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Ideals and the binary discriminator in universal algebra

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Abstract. We introduce the binary discriminator and the dual binary discriminator and the corresponding universal algebras with 0. The latter are related to permutability and distributivity at 0. For A finite the dual binary discriminator is in the intersection of all maximal subclones of the clone of all f satisfying $f(0,\ldots,0) = 0$ (except certain maximal subclones if A is of prime power cardinality). An algebra with a special binary term function and a special unary term function is a dual binary discriminator algebra if and only if it is ideal-free. Finally we characterize binary and dual binary discriminator varieties.

Pixley's well-known result shows that the ternary discriminator, introduced in [12], is a term function on an algebra $\mