Conversely assume (2); let $\mathbf{A}\in\mathscr{K}$ and $(I_\lambda)_{\lambda\in\Lambda}\subseteq \mathrm{I}(\mathbf{A})$. Let $\theta = (\bigwedge_{\lambda\in\Lambda} I_\Lambda)^\delta$; then by Proposition 2.5(2) $I \mapsto I/\theta$ is a complete lattice isomorphism between $\mathrm{I}(\mathbf{A}/\theta)$ and $\{K\in \mathrm{I}(\mathbf{A})\colon \bigwedge_{\lambda\in\Lambda} I_\lambda \subseteq K\}$. Hence $(\bigwedge_{\lambda\in\Lambda} I_\lambda)/\theta = \bigwedge_{\lambda\in\Lambda}(I_\lambda/\theta) = (0)/\theta$ and so by hypothesis $\bigwedge_{\lambda\in\Lambda}(I_\lambda/\theta)^\delta = 0_{\mathbf{A}/\theta} = \theta/\theta$. But by Proposition 2.5(3), since $I_\lambda^\delta \geq \theta$

$$\theta/\theta = \bigwedge_{\lambda\in\Lambda}(I_\lambda/\theta)^\delta = \bigwedge_{\lambda\in\Lambda} I_\lambda^\delta/\theta = \left(\bigwedge_{\lambda\in\Lambda} I_\lambda^\delta\right)\Big/\theta.$$

Hence $\bigwedge_{\lambda\in\Lambda} I_\lambda^\delta = \theta = (\bigwedge_{\lambda\in\Lambda} I_\lambda)^\delta$ which, in view of Theorem 2.4, gives (1).

Assume now (2) and let $\mathbf{A}$ be a subdirectly irreducible algebra in $\mathscr{K}$. Suppose $(I_\lambda)_{\lambda\in\Lambda}\subseteq \mathrm{I}(\mathbf{A})$ with $\bigwedge_L I_\lambda = (0)$; then by hypothesis $\bigwedge_{\lambda\in\Lambda} I_\lambda^\delta = 0_{\mathbf{A}}$ and since $\mathbf{A}$ is subdirectly irreducible $I_{\lambda_0}^\delta = 0_{\mathbf{A}}$ for some I_{λ_0}. It follows that $I_{\lambda_0} = (0)$ and $\mathbf{A}$ is ideal irreducible. □

By a similar proof we get:

PROPOSITION 2.7. *Let* $\mathscr{K}$ *be a class of subtractive algebras closed under homomorphic images*; *then the following are equivalent*.

(1) $(\,)^\delta$ *is a normalizer for* $\mathscr{K}$.

(2) *For any* $\mathbf{A}\in\mathscr{K}$ *and* $I, J\in \mathrm{I}(\mathbf{A})$

$$I \wedge J = (0) \quad \textit{implies} \quad I^\delta \wedge J^\delta = 0_{\mathbf{A}}.$$

Moreover each of (1) *and* (2) *implies*:

(3) *Any finitely subdirectly irreducible algebra in* $\mathscr{K}$ *is also finitely ideal irreducible*.

The problem whether (3) implies (2) in Theorem 2.6 and Proposition 2.7 remains open. A first step would be to find a semisimple subtractive variety which is not ideal determined; in that setting then one might be able to select the proper counterexample. However Remarks (2) in Section 3 seems to indicate that such a variety is hard to find.

3. The greatest congruence

The problem of getting a reasonable algebraic description of the operators $\theta \mapsto \theta_1$ and $I \mapsto I^\varepsilon$ is harder to solve. We can give a description of them, albeit very generic, in case of algebras with normal ideals.

PROPOSITION 3.1. *Let* **A** *be any algebra and* $\theta \in \mathrm{Con}(\mathbf{A})$; *then* $(a, b) \in \theta_1$ *iff for every unary polynomial* $p(x)$

$$p(a)\ \theta_1\ 0 \quad \textit{iff} \quad p(b)\ \theta_1\ 0.$$

Proof. Let ψ be the relation defined by: $(a, b) \in \psi$ iff they satisfy the conclusion of the statement above. It is straightforward to check that it is a congruence and $\theta \subseteq \psi$; Moreover $0/\theta = 0/\psi$, hence $\psi \subseteq \theta_1$. Let now $\varphi \in \mathrm{Con}(\mathbf{A})$ be such that $0/\varphi = 0/\psi$; then of course $\varphi \subseteq \psi$. □

COROLLARY 3.2. *If* **A** *has normal ideals and* $I \in \mathrm{I}(\mathbf{A})$, *then* $(a, b) \in I^\varepsilon$ *iff for every unary polynomial* p, $p(a) \in I$ *iff* $p(b) \in I$.

PROPOSITION 3.3. *Let* **A**, **B** *two algebras and* h *an epimorphism of* **A** *onto* **B**. *Then, for any* $\theta \in \mathrm{Con}(\mathbf{B})$,

$$h^{-1}(\theta_1) = (h^{-1}(\theta))_1.$$

Proof. It is easy to see that

$$0/h^{-1}(\theta_1) = h^{-1}(0/\theta)$$

and

$$0/h^{-1}(\theta_1) = 0/h^{-1}(\theta);$$

hence $h^{-1}(\theta_1) \subseteq (h^{-1}(\theta))_1$.

To prove that $(h^{-1}(\theta))_1 \subseteq h^{-1}(\theta_1)$ it is enough to show that $h((h^{-1}(\theta))_1) \subseteq \theta_1$. Let $q(x, y_1, \ldots, y_n)$ be any term and let

$$p(x) = q(x, h(a_1), \ldots, h(a_n))$$

(for suitable $a_1, \ldots, a_n \in A$) be a unary polynomial of $\mathbf{B}$. Let $(u, v) \in h^{-1}(\theta)_1$ and assume $p(h(u))\,\theta\,0$; observe that $p(h(u)) = h(p(u, a_1, \ldots, a_n))$ and $p(h(v))) = h(p(v, a_1, \ldots, a_n))$. Thus $(p(u, a_1, \ldots, a_n), 0) \in h^{-1}(\theta)$ and so $(p(v, a_1, \ldots, a_n), 0) \in h^{-1}(\theta)$ and finally $p(h(v))\theta 0$. □

COROLLARY 3.4. *If* $\mathbf{A}$, $\mathbf{B}$ *have normal ideals and* $h : \mathbf{A} \to \mathbf{B}$ *is an epimorphism, then for any* $I \in \mathrm{I}(\mathbf{B})$

$$h^{-1}(I^{\varepsilon}) = (h^{-1}(I))^{\varepsilon}.$$

Proof. We have that $I^{\varepsilon} = \varphi_1$ for some $\varphi \in \mathrm{Con}(\mathbf{B})$ such that $0/\varphi = I$. But then $h^{-1}(I) = h^{-1}(0/\varphi) = 0/h^{-1}(\varphi)$ and Proposition 3.3 applies. □

In the sequel we will deal with a family $D = \{d_\lambda(x, y, \vec{z}^{\lambda}) : \lambda \in \Lambda\}$ of terms $d_\lambda(x, y, \vec{z}^{\lambda})$ in the shown variables in the algebraic language considered. For any algebra $\mathbf{A}$, $I \in \mathrm{I}(\mathbf{A})$ we define $I^D \subseteq A \times A$ as follows

$$(a, b) \in I^D \quad \text{iff} \quad \forall \lambda \in \Lambda, \forall \vec{c}^{\lambda} \in A, d_\lambda(a, b, \vec{c}^{\lambda}) \in I.$$

If $\theta \in \mathrm{Con}(\mathbf{A})$ we define $\theta^D = (0/\theta)^D$.

PROPOSITION 3.5. *If* h *is an epimorphism from* $\mathbf{A}$ *onto* $\mathbf{B}$ *and* $I \in \mathrm{I}(\mathbf{B})$, *then*

$$(h^{-1}(I))^D = h^{-1}(I^D).$$

Proof. We have

$$\begin{aligned} (a, b) \in (h^{-1}(I))^D \quad & \text{iff} \quad \text{for all } \lambda \in \Lambda \text{ and } \vec{c}^{\lambda} \in A, d_\lambda(a, b, \vec{c}^{\lambda}) \in h^{-1}(I) \\ & \text{iff} \quad \text{for all } \lambda \in \Lambda \text{ and } \vec{d}^{\lambda} \in B, d_\lambda(h(a), h(b), \vec{d}^{\lambda}) \in I \\ & \text{iff} \quad (h(a), h(b)) \in I^D. \end{aligned}$$ □

We say that $D = \{d_\lambda(x, y, \vec{z}^{\lambda}) : \lambda \in \Lambda\}$ is a *system of* 0-*terms with parameters for a class* $\mathscr{K}$ of algebras if

(1) $\mathscr{K} \models d_\lambda(x, x, \vec{z}^{\lambda}) \approx 0$, for $\lambda \in \Lambda$;
(2) for $\mathbf{A} \in \mathscr{K}$ and $a \in A$, if $d_\lambda(0, a, \vec{c}^{\lambda}) = 0$ for all $\lambda \in \Lambda$, $\vec{c}^{\lambda} \in A$, then $a = 0$.

PROPOSITION 3.6. *Let $\mathscr{K}$ be a class of algebras with normal ideals and let D be a system of* 0-*terms with parameters for* $\boldsymbol{H}K$. *Then for every* $\mathbf{A} \in \mathscr{K}$

(1) *if* $U \subseteq A$, *then*

$$\langle U \rangle_{\mathbf{A}} = \langle d_\lambda(0, u, \vec{c}^{\lambda}) \colon \lambda \in \Lambda, \vec{c}^{\lambda} \in A, u \in U \rangle$$

(*in particular, for* $a \in A$, $(a) = \langle d_\lambda(0, a, \vec{c}^{\lambda}) \colon \lambda \in \Lambda, \vec{c}^{\lambda} \in A \rangle$);

(2) *the congruences of* $\mathbf{A}$ *permute at* 0.

Proof.

(1) Let $I = \langle d_\lambda(0, u, \vec{c}^{\lambda}) \colon \lambda \in \Lambda, \vec{c}^{\lambda} \in A, u \in U \rangle$; since $\mathscr{K} \vDash d_\lambda(0, 0, \vec{z}^{\lambda}) \approx 0$ we get $d_\lambda(0, u, \vec{c}^{\lambda}) \in \langle U \rangle$ for $u \in U$ and $\vec{c}^{\lambda} \in A$, so that $I \subseteq \langle U \rangle$. On the other hand let $I = 0/\varphi$ for some $\varphi \in \mathrm{Con}(\mathbf{A})$; in $\mathbf{A}/\varphi$, $d_\lambda(0/\varphi, u/\varphi, \vec{c}^{\lambda}/\varphi) = 0/\varphi$, hence $u/\varphi = 0/\varphi$, i.e. $u \in I$.

(2) Now let $\varphi, \psi \in \mathrm{Con}(\mathbf{A})$ and $0 \varphi a \, \psi \, b$. For $\vec{c}^{\lambda} \in A$, $\lambda \in \Lambda$

$$d_\lambda(0, b, \vec{c}^{\lambda}) \; \varphi \; d_\lambda(a, b, \vec{c}^{\lambda}) \; \psi \; d_\lambda(b, b, \vec{c}^{\lambda}) = 0,$$

hence $d_\lambda(0, b, \vec{c}^{\lambda}) \in 0/\psi \circ \varphi$. By (1), since $0/\psi \circ \varphi \in \mathrm{I}(\mathbf{A})$, we conclude that $b \in 0/\psi \circ \varphi$ i.e. for some c, $0 \; \psi \; c \; \varphi \; b$. □

For a class $\mathscr{K}$ of algebras, we say that $D = \{d_\lambda(x, y, \vec{z}^{\lambda}) \colon \lambda \in \Lambda\}$ is a *system of ideal congruence terms with parameters for* $\mathscr{K}$ (shortly: an *IC-system with parameters for* $\mathscr{K}$) if $\theta^D \in \hat{\theta}$ for all $\mathbf{A} \in \mathscr{K}$ and $\theta \in \mathrm{Con}(\mathbf{A})$ (i.e. $\theta^D \in \mathrm{Con}(\mathbf{A})$ and $0/\theta^D = 0/\theta$). If $\mathscr{K}$ has normal ideals, then D is an IC-system for $\mathscr{K}$ iff for all $\mathbf{A} \in \mathscr{K}$ and $I \in \mathrm{I}(\mathbf{A})$, $I^D \in \mathrm{CON}(I)$ (i.e. $I^D \in \mathrm{Con}(\mathbf{A})$ and $0/I^D = I$).

PROPOSITION 3.7. *Let D be an IC-system with parameters for a class $\mathscr{K}$. Then D is a system of* 0-*terms for $\mathscr{K}$ and moreover for* $\mathbf{A} \in \mathscr{K}$, $\varphi \in \mathrm{Con}(\mathbf{A})$, $\varphi \subseteq \varphi^D$.

Proof. For any $\mathbf{A} \in \mathscr{K}$ and $a \in A$, $0/(0_{\mathbf{A}})^D = (0)$ and $(a, a) \in 0_{\mathbf{A}}$; then $d_\lambda(a, a, \vec{c}^{\lambda}) = 0$ for all $\vec{c}^{\lambda} \in A$. By the same token if $d_\lambda(0, a, \vec{c}^{\lambda})$ for all $\lambda \in \Lambda$ and $\vec{c}^{\lambda} \in A$, then $a \in 0/(0_{\mathbf{A}})^D$, so $a = 0$.

Finally if $(a, b) \in \varphi$, then $d_\lambda(a, b, \vec{c}^{\lambda}) \; \varphi \; d_\lambda(a, a, \vec{c}^{\lambda}) = 0$, thus $d_\lambda(a, b, \vec{c}^{\lambda}) \in 0/\varphi$ for $\lambda \in \Lambda$ and $\vec{c}^{\lambda} \in A$. Thus $(a, b) \in \varphi^D$. □

Our interest in IC-systems stems from the following:

PROPOSITION 3.8. *For a family D of terms and a class $\mathscr{K}$ of algebras the following are equivalent.*

(1) *D is an IC-system for* $\mathscr{K}$.

(2) *For* $\mathbf{A} \in \mathscr{K}$ *and* $\theta \in \mathrm{Con}(\mathbf{A})$, $\theta^D = \theta_1$

If moreover $\mathscr{K}$ *has normal ideals then they are also equivalent to*

(3) *For* $\mathbf{A} \in \mathscr{K}$ *and* $I \in \mathrm{I}(\mathbf{A})$, $I^D = I^\varepsilon$.

Proof. Assume (1); then $\theta \subseteq \theta^D \subseteq \theta_1$ by Proposition 3.7. Since $\theta \in \hat{\theta}$, if $\varphi \in \mathrm{Con}(\mathbf{A})$ and $0/\varphi = 0/\theta$, then $\varphi \subseteq \varphi^D = \theta^D$; in particular $\theta_1 \subseteq \theta^D$.

Since $\theta_1 \in \hat{\theta}$, (2) trivially implies (1). That (1) and (3) are equivalent is shown in a similar way. □

PROPOSITION 3.9. *Let* $\mathscr{K}$ *be a class of algebras with normal ideals and let D be a system of* 0*-terms with parameters for* $\boldsymbol{H}\mathscr{K}$; *then the following are equivalent*.

(1) *For all* $\mathbf{A} \in \mathscr{K}$, $I \in \mathrm{I}(\mathbf{A})$, I^D *is a subalgebra of* $\mathbf{A} \times \mathbf{A}$.

(2) *For all* $\mathbf{A} \in \mathscr{K}$, $I \in \mathrm{I}(\mathbf{A})$, $I = 0/\mathrm{Sub}_{\mathbf{A}\times\mathbf{A}}(I^D)$.

(3) *D is an IC-system with parameters for* $\mathscr{K}$.

(4) *D is an IC-system with parameters for* $\boldsymbol{H}\mathscr{K}$.

(5) *For* $\mathbf{A} \in \boldsymbol{H}\mathscr{K}$, $(0)^D \in \mathrm{Con}(\mathbf{A})$.

(6) *For* $\mathbf{A} \in \boldsymbol{H}\mathscr{K}$, $(0)^D$ *is a subalgebra of* $\mathbf{A} \times \mathbf{A}$.

Proof. We will first show that (1) and (2) are equivalent. Assuming (1), Proposition 3.6(1) yields $I = 0/I^D$ and hence (2) holds. Conversely assume (2); from $(a, b) \in \mathrm{Sub}_{\mathbf{A}\times\mathbf{A}}(I^D)$ we get $(0, d_\lambda(a, b, \vec{c}^\lambda)) \in \mathrm{Sub}_{\mathbf{A}\times\mathbf{A}}(I^D)$ for $\vec{c}^\lambda \in A$ and $\lambda \in \Lambda$, hence $d_\lambda(a, b, \vec{c}^\lambda) \in I$ for $\vec{c}^\lambda \in A$ and $\lambda \in \Lambda$ and finally $(a, b) \in I^D$.

Now assume (1) (and (2)). To prove (3) observe that for $\mathbf{A} \in \mathscr{K}$ and $I \in \mathrm{I}(\mathbf{A})$ we have $I = 0/I^D$. Thus we need only to show that I^D is an equivalence. Reflexivity is obvious; from $(u, v) \in I^D$ and $(u, u) \in I^D$ we get $(d_\lambda(u, u, \vec{c}^\lambda), d_\lambda(v, u, \vec{c}^\lambda)) \in I^D$ for $\lambda \in \Lambda$ and $\vec{c}^\lambda \in A$. Thus $(0, d_\lambda(v, u, \vec{c}^\lambda)) \in I^D$ i.e. $d_\lambda(v, u, \vec{c}^\lambda) \in 0/I^D = I$ hence $(v, u) \in I^D$. Finally if $(u, v), (v, w) \in I^D$, then $(v, u) \in I^D$, so $(d_\lambda(v, v, \vec{c}^\lambda), d_\lambda(u, w, \vec{c}^\lambda)) \in I^D$ and so eventually $(u, w) \in I^D$.

Assume now (3); to prove (4) let $\mathbf{A} \in \mathscr{K}$, $\theta \in \mathrm{Con}(\mathbf{A})$, $\mathbf{B} = \mathbf{A}/\theta$ and $J \in \mathrm{I}(\mathbf{B})$. Let h be the natural epimorphism of $\mathbf{A}$ onto $\mathbf{B}$; by Corollary 3.4 and Proposition 3.8 we have that

$$h^{-1}(J^D) = (h^{-1}(J))^D = (h^{-1}(J))^\varepsilon = h^{-1}(J^\varepsilon)$$

whence $J^D = J^\varepsilon$ which of course implies $J^D \in \mathrm{CON}(J)$.

Trivially (4) implies (5) and (5) implies (6). Assume then (6) and let $\mathbf{A} \in \mathscr{K}$, $I \in \mathrm{I}(\mathbf{A})$, $I = 0/\varphi$ for some $\varphi \in \mathrm{Con}(\mathbf{A})$. If $\mathbf{B} = \mathbf{A}/\varphi$ and h is the natural epimorphism from $\mathbf{A}$ onto $\mathbf{B}$, then $I = h^{-1}(0)$ and by Proposition 3.5 $I^D = h^{-1}((0)^D)$. So clearly I^D is a subalgebra of $\mathbf{A} \times \mathbf{A}$ and (1) holds. □

THEOREM 3.10. *For a variety* $\mathscr{V}$ *with normal ideals the following are equivalent*:

(1) $\mathscr{V}$ *is subtractive*;

(2) $\mathscr{V}$ *has an IC-system with parameters*.

(3) $\mathscr{V}$ *has a system of* 0*-terms with parameters*.

Proof. Since (2) implies (3), by Proposition 3.7 and (3) implies (1) by Proposition 3.6, we need only to prove that (1) implies (2). Let $s(x, y)$ be a subtraction term for $\mathscr{V}$ and let

$$D_{\mathscr{V}} = \{d(x, y, \vec{u})\text{: for some term } t \text{ of } \mathscr{V}\, d(x, y, \vec{u}) = s(t(y, \vec{u}), t(x, \vec{u}))\}.$$

Observe that $\mathscr{V} \models d(x, x, \vec{u}) \approx 0$, and if $d(0, a, \vec{c}) = 0$ for any such d, then in particular $a = s(a, 0) = 0$. Thus $D_{\mathscr{V}}$ is a system of 0-terms for $\mathscr{V}$ and by Proposition 3.9 it is enough to show that $I^{D_{\mathscr{V}}} = I^{\varepsilon}$. By Corollary 3.2 this reduces to proving that for $\mathbf{A} \in \mathscr{V}$, $a, b \in A$ and $I \in \mathrm{I}(\mathbf{A})$

$$(a, b) \in I^{D_{\mathscr{V}}} \quad \text{iff} \quad \text{for every unary polynomial } p(x), p(a) \in I \text{ iff } p(b) \in I.$$

Suppose $(a, b) \in I^{D_{\mathscr{V}}}$ and let $p(x) = t(x, a_1, \ldots, a_n)$ for $a_1, \ldots, a_n \in A$; then $s(p(a), p(b)) = s(t(a, a_1, \ldots, a_n), t(b, a_1, \ldots, a_n)) \in I^{D_{\mathscr{V}}}$ and similarly $s(p(b), p(a)) \in I^{D_{\mathscr{V}}}$. If $p(a) \in I$ then (the term $u(x, y, z)$ was defined above Proposition 1.5)

$$p(b) = u(p(b), p(a), s(p(b), p(a)) \in I$$

and similarly if $p(b) \in I$, then $p(a) \in I$.

Conversely for any $d(x, y, \vec{z}) = s(t(y, \vec{z}), t(x, \vec{z})) \in D^{\mathscr{V}}$ let

$$p_{d,\vec{c}}(x) = s(t(b, \vec{c}), t(x, \vec{c})).$$

Then $p_{d,\vec{c}}(b) = s(t(b, \vec{c}), t(b, \vec{c})) = 0 \in I$ and hence

$$p_{d,\vec{c}}(a) = s(t(b, \vec{c}), t(a, \vec{c})) = d(a, b, \vec{c}) \in I$$

for any $d \in D^{\mathscr{V}}$ and $\vec{c} \in A$. So $(a, b) \in I^{D_{\mathscr{V}}}$. □

REMARKS. (1) Theorem 3.10 shows that the existence of an IC-system with parameters characterizes subtractive varieties among varieties with normal ideals.

(2) If $\mathscr{V}$ is a subtractive variety the set $D_{\mathscr{V}}$ is the largest set of terms from which we can get an IC-system for $\mathscr{V}$. In fact suppose D is another such set and let $d'(x, y, \vec{u}) \in D$; then

$$d'(x, y, \vec{u}) = s(d'(x, y, \vec{u}), 0) = s(d'(x, y, \vec{u}), d'(y, y, \vec{u})) \in D_{\mathscr{V}}.$$

To obtain more information on the operator $(\ )^{\varepsilon}$ one has to make a connection with Blok and Pigozzi's work in abstract algebraic logic (see [4], [6] and the bibliographies therein).

A *deductive system* C_{S} in a language $\mathscr{L}$ is a structural and finitary closure operator on the term algebra $\mathrm{FM}_{\mathscr{L}}$. If $\Gamma \cup \{\varphi\} \subseteq FM_{\mathscr{L}}$ then it is customary to write $\Gamma \vdash_{\mathsf{S}} \varphi$, for $\varphi \in C_{\mathsf{S}}(\Gamma)$. If $\mathbf{A}$ is an algebra in the same language as $\mathscr{L}$ and $F \subseteq A$, then $\langle \mathbf{A}, F \rangle$ is called an *$\mathscr{L}$-matrix* or simply a *matrix*. A congruence $\theta \in \mathrm{Con}(\mathbf{A})$ is *compatible* with F if F is a union of θ-blocks; the largest congruence compatible with F (which always exists) is called the *Leibniz congruence* and it is denoted by $\Omega_{\mathbf{A}}(F)$. If C_{S} is a deductive system and $\mathbf{A}$ an algebra, then $F \subseteq A$ is an S-*filter* if for any $\Gamma \cup \{\varphi\} \subseteq \mathrm{FM}_{\mathscr{L}}$

$$\Gamma \vdash_{\mathsf{S}} \varphi \quad \text{iff} \quad \forall a_1, \ldots, a_n \in A, \qquad \Gamma(a_1, \ldots, a_n) \in F \quad \Rightarrow \quad \varphi(a_1, \ldots, a_n) \in F.$$

A matrix $\langle \mathbf{A}, F \rangle$ is a *matrix model* for C_{S} if F is an S-filter; a matrix model is *reduced* if $\Omega_{\mathbf{A}}(F) = 0_{\mathbf{A}}$.

If $\mathbb{M}$ is a class of matrices then one can define a deductive system $C_{\mathbb{M}}$ associated with $\mathbb{M}$ in the following way: for $\Gamma \cup \{\varphi\} \subseteq \mathrm{FM}_{\mathscr{L}}$

$$\Gamma \vdash_{\mathbb{M}} \varphi \quad \text{iff} \quad \forall \langle \mathbf{A}, F \rangle \in \mathbb{M}, \forall a_1, \ldots, a_n \in A,$$

$$\Gamma(a_1, \ldots, a_n) \in F \quad \Rightarrow \quad \varphi(a_1, \ldots, a_n) \in F.$$

If $\mathscr{V}$ is a pointed variety then the *assertional logic* of $\mathscr{V}$, in symbols $\mathsf{AL}_{\mathscr{V}}$, is the deductive system defined by the class of matrices

$$\mathbb{M}_{\mathscr{V}} = \{\langle \mathbf{A}, \{0\} \rangle : \mathbf{A} \in \mathscr{V}\}.$$

If $\mathscr{V}$ has normal ideal one sees easily that the $\mathsf{AL}_{\mathscr{V}}$-filters are exactly the ideals and that for $\mathbf{A} \in \mathscr{V}$ and $I \in \mathrm{I}(\mathbf{A})$, $\Omega_{\mathbf{A}}(I) = I^{\varepsilon}$. Keeping [6] at hand the reader can easily determine the class of reduced matrix models of $\mathsf{AL}_{\mathscr{V}}$; accordingly we will say that an algebra $\mathbf{A} \in \mathscr{V}$ *reduced* if $(0)^{\varepsilon}_{\mathbf{A}} = 0_{\mathbf{A}}$. We denote by $\mathscr{V}_{\varepsilon}$ the class of reduced algebras of $\mathscr{V}$ and of course

$$\mathscr{V}_\varepsilon = \{\mathbf{A}/(0)^\varepsilon_{\mathbf{A}}: \mathbf{A} \in \mathscr{V}\}.$$

It is a trivial exercise to show that $\mathbf{A}/\theta$ is reduced iff $(0/\theta)^\varepsilon = \theta$; it follows that $\mathbf{A}/\theta \in \mathscr{V}_\varepsilon$ iff $\theta = I^\varepsilon$ for some $I \in \mathrm{I}(\mathbf{A})$.

In [6] Blok and Pigozzi classified deductive systems according to the behavior of the Leibniz congruence; it turns out that this depends on the existence of what they call *equivalence systems with parameters*. Again one easily shows that in case $\mathscr{V}$ has normal ideals an equivalence system with parameters for $\mathsf{AL}_{\mathscr{V}}$ is just an IC-system with parameters for $\mathscr{V}$. With this in mind we can start the translation of Blok and Pigozzi's results in our framework.

THEOREM 3.11. *For a variety with normal ideals the following are equivalent.*
(1) *$\mathscr{V}$ is subtractive;*
(2) *for all $\mathbf{A} \in \mathscr{V}$ and $I \in \mathrm{I}(\mathbf{A})$, $(0)^\varepsilon_{\mathbf{A}} \subseteq I^\varepsilon$;*
(3) *$(\ )^\varepsilon$ is monotonic in $\mathscr{V}$, i.e. for any $\mathbf{A} \in \mathscr{V}$ and any $I, J \in \mathrm{I}(\mathbf{A})$*

$$I \subseteq J \quad \textit{implies} \quad I^\varepsilon \subseteq J^\varepsilon.$$

(4) *$\mathsf{AL}_{\mathscr{V}}$ is* protoalgebraic ([6], 7.1).
(5) *$\mathscr{V}_\varepsilon$ is closed under subdirect products.*

Proof. (1) implies (2) by Proposition 1.5(2). Assume then (2) and let $\mathbf{A} \in \mathscr{V}$ and $I, J \in \mathrm{I}(\mathbf{A})$ with $I \subseteq J$; if $\theta = \sigma(I)$, then, by (3) before Proposition 1.3,

$$J = I \vee J = J/\theta.$$

Let now $h: \mathbf{A} \to \mathbf{A}/\theta$ be the natural homomorphism; by Corollary 3.4 we have

$$h^{-1}((hI)^\varepsilon) = (I/\theta)^\varepsilon = I^\varepsilon$$

$$h^{-1}((hJ)^\varepsilon) = (J/\theta)^\varepsilon = J^\varepsilon.$$

Now $hI = (0/\theta)_{\mathbf{A}/\theta}$ hence by assumption, $(hI)^\varepsilon \subseteq (hJ)^\varepsilon$; but then $I^\varepsilon \subseteq J^\varepsilon$ and hence $(\ )^\varepsilon$ is monotonic in $\mathscr{V}$. Assume now (3), let $\mathbf{A} \in \mathscr{V}$ and $\theta \in \mathrm{Con}(\mathbf{A})$ with $0/\theta \subseteq I$. Then $(0/\theta)^\varepsilon \subseteq I^\varepsilon$ which implies (in view the above remarks about $(\ )^\varepsilon$) that I is a union of θ-blocks. From now on a standard Mal'cev argument on the free 2-generated algebra in $\mathscr{V}$ yields terms to guarantee subtractivity (but see [1], Theorem 2.4, for a complete proof). Therefore (1), (2) and (3) are equivalent.

That (3) and (4) are equivalent follows from the definition of protoalgebraic deductive system in [6] and our remarks previous to the theorem.

Assume now that (3) holds. Let $\mathbf{A}$ be a subdirect product of algebras in $\mathscr{V}_\varepsilon$; this means that there is a family $(\theta_\lambda : \lambda \in \Lambda)$ of congruences of $\mathbf{A}$ such that $\bigcap_{\lambda\in\Lambda} \theta_\lambda = 0_\mathbf{A}$ and $(0/\theta_\lambda)^\varepsilon = \theta_\lambda$. Since $(0)_\mathbf{A} = 0/0_\mathbf{A} = 0/(\bigcap_{\lambda\in\Lambda}\theta_\lambda)$ by monotonicity we get

$$(0)^\varepsilon_\mathbf{A} = \left(0\Big/\left(\bigcap_{\lambda\in\Lambda}\theta_\lambda\right)\right)^\varepsilon \subseteq \bigcap_{\lambda\in\Lambda}(0/\theta_\lambda)^\varepsilon = \bigcap_{\lambda\in\Lambda}\theta_\lambda = 0_\mathbf{A}.$$

Therefore $\mathbf{A} \in \mathscr{V}_\varepsilon$ and (5) holds.

Assume now (5); let $\mathbf{A} \in \mathscr{V}$, $I, J \in \mathrm{I}(\mathbf{A})$ with $I \subseteq J$ and let $\theta = I^\varepsilon \cap J^\varepsilon$. Clearly $\mathbf{A}/\theta$ is a subdirect product of $\mathbf{A}/I^\varepsilon, \mathbf{A}/J^\varepsilon \in \mathscr{V}_\varepsilon$, hence $\mathbf{A}/\theta \in \mathscr{V}_\varepsilon$. It follows that $(0/\theta)^\varepsilon = \theta$ and moreover $0/\theta = 0/(I^\varepsilon \cap J^\varepsilon) = 0/I^\varepsilon \cap 0/J^\varepsilon = I \cap J = I$. We conclude that $I^\varepsilon = \theta = I^\varepsilon \cap J^\varepsilon$ and so $I^\varepsilon \subseteq J^\varepsilon$. Thus $(\ )^\varepsilon$ is monotonic in $\mathscr{V}$ and (3) holds. □

A variety $\mathscr{V}$ is (*finitely*) *congruential* if it has a (finite) IC-system without parameters. In other words $\mathscr{V}$ is congruential iff there is a set $D = \{d_\lambda(x, y): \lambda \in \Lambda\}$ of binary terms such that for any $\mathbf{A} \in \mathscr{V}$ and any $\theta \in \mathrm{Con}(\mathbf{A})$

$$\theta_1 = \theta^D;$$

it is finitely congruential iff $|\Lambda| < \omega$. For varieties with normal ideals being (finitely) congruential is equivalent to saying that

$$I^\varepsilon = \{(a, b): d_\lambda(a, b) \in I, \lambda \in \Lambda\};$$

a (finitely) congruential variety with normal ideals is always subtractive by Theorem 3.10. We remark the obvious fact that Proposition 3.9 continues to hold for IC-systems (and sets of 0-terms) without parameters.

THEOREM 3.12. *For a variety $\mathscr{V}$ with normal ideals the following are equivalent.*

(1) *$\mathscr{V}$ is congruential*;
(2) $\mathsf{AL}_\mathscr{V}$ *is weakly* congruential ([6], 13.11);
(3) $(\ )^\varepsilon$ *is monotonic on $\mathscr{V}$ and for any $\mathbf{A} \in \mathscr{V}$, any $\mathbf{B}$ subalgebra of $\mathbf{A}$ and any* $I \in \mathrm{I}(\mathbf{A})$

$$(I \cap B)^\varepsilon = I^\varepsilon \cap B^2.$$

(4) *$\mathscr{V}_\varepsilon$ is closed under subalgebras and direct products*.

Proof. (1) and (2) are equivalent by the definition of weakly congruential in [6] and our remarks. The equivalence of (1), (3) and (4) comes really from Theorems 13.12 and 13.13 in [6]; however we display a complete argument in this case, which can help the reader to prove some parts of Theorem 3.13 below.

Assume (1); by Theorem 3.11 $\mathcal{V}_\varepsilon$ is closed under subdirect products and hence direct products. Let $D = \{d_\lambda(x, y) \colon \lambda \in \Lambda\}$ be the set of terms defining the equational IC-system for $\mathcal{V}$ and let $\mathbf{A} \in \mathcal{V}_\varepsilon$. Let $\mathbf{B} \leq \mathbf{A}$ and observe that

$$(0)^\varepsilon_{\mathbf{B}} = \{(a, b) \in B^2 \colon d_\lambda(a, b) = 0, \lambda \in \Lambda\};$$

since $\mathbf{B}$ is a subalgebra of $\mathbf{A}$ we have

$$(0)^\varepsilon_{\mathbf{B}} \subseteq 0_{\mathbf{A}} \cap B^2 = 0_{\mathbf{B}},$$

since $\mathbf{A}$ is reduced. But then $\mathbf{B}$ is reduced as well, $\mathcal{V}_\varepsilon$ is closed under $\boldsymbol{S}$ and (4) holds.

Assume now (4); then $\mathcal{V}_\varepsilon$ is closed under subdirect products and so by Theorem 3.11 $(\ )^\varepsilon$ is monotonic on $\mathcal{V}$. Next let $\mathbf{A} \in \mathcal{V}$, $\mathbf{B} \leq \mathbf{A}$ and $I \in \mathrm{I}(\mathbf{A})$; then $I^\varepsilon \cap B^2 \in \mathrm{Con}(\mathbf{B})$ and $\mathbf{B}/(I^\varepsilon \cap B^2)$ is (isomorphic to) a subalgebra of $\mathbf{A}/I^\varepsilon$. Since the latter belongs to $\mathcal{V}_\varepsilon$ by assumption $\mathbf{B}/(I^\varepsilon \cap B^2)$ is reduced as well; it follows that

$$I^\varepsilon \cap B^2 = (0/(I^\varepsilon \cap B^2))^\varepsilon = (I \cap B)^\varepsilon.$$

Finally assume (3); then since $(\ )^\varepsilon$ is a monotonic on $\mathcal{V}$, by Theorem 3.11 $\mathcal{V}$ has an equational IC-system defined by $\{d_\lambda(x, y, \vec{u}) \colon \lambda \in \Lambda\}$. Let $\mathbf{A} \in \mathcal{V}$, $I \in \mathrm{I}(\mathbf{A})$, $a, b \in A$ and let $\mathbf{B} = \mathrm{Sub}_{\mathbf{A}}(a, b)$. Then

$$\begin{aligned} (a, b) \in I^\varepsilon \quad &\text{iff} \quad (a, b) \in I^\varepsilon \cap B^2 \\ &\text{iff} \quad (a, b) \in (I \cap B)^\varepsilon \\ &\text{iff} \quad d_\lambda(a, b, \vec{c}) \in \mathrm{I} \quad \text{for all } \lambda \in \Lambda \text{ and } \vec{c} \in B \\ &\text{iff} \quad d_\lambda(a, b, t_1(a, b), \ldots, t_{n_\lambda}(a, b)) \in I \quad \text{for all } \lambda \in \Lambda \text{ and all} \\ &\qquad\qquad \text{binary terms } t_1, \ldots, t_{n_\lambda}. \end{aligned}$$

it follows that if one sets

$$\begin{aligned} D' = \{d'(x, y) \colon d'(x, y) = d_\lambda(x, y, t_1(x, y), \ldots, t_{n_\lambda}(x, y)), \lambda \in \Lambda \\ \text{and } t_1, \ldots, t_{n_\lambda} \text{ binary terms}\} \end{aligned}$$

then D' defines an equational IC-system without parameters for $\mathscr{V}$, which is then congruential. □

THEOREM 3.13. *For a variety $\mathscr{V}$ with normal ideals the following are equivalent.*

(1) *$\mathscr{V}$ is finitely congruential.*
(2) $\mathsf{AL}_{\mathscr{V}}$ *is* algebraizable ([6], 13.12).
(3) $(\ )^\varepsilon$ *is* continuous, *i.e. for any $\mathbf{A} \in \mathscr{V}$ and any upward directed family $(I_\gamma : \gamma \in \Gamma)$ of ideals of* $\mathbf{A}$

$$\left(\bigcup_{\gamma \in \Gamma} I_\gamma\right)^\varepsilon = \bigcup_{\gamma \in \Gamma} I_\gamma^\varepsilon.$$

(4) *$\mathscr{V}_\varepsilon$ is a quasivariety, i.e. it is closed under subalgebras, direct products and ultraproducts.*
(5) *There are binary terms $\{d_1, \ldots, d_n\}$, an $n+3$-ary term q and, for every m-ary operation f and $i = 1, \ldots, n$ there is a $(2+n)m$-ary term r_f^i such that the identities*

$$d_i(x, x) \approx 0 \quad (i = , \ldots, n)$$

$$q(x, y, 0, \ldots, 0) \approx 0$$

$$q(x, y, y, d_1(x, y), \ldots, d_n(x, y)) \approx x$$

$$d_i(f(\vec{x}), f(\vec{y})) \approx r_f^i(\vec{x}, \vec{y}, \vec{d}(x_1, y_1), \ldots, \vec{d}(x_m, y_m))$$

$$r_f^i(\vec{x}, \vec{y}, \vec{0}, \ldots, \vec{0}) \approx 0$$

hold in $\mathscr{V}$.

Proof. Assume (1); since $\mathscr{V}$ is finitely congruential there is a finite set $D = \{d_1, \ldots, d_n\}$ of binary terms such that $I^D = I^\varepsilon$ for any $I \in \mathrm{I}(\mathbf{A})$ and $\mathbf{A} \in I$. It follows that

$$\begin{aligned} \mathbf{A} \in \mathscr{V}_\varepsilon \quad &\text{iff} \quad (0)^\varepsilon_{\mathbf{A}} = 0_{\mathbf{A}} \\ &\text{iff} \quad d_1(a, b) = \cdots = d_n(a, b) = 0 \quad \text{implies} \quad a = b. \end{aligned}$$

Hence $\mathscr{V}_\varepsilon$ is axiomatized (relative to $\mathscr{V}$) by the quasi-equation

$$d_1(x, y) \approx \cdots \approx d_n(x, y) \approx 0 \to x \approx y.$$

This shows that $\mathscr{V}_\varepsilon$ is a quasivariety and (4) holds. The reader can supply the proofs that (4) implies (1) and that (1) and (3) are equivalent by looking at Theorems 13.12 and 13.13 in [6]. Moreover, since the operator $(\ )^\varepsilon$ is always injective, from Theorem 13.15 in [6] we get that (2) and (3) are equivalent.

Assume again (1) and work in the algebra freely generated in $\mathscr{V}$ by x, y. From Proposition 3.9 we get the term q and its identities. The consider the algebra freely generated in $\mathscr{V}$ by $\vec{x}, \vec{y}$; again from Proposition 3.9 we get the terms r_f^i and their identities.

Finally assume (5) and observe that the set $D = \{d_1(x, y), \ldots, d_n(x, y)\}$ is a system of 0-terms without parameters for $\mathscr{V}$. In fact $d_i(x, x) \approx 0$ for $i = 1, \ldots, n$ and if $d_i(x, 0) \approx 0$ for $i = 1, \ldots, n$, then $x \approx q(x, 0, 0, 0, \ldots, 0) \approx 0$. By using the terms in an appropriate way, one shows again that $(0)_{\mathbf{A}}^D$ is a subalgebra of $\mathbf{A} \times \mathbf{A}$, for any $\mathbf{A} \in \mathscr{V}$. Hence (1) holds and the proof is finished. □

We are now ready to answer the question: when is $(\ )^\varepsilon$ a normalizer for a variety $\mathscr{V}$ with normal ideals? According to Section 1, if D is an IC-system for $\mathscr{V}$, then for any $\mathbf{A} \in \mathscr{V}$

$$\mathrm{Con}_\varepsilon(\mathbf{A}) = \{(0/\theta)^D \colon \theta \in \mathrm{Con}(\mathbf{A})\} = \{\theta \colon \mathbf{A}/\theta \in \mathscr{V}_\varepsilon\}.$$

So $\mathrm{Con}_\varepsilon(\mathbf{A})$ is the set of the so called *relative congruences* of $\mathscr{V}_\varepsilon$. It is folklore that in general the set of relative congruences of a class of algebras $\mathscr{K}$ can be given an algebraic lattice structure if $\mathscr{K}$ happens to be a quasivariety. In this particular case the relative congruence lattice coincides with the ideal lattice.

PROPOSITION 3.14. *Let $\mathscr{V}$ be a variety and let $D = \{d_1, \ldots, d_n\}$ be a finite IC-system without parameters for $\mathscr{V}$. Then for any $\mathbf{A} \in \mathscr{V}$*

$$\mathrm{Con}_\varepsilon(\mathbf{A}) \cong \mathrm{I}(\mathbf{A}).$$

Proof. Consider the mapping from $\mathrm{Con}_\varepsilon(\mathbf{A})$ to $\mathrm{I}(\mathbf{A})$ defined by $\theta \mapsto 0/\theta$; the mapping is onto, since for $I \in \mathrm{I}(\mathbf{A})$, $I = 0/I^D$ and $I^D \in \mathrm{Con}_\varepsilon(\mathbf{A})$. The mapping is one-to-one, since from $0/\theta = 0/\varphi$, with $\theta, \varphi \in \mathrm{Con}_\varepsilon(\mathbf{A})$ one deduces

$$\begin{aligned}(a, b) \in \theta \quad &\text{iff} \quad d_i(a, b) \in (0/\theta)^D \quad i = 1, \ldots, n\\ &\text{iff} \quad d_i(a, b) \in (0/\varphi)^D \quad i = 1, \ldots, n\\ &\text{iff} \quad (a, b) \in \varphi.\end{aligned}$$

Finally the map is clearly order preserving, as well as its inverse $I \mapsto I^D$; this is enough to conclude that it is a lattice homomorphism. □

The following proposition mirrors Theorem 2.4.

PROPOSITION 3.15. *For a subtractive algebra* **A** *the following are equivalent*:
(1) $(\ )^\varepsilon$ *is a normalizer for* **A**;
(2) $(\ )^\varepsilon$ *is a complete normalizer for* **A**;
(3) $(\ )^\varepsilon$ *preserves joins in* $\mathrm{I}(\mathbf{A})$;
(4) $(\ )^\varepsilon$ *is a lattice homomorphism*;
(5) $\mathrm{Con}_\varepsilon(\mathbf{A})$ *is a sublattice of* $\mathrm{Con}(\mathbf{A})$;
(6) $\mathrm{Con}_\varepsilon(\mathbf{A})$ *is closed under joins of* $\mathrm{Con}(\mathbf{A})$.

Proof. (1) is equivalent to (2) simply because ideal generation is an algebraic closure operator. (1) and (4) are equivalent by Proposition 1.6 and clearly (4) implies (5) and (5) implies (6). Next note that, by Theorem 3.11, both $(\ )^\varepsilon$ and $(\ )_1$ are monotonic. Assume then (5) and note that this is really equivalent to saying that $(\ )_1$ is a join endomorphism. Let then $I, J \in \mathrm{I}(\mathbf{A})$; to show (2) compute

$$\begin{aligned} I^\varepsilon \vee J^\varepsilon &= (I^\varepsilon)_1 \vee (J^\varepsilon)_1 = (I^\varepsilon \vee J^\varepsilon)_1 \\ &= (0/(I^\varepsilon \vee J^\varepsilon))^\varepsilon = (0/I^\varepsilon \vee 0/J^\varepsilon)^\varepsilon = (I \vee J)^\varepsilon. \end{aligned}$$

Assume now (2) and note that this is equivalent to saying that $(\ )^\varepsilon$ is a join homomorphism; since it is clearly a meet homomorphism we get (1). □

THEOREM 3.16. *For a variety* $\mathscr{V}$ *with normal ideals the following are equivalent*.
(1) $(\ )^\varepsilon$ *is a lattice homomorphism for any algebra in* $\mathscr{V}$.
(2) $(\ )^\varepsilon$ *is a normalizer for* $\mathscr{V}$.
(3) $(\ )^\varepsilon$ *is a complete lattice homomorphism for any algebra in* $\mathscr{V}$.
(4) $(\ )^\varepsilon$ *is a complete normalizer for* $\mathscr{V}$.
(5) $\mathscr{V}_\varepsilon$ *is a variety*.
(6) *For any* $\mathbf{A} \in \mathscr{V}$, $\mathrm{Con}_\varepsilon(\mathbf{A}) = [(0)^\varepsilon_{\mathbf{A}}, 1_{\mathbf{A}}]$.
(7) $\mathsf{AL}_{\mathscr{V}}$ *is* strongly algebraizable ([3]).

Proof. Note that $0/I^\varepsilon = I$ by definition; moreover each of (1)–(4) (via Theorem 3.11) implies that $\mathscr{V}$ is subtractive. Hence, using Proposition 3.15 when necessary, we quickly conclude that (1)–(4) are equivalent.

Assume then that $(\ )^\varepsilon$ is a complete normalizer; in particular $(\ )^\varepsilon$ must be a complete lattice homomorphism, so by Theorem 3.13 $\mathscr{V}$ is finitely congruential.

Hence there is a finite set $\{d_1, \ldots, d_n\}$ of binary terms such that $\mathscr{V}$ witnessing this fact. By Proposition 3.15 this means that for any $\mathbf{A} \in \mathscr{V}$ $\mathrm{Con}_\varepsilon(\mathbf{A})$ is closed under joins (evaluated in $\mathrm{Con}(\mathbf{A})$). Let hence $\mathbf{A} \in \mathscr{V}_\varepsilon$ and let $\theta \in \mathrm{Con}(\mathbf{A})$; let $B = \theta$ regarded as a subalgebra of $\mathbf{A} \times \mathbf{A}$. Since $\mathscr{V}$ is finitely congruential, $\mathscr{V}_\varepsilon$ is a quasivariety, hence $\mathbf{B} \in \mathscr{V}_\varepsilon$. Let π_1, π_2 be the canonical projections of $\mathbf{B}$ onto $\mathbf{A}$; then $\mathbf{B}/\pi_1 \cong \mathbf{B}/\pi_2 \cong \mathbf{A}$, hence $\pi_1, \pi_2 \in \mathrm{Con}_\varepsilon(\mathbf{A})$ and by assumption $\pi_1 \vee \pi_2 \in \mathrm{Con}_\varepsilon(\mathbf{A})$. It follows that

$$\mathbf{A}/\theta \cong \mathbf{B}/(\pi_1 \vee \pi_2) \in \mathscr{V}_\varepsilon$$

and so $\mathscr{V}_\varepsilon$ is a variety[1].

Next assume that $\mathscr{V}_\varepsilon$ is a variety; then by Theorem 3.11 $(\ )^\varepsilon$ is monotonic. This clearly yields $\mathrm{Con}_\varepsilon(\mathbf{A}) \subseteq [(0)^\varepsilon_{\mathbf{A}}, 1_{\mathbf{A}}]$. Conversely if $\theta \geq (0)^\varepsilon$, then $\theta/(0)^\varepsilon_{\mathbf{A}} \in \mathrm{Con}(\mathbf{A}/(0)^\varepsilon)$ and by the second homomorphism theorem

$$(\mathbf{A}/(0)^\varepsilon_{\mathbf{A}})(\theta/(0)^\varepsilon_{\mathbf{A}}) \cong \mathbf{A}/\theta.$$

But the left hand side belongs to $\mathscr{V}_\varepsilon$, since $\mathbf{A}/(0)^\varepsilon_{\mathbf{A}} \in \mathscr{V}_\varepsilon$ and $\mathscr{V}_\varepsilon$ is a variety; hence $\theta \in \mathrm{Con}_\varepsilon(\mathbf{A})$ and (6) holds.

Assume now (6); then $\mathscr{V}$ is subtractive via Theorem 3.11(2) and $\mathrm{Con}_\varepsilon(\mathbf{A})$ is closed under joins; applying Proposition 3.15 we conclude that $(\ )^\varepsilon$ is a normalizer.

Finally the equivalence of (7) and (5) is again a consequence of the definition of strong algebraizability and our remarks about $\mathsf{AL}_{\mathscr{V}}$. □

Note that from the proof one easily sees that the variety $\mathscr{V}_\varepsilon$ is in fact ideal determined. Hence as a corollary we get a further characterization of ideal determinacy.

COROLLARY 3.17. *For a variety $\mathscr{V}$ with normal ideals the following are equivalent*.

(1) *$(\ )^\varepsilon$ is a lattice isomorphism for any $\mathbf{A} \in \mathscr{V}$.*
(2) *$\mathscr{V}$ is ideal determined.*
(3) *$\mathscr{V} = \mathscr{V}_\varepsilon$.*
(4) *$\mathsf{AL}_{\mathscr{V}}$ is strongly algebraizable and its* equivalent algebraic semantics ([4]) *is exactly $\mathscr{V}$.*

[1] This part of the proof is a corollary of a more general result which will appear in [8]; we thank K. Kearnes for making us aware of this fact.

REMARKS. (1) In our classification of IC-systems we did not consider finite IC-systems with parameters. The reason lies in the fact that any variety with normal ideals having a finite IC-system with parameters has in fact a finite IC-system *without* parameters. The proof of this claim is left to the reader.

(2) Our second remark deals with one more characterization of IC-systems: we claim that, if $\mathscr{V}$ is a variety with normal ideals, then $D = \{d_\lambda(x, y, \vec{z}^\lambda): \lambda \in \Lambda\}$ is an IC-system if and only if for any $\mathbf{A} \in \mathscr{V}$ and any $a, b \in A$

$$0/\vartheta(a, b) = \langle d_\lambda(a, b, \vec{c}^\lambda): \lambda \in \Lambda, \vec{c}^\lambda \in A \rangle_{\mathbf{A}}.$$

In fact if the above equation holds, then one easily checks that D is a system of 0-terms with parameters for $\mathscr{V}$. Suppose now that $(a, b) \in (0)^D_{\mathbf{A}}$ and $p(x)$ be a unary polynomial of $\mathbf{A}$; then $d_\lambda(a, b, \vec{c}^\lambda) = 0$ for $\lambda \in \Lambda$ and $\vec{c}^\lambda \in A$. This of course implies that $0/\vartheta(a, b) = (0)_{\mathbf{A}}$. Now for any $\lambda \in \Lambda$ and $\vec{c}^\lambda \in A$

$$d_\lambda(p(a), p(b), \vec{c}^\lambda)\ \vartheta(a, b)\ d_\lambda(p(a), p(a), \vec{c}^\lambda) = 0$$

which implies $(p(a), p(b)) \in (0)^D_{\mathbf{A}}$. Hence $(0)^D_{\mathbf{A}}$ is a subalgebra of $\mathbf{A} \times \mathbf{A}$. Thus by Proposition 3.9 D is an IC-system for $\mathscr{V}$.

Conversely let D be an IC-system for $\mathscr{V}$, $\mathbf{A} \in \mathscr{V}$, $a, b \in A$ and $I = \langle d_l(a, b, \vec{c}^\lambda): \lambda \in \Lambda, \vec{c}^\lambda \in A \rangle_{\mathbf{A}}$. Then $d_\lambda(a, b, \vec{c}^\lambda)\ \vartheta_{\mathbf{A}}(a, b)\ d_\lambda(a, a, \vec{c}^\lambda) = 0$ so $I \subseteq 0/\vartheta(a, b)$; on the other hand $(a, b) \in I^D$ by definition so $0/\vartheta(a, b) \subseteq 0/I^D = I$.

(3) As a consequence of (2) we get that an IC-system $D = \{d_\lambda(x, y, \vec{z}^\lambda)c: \lambda \in \Lambda\}$ for a variety $\mathscr{V}$ behaves like a set of ‘‘implications’’ satisfying ‘‘modus ponens’’: for any $\mathbf{A} \in \mathscr{V}$ and $I \in \mathrm{I}(\mathbf{A})$, if $b \in I$ and $d_\lambda(a, b, \vec{c}^\lambda) \in I$ for $\lambda \in \Lambda$ and $\vec{c}^\lambda \in A$, then $a \in I$. In fact by (2) $0/\vartheta(a, b) \subseteq I$ and thus

$$\begin{aligned} a \in 0/\vartheta(a, 0) &\subseteq 0/(\vartheta(a, b) \circ \vartheta(b, 0)) = 0/\vartheta(a, b) \vee 0/\vartheta(b, 0) \\ &= 0/\vartheta(a, b) \vee (b)_{\mathbf{A}} \subseteq I. \end{aligned}$$

(4) The above remarks imply that, for a subtractive variety, having an IC-system can be seen as a generalization of having 0-regular congruences. The connection is made once one recalls that, if $\mathscr{V}$ is ideal-determined and the terms $t_1(x, y), \ldots, t_n(x, y)$ witness 0-regularity for $\mathscr{V}$, then for any $\mathbf{A} \in \mathscr{V}$ and $a, b \in A$

$$\vartheta(a, b) = \vartheta(t_1(a, b), 0) \vee \cdots \vee \vartheta(t_n(a, b), 0).$$

Then ideal determinacy implies

$$0/\vartheta(a, b) = \langle t_1(a, b), \ldots, t_n(a, b)\rangle_{\mathbf{A}}.$$

(5) If $\mathscr{V}$ is congruential, witness $D = \{d_\lambda(a, x): \lambda \in \Lambda\}$, then for any $\mathbf{A} \in \mathscr{V}$ and $a, b \in A$

$$(a) = (b) \quad \text{iff} \quad d_\lambda(a, b) \in (a) \cap (b) \quad \lambda \in \Lambda.$$

In fact, from $d_\lambda(0, 0) = 0$ and $(a) = (b)$ we get $d_\lambda(a, b) \in (a) \cap (b)$ for $\lambda \in \Lambda$. Conversely from $d_\lambda(a, b) \in (a) \cap (b)$ for $\lambda \in \Lambda$, we get

$$0/\vartheta(a, b) \subseteq (a) \cap (b).$$

If s is a subtractive term for $\mathscr{V}$, from $(a, b) \in \vartheta(a, b)$ and $(a, a) \in \vartheta(a, b)$ we get $(0, s(a, b)) \in \vartheta(a, b)$, i.e. $s(a, b) \in (a) \cap (b)$ and similarly $s(b, a) \in (a) \cap (b)$. Then $a = u(a, b, s(a, b)) \in (b)$ and $b = u(b, a, s(b, a)) \in (a)$.

4. Examples

4.1. The s-subtractive varieties for which $D = \{s(x, y)\}$ happens to be an IC-system have been investigated in [15] and [2] under the name of *d-subtractive varieties*. Such varieties have the interesting property that the congruence lattices of algebras therein are in fact arguesian (Corollary 1.9 of [15]). We remark also that Theorem 3.13 clearly generalizes 1.7 in [15].

Many examples of d-subtractive varieties come either from classical algebras or the algebraization of logical systems. Clearly all these varieties are ideal determined: we will produce later on an example of a non ideal determined d-subtractive variety. In the first class we quote groups, rings, Lie algebras. Banach algebras and in general any variety of algebras which is *classically ideal determined* [15].

For the second class we can start with left-complemented monoids: a *left-complemented monoid* [7] is an algebra $\langle A, \rightarrow, \cdot, 1\rangle$ such that $\langle A, \cdot, 1\rangle$ is a monoid and moreover for $a, b, c \in A$

$$a \rightarrow a = 1$$

$$(a \rightarrow b)a = (b \rightarrow a)b$$

$$ab \rightarrow c = a \rightarrow (b \rightarrow c).$$

One sees easily that $1 \to a = a$ holds as well so the variety $\mathscr{L}\mathscr{M}$ of left complemented monoids is subtractive with witness term $x \to y$. The fact that $\mathscr{L}\mathscr{M}$ is d-subtractive is essentially contained in [7].

A *hoop* is a commutative left complemented monoid; the variety of hoops is d-subtractive and so is any variety consisting of algebras having a hoop reduct: Brouwerian semilattices, relatively pseudocomplemented lattices, Heyting algebras, Wajsberg algebras (sometimes called MV-algebras) [10], Post algebras, Boolean algebras, interior algebras, Magari algebras (former diagonalizable algebras), monadic algebras, cylindric algebras etc.

A large class of varieties coming from logic not sharing the above property stems from BCK-algebras; a *BCK-algebra* is an algebra $\langle A, \to, 1\rangle$ satisfying the following identities and quasi identities:

$$x \to x = 1$$

$$x \to 1 = 1$$

$$1 \to x = x$$

$$(x \to y) \to ((z \to x) \to (z \to y)) = 1$$

$$x \to (y \to z) = y \to (x \to z)$$

$$x \to y = 1 \quad \text{and} \quad y \to x = 1 \quad \text{imply} \quad x = y.$$

These algebras arise from the algebraization of the pure implicative logic BCK [12]; it is not trivial to see that the quasi variety of BCK-algebras is not a variety [16]. However any nontrivial variety of BCK-algebras is clearly ideal determined; we will show that no variety of BCK-algebras is d-subtractive for any binary term $s(x, y)$. Let $\mathscr{V}$ be such a variety and let $\mathbf{A} \in \mathscr{V}$; we will prove that for no $s(x, y)$ do we have that $s(a, b) = 1$ implies $a = b$ for all $a, b \in A$; in view of the (3) of the Remarks in Section (3) this will be enough. We will proceed by induction on the number of $\to$ appearing in $s(x, y)$.

The initial step is easily proven by cases. Suppose now we have proved the statement for any terms containing less than n arrows and let

$$s(x, y) = r(x, y) \to t(x, y).$$

Since $t(x, y)$ contains less than n arrows, there must be $a, b \in A$, with $a \neq b$ and $t(a, b) = 1$. Therefore

$$s(a, b) = r(a, b) \to t(a, b) = r(a, b) \to 1 = 1$$

and the induction step is complete.

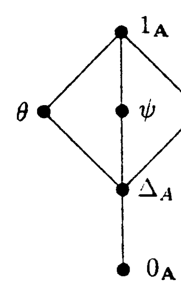
Many interesting varieties of BCK-algebras may be obtained in the following way; let $\mathscr{V}$ be any variety of hoops (possibly with dual normal operators) and let $\boldsymbol{S}\mathscr{V}^{\rightarrow}$ be the class of all subalgebras of $\{\rightarrow, 1\}$-reducts of algebras in $\mathscr{V}$. It is easy to see that $\boldsymbol{S}\mathscr{V}^{\rightarrow}$ is a class of BCK-algebras; it is harder to prove, but still true [5], that $\boldsymbol{S}\mathscr{V}^{\rightarrow}$ is a variety of BCK-algebras for any variety $\mathscr{V}$ of hoops. There are two important varieties of this kind arising from the algebraization of pure implicational logic, where for $\mathscr{V}$ we take the variety $\mathscr{BS}$ of Brouwerian semilattices and the variety $\mathscr{BA}$ of Boolean algebras respectively. The algebras in $\boldsymbol{S}\mathscr{BS}^{\rightarrow}$ are called *Hilbert algebras* [9] and arise from the algebraization of Hilbert and Bernay's positive implicative logic. The algebras in $\boldsymbol{S}\mathscr{BA}^{\rightarrow}$ are called *Tarski algebras* (or implication algebras [14]) and arise from the algebraization of classical implicative logic. It is obvious that all these varieties are finitely congruential witness $\{x \rightarrow y, y \rightarrow x\}$.

Varieties of BCK-algebras with lattice operations added are also not d-subtractive; for instance the variety of lower BCK-semilattices [13] is ideal determined but not d-subtractive (this can be shown with an inductive argument similar to the one for BCK-algebras).

4.2. Consider the algebra $\mathbf{A} = \langle \{0, a, b, c, d\}, + \rangle$ where $+$ is defined by the table

$+$	0	a	b	c	d
0	0	a	b	c	d
a	a	0	c	b	b
b	b	c	0	a	a
c	c	b	a	0	0
d	d	b	a	0	0

This algebra was presented also in [2] (Example 6.3) in a slightly different context. The reader can easily check that Con($\mathbf{A}$) is



where, denoting the congruences by the associated partitions,

$$\Delta_A = (0)(a)(b)(cd) \qquad \theta = (0a)(bcd) \qquad \psi = (0cd)(ab) \qquad \varphi = (0b)(acd).$$

This shows at once that $\boldsymbol{V}(\mathbf{A})$ is not ideal determined, since it is not 0-regular, being $0/\Delta_A = (0)$. Moreover $\mathbf{A}$ is subdirectly irreducible but, if $I = \{0, a\}$ and $J = \{0, b\}$, then $I \wedge J = (0)$. Hence $\mathbf{A}$ is not ideal irreducible and so by Theorem 2.6 $(\,)^\delta$ is not a normalizer for $\boldsymbol{V}(\mathbf{A})$. However $\boldsymbol{V}(\mathbf{A})$ has strongly normal ideals; in fact it is d-subtractive witness $x + y$ (see [2]).

4.3. Let $\mathscr{V}$ be the variety of algebras having a single binary operation $s(x, y)$ and a constant 0 satisfying the equations

$$s(x, x) \approx 0 \qquad s(x, 0) \approx x.$$

$\mathscr{V}$ is clearly subtractive; let $\mathbf{A} = \langle \{0, a, b, 1\}, s \rangle$ be the algebra whose s-table is

s	0	a	b	1
0	0	0	0	0
a	a	0	1	0
b	b	0	0	1
1	1	0	0	0

One easily sees that $\mathbf{A}$ is simple, hence reduced; however $\mathbf{B} = \langle \{0, a, 1\}, s \rangle$ is a subalgebra of $\mathbf{A}$ and $0/\vartheta_{\mathbf{B}}(a, 1) = \{0\}$. Hence $\mathbf{B}$ is not reduced, so $\mathscr{V}_\varepsilon$ is not closed under $\boldsymbol{S}$; it follows that $\mathscr{V}$ is not congruential.

4.4. Let $\mathbf{L}$ be a meet semilattice with a bottom element 0; $\mathbf{L}$ is pseudocomplemented if for any $a \in L$ there is an $a^* \in L$ such that

$$b \leq a^* \quad \text{iff} \quad a \wedge b = 0.$$

Pseudocomplemented semilattices are well-known structures; for the properties below and for any other claim we refer the reader to [11], Chapter I.6 and to the extensive bibliography therein. If $\mathbf{A}$ is a pseudocomplemented semilattice and $a, b \in A$, then

(1) $a \leq a^{**}$
(2) $a^* = a^{***}$
(3) $a \leq b \Rightarrow b^* \leq a^*$
(4) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$.

A pseudocomplemented semilattice has always a top element in the order, namely 0^*; we will denote this element by 1. In a pseudocomplemented semilattice one can define a binary operation $a \oplus b$ by

$$a \oplus b = (a^* \wedge b^*)^*.$$

The *skeleton* of $\mathbf{L}$ is $\mathrm{S}(\mathbf{L}) = \{a^*: a \in L\}$; it is well-known that $a \in \mathrm{S}(\mathbf{L})$ iff $a^{**} = a$ and that $\mathbf{S}(\mathbf{L}) = \langle \mathrm{S}(\mathbf{L}), \oplus, \wedge, ^*, 0, 1 \rangle$ is a Boolean algebra. However $\oplus$ is "almost" a join even in $\mathbf{L}$; the following proposition shows some of its properties.

PROPOSITION 4.1. *For any pseudocomplemented semilattice* $\mathbf{L}$ *and* $a, b, c \in L$ *we have*:

(1) $a \oplus b = b \oplus a$
(2) $a \oplus (b \oplus c) = (a \oplus b) \oplus c$
(3) $a \oplus a = a^{**}$
(4) $a \oplus 0 = a^{**}$
(5) $a \oplus 1 = 1$
(6) $a \oplus a^* = 1$
(7) $(a \oplus b)^{**} = a^{**} \oplus b = a \oplus b^{**} = a^{**} \oplus b^{**} = a \oplus b$
(8) *if* $b \leq c$ *then* $a \oplus b \leq a \oplus c$
(9) *if* $a, b \leq c$ *then* $a \oplus b \leq c^{**}$
(10) $a^{**} \leq a \oplus b$ *and hence* $a \leq a \oplus b$
(11) $(a \oplus b) \wedge c \leq a \oplus (b \wedge c)$
(12) $a \oplus (b \wedge c) \leq (a \oplus b) \wedge (a \oplus c)$
(13) $a \oplus (b \wedge c) = (a \oplus b) \wedge (a \oplus c)$
(14) $a^{**} \wedge (b \oplus c) = (a \wedge b) \oplus (a \wedge c)$.

Proof. (1)–(7) are obvious. For (8) let $b \leq c$; then $c^* \geq b^*$ implying $a^* \wedge c^* \geq b^* \wedge c^*$. Hence $a \oplus b = (a^* \wedge b^*)^* \leq (a^* \wedge c^*)^* = a \oplus c$. (9) and (10) and (12) are consequences of (8).

For (11) we have by (10)

$$a \wedge c \leq a \oplus (b \wedge c)$$

$$b \wedge c \leq a \oplus (b \wedge c)$$

therefore

$$a \wedge c \wedge (a \oplus (b \wedge c))^* = 0 \quad \Rightarrow \quad c \wedge (a \oplus (b \wedge c))^* \leq a^*$$
$$b \wedge c \wedge (a \oplus (b \wedge c))^* = 0 \quad \Rightarrow \quad c \wedge (a \oplus (b \wedge c))^* \leq b^*$$

and hence

$$c \wedge (a \oplus (b \wedge c))^* \subseteq a^* \wedge b^*$$
$$c \wedge (a \oplus (b \wedge c))^* \wedge (a^* \wedge b^*)^* = 0$$
$$(a \oplus b) \wedge c \leq (a \oplus (b \wedge c))^{**} = a \oplus (b \wedge c).$$

For (13) apply (11) (in x, y, z) with $a = x$, $b = y$ and $a \oplus c = z$, to get

$$\begin{aligned}(a \oplus b) \wedge (a \oplus c) &\leq a \oplus (b \wedge (a \oplus c))\\ &\leq a \oplus (a \oplus (b \wedge c))\\ &= a^{**} \oplus (b \wedge c) = a \oplus (b \wedge c).\end{aligned}$$

Hence by (12) we are done.

For (14) apply (12) (in x, y, z) with $x = a \wedge b$, $y = a$ and $z = c$ to get

$$\begin{aligned}(a \wedge b) \oplus (a \wedge c) &= ((a \wedge b) \oplus a) \wedge ((a \wedge b) \oplus c))\\ &= (a \oplus a) \wedge (a \oplus b) \wedge (a \oplus c) \wedge (b \oplus c)\\ &= a^{**} \wedge (b \oplus c).\end{aligned}$$ □

The variety $\mathcal{PS}$ of type $\langle \wedge, *, 0 \rangle$ is defined by the following identities

(1) a set of identities defining meet semilattices;
(2) $x \wedge (x \wedge y)^* = x \wedge y^*$;
(3) $x \wedge 0^* = x$;
(4) $0^{**} = 0$.

Note that by (3) $1 = 0^*$ is the top element in the semilattice ordering. It is easy to see that the variety $\mathcal{PS}$ is a subtractive variety and every semilattice in $\mathcal{PS}$ is pseudocomplemented. Conversely any pseudocomplemented semilattice, once we throw in * as a operation, belongs to $\mathcal{PS}$.

The term witnessing subtractivity is $x \wedge y^*$; in fact $x \wedge 0^* = x$ and

$$x \wedge x^* = x \wedge (x \wedge 0^*)^* = x \wedge 0^{**} = x \wedge 0 = 0.$$

Moreover if $\mathbf{L} \in \mathscr{PS}$ and $a, b \in L$; then

$$b \leq a^* \quad \Rightarrow \quad 0 = a \wedge a^* \geq a \wedge b$$

and

$$a \wedge b = 0 \quad \Rightarrow \quad b = b \wedge 0^* = b \wedge (b \wedge a)^* = b \wedge a^* \quad \Rightarrow \quad b \leq a^*.$$

PROPOSITION 4.2. *Let* $\mathbf{L} \in \mathscr{PS}$; *a subset* $I \subseteq L$ *is an ideal iff*
(a) *I is a semilattice ideal*;
(b) *$a \in I$ implies $a^{**} \in I$*;
(c) *$a, b \in I$ implies $a \oplus b \in I$.*

Proof. That an ideal of $\mathbf{L}$ has to satisfy (a), (b), (c) follows from the identities

$$0 \wedge x = 0 \wedge 0 = 0^{**} = 0 \oplus 0 = 0.$$

On the other hand let I be a subset of L satisfying (a), (b), (c) and let

$$\theta_I = \{(a, b): a \wedge b^*, a^* \wedge b \in I\}.$$

θ_I is clearly reflexive, since $0 \in I$; suppose $(a, b) \in \theta_I$. Then $a \wedge b^* \in I$ implies $(a \wedge b^*)^{**} \in I$, hence $a^{**} \wedge b^* \in I$. Similarly $a^* \wedge b \in I$ implies $a^* \wedge b^{**} \in I$. Hence $(a^*, b^*) \in I$.

Let now $(a, b), (c, d) \in \theta_I$, then

$$\begin{aligned} (a \wedge c)^* \wedge (b \wedge d) &\leq (a^{**} \wedge c^{**})^* \wedge (b \wedge d)^{**} \\ &= (a^* \oplus c^*) \wedge (b \wedge d)^{**} \\ &= (a^* \wedge b \wedge d) \oplus (c^* \wedge d \wedge b) \in I. \end{aligned}$$

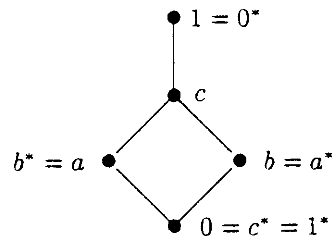
Symmetrically we obtain $(a \wedge c) \wedge (b \wedge d)^* \in I$ and hence $(a \wedge c, b \wedge d) \in \theta_I$.

We conclude that θ_I is a semicongruence and hence $I = 0/\theta_I$ is an ideal. □

Since $D = \{x \wedge y^*, y \wedge x^*\}$ is a set of 0-terms for $\mathscr{PS}$ by Proposition 3.9 we have:

COROLLARY 4.3. *$\mathscr{PS}$ is finitely congruential, witness $D = \{x \wedge y^*, y \wedge x^*\}$.*

Observe also that the variety $\mathscr{PS}$ is not ideal determined; consider the pseudocomplemented (semi)lattice below:



Then $0/\theta(1, c) = \{0\}$, hence $\mathscr{PL}$ is not 0-regular and hence not ideal determined.

PROPOSITION 4.4. *For any $\mathbf{L} \in \mathscr{PS}$ and $a, b \in L$*
(1) *$(a, b) \in (0)^D_{\mathbf{L}}$ iff $a^* = b^*$;*
(2) *$\langle L/(0)^D_{\mathbf{L}}, \oplus, \wedge, 0, 1\rangle$ is a Boolean algebra isomorphic with $\mathbf{S}(\mathbf{L})$.*

Proof. (1) If $a^* = b^*$, then from $b^* \leq a^*$ we get $a \wedge b^* = 0$; similarly $b \wedge a^* = 0$ and $(a, b) \in (0)^D_{\mathbf{L}}$. Conversely if $a \wedge b^* = b \wedge a^* = 0$ we get $a^* = b^*$.

(2) Consider the mapping $a/(0)^D_{\mathbf{L}} \mapsto a^{**}$; it is clearly well-defined; it is onto since if $c \in \mathrm{S}(\mathbf{L})$, then $c = a^* = a^{***}$ for some $a \in L$ and $a/(0)^D_{\mathbf{L}} = a^*/(0)^D_{\mathbf{L}}$. It is one-to-one by (1), since from $a^{**} = b^{**}$ we get $a^* = b^*$ and it preserves meets and * because of the axioms of pseudocomplemented semilattices. Finally it preserves $\oplus$ because of Proposition 4.1(7). □

By the above proposition and the trivial fact that any Boolean algebra $\mathbf{B}$ is a pseudocomplemented semilattice where $\Delta_B = 0_{\mathbf{B}}$, we conclude that $\mathscr{PS}_D$ is the variety of Boolean algebras. By Theorem 3.16 we conclude that $\mathscr{PS}$ is strongly normal with normalizer $(\ )^D$.

4.5. For the next two examples consider the following equations in the language $\langle \rightarrow, 1\rangle$:

(1) $x \rightarrow x \approx 1$
(2) $x \rightarrow 1 \approx 1$
(3) $1 \rightarrow x \approx x$
(4) $(x \rightarrow (y \rightarrow z)) \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow z)) \approx 1$
(5) $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) \approx 1$

(6) $(y \to z) \to ((x \to y) \to (x \to z)) \approx 1$
(7) $x \to (y \to z) \approx y \to (x \to z)$.

Observe that (2), (3) and (4) imply (1) and that (7) and any of (5) or (6) imply the other; any variety satisfying (1) and (3) is subtractive with respect to 1.

PROPOSITION 4.5. *Let* $\mathscr{V}$ *be the variety axiomatized by* (1), (2), (3), (5) *and* (6). *Then* $\mathscr{V}$ *is finitely congruential witness* $D = \{x \to y, y \to x\}$.

Proof. Let $\mathbf{A} \in \mathscr{V}$; we will show that a subset I of A is an ideal of $\mathbf{A}$ iff
(i) $1 \in I$;
(ii) if $a \in I$ and $a \to b \in I$, then $b \in I$.
Since $\mathscr{V}$ is subtractive, witness $y \to x$, any ideal has the above properties.
For the converse, suppose that $I \subseteq A$ satisfies (i) and (ii) and let

$$I^D = \{(a, b): a \to b, b \to a \in I\}.$$

Since $D = \{x \to y, y \to x\}$ is a set of 0-terms for $\mathscr{V}$, by Proposition 3.9 we have only to show that I^D is a subalgebra of $\mathbf{A} \times \mathbf{A}$. First we show that I^D is transitive; if $(a, b), (b, c) \in I^D$ by (5).

$$(a \to b) \to ((b \to c) \to (a \to c)) = 1 \in I.$$

Then by (ii) we get that $(b \to c) \to (a \to c) \in I$ and again $a \to c \in I$; $c \to a \in I$ is proven similarly and we conclude that $(a, c) \in I^D$.
Suppose now that $(a, b), (a', b') \in I^D$. Then by (6)

$$(a' \to b') \to ((a \to a') \to (a \to b')) = 1 \in I$$

and by (ii) $(a \to a') \to (a \to b') \in I$. Reversing the role of a' and b' above we get $(a \to b') \to (a \to a') \in I$ and hence $(a \to a', a \to b') \in I^D$. Similarly by (5)

$$(b \to a) \to ((a \to b') \to (b \to b')) = 1 \in I$$

and applying (ii) we get $(a \to b') \to (b \to a') \in I$. Reversing the role of a and b we get $(b \to b') \to (a \to b') \in I$ and hence $(a \to b', b \to b') \in I$. By transitivity we conclude that $(a \to a', b \to b') \in I^D$ which is therefore compatible; we have thus proved that $\mathscr{V}$ is finitely congruential. Finally observe that

$$1/I^D = \{a: a \to 1, 1 \to a \in I\} = I,$$

hence I is an ideal. □

Let now $\mathscr{W}$ be the variety axiomatized by (2), (3), (4), (5) and (6); we claim that $\mathscr{W}_D$ is the variety of Hilbert algebras. In fact the variety of Hilbert algebras is clearly contained in $\mathscr{W}_D$; in any Hilbert algebra $\mathbf{H}$, $(0)^D_{\mathbf{H}} = 0_{\mathbf{H}}$. On the other hand suppose that $\mathbf{A} \in \mathscr{W}_D$; to prove that it is an Hilbert algebra we only have to show that

$$x \to (y \to x) = 1 \tag{W}$$

holds in $\mathbf{A}$. Of course $(x \to (y \to x)) \to 1 = 1$ always; on the other hand

$$1 \to (x \to (y \to x)) = (y \to 1) \to ((1 \to x) \to (y \to x)) = 1$$

by (5), hence $(1, x \to (y \to x)) \in (0)^D_{\mathbf{A}}$; but $\mathbf{A} \in \mathscr{W}_D$, hence (W) holds in $\mathbf{A}$. By Theorem 3.16 we conclude that the variety $\mathscr{W}$ is strongly normal with normalizer $(\ )^{\varepsilon}$.

Let now $\mathscr{U}$ be the variety axiomatized by (1), (2), (3), (5), (7); the reader can see at once that $\mathscr{U}_D$ coincides with the quasivariety of BCK-algebras. Since it is a quasivariety that is not a variety we conclude again by Theorem 3.16 that $(\ )^{\varepsilon}$ is not a normalizer for $\mathscr{U}$.

Note that $\mathscr{W}, \mathscr{U}$ are not ideal determined. Consider the algebra $\mathbf{A} = \langle \{1, a, b\}, \to, 1 \rangle$ where

$$a \to b = \begin{cases} b & \text{if } a = 1 \\ 1 & \text{otherwise.} \end{cases}$$

We leave it to the reader to check that (1)–(7) hold in $\mathbf{A}$ and a congruence of $\mathbf{A}$ is simply a partition to which $\{1\}$ belongs. Hence $\boldsymbol{V}(\mathbf{A})$ is not ideal determined, since it is not 1-regular.

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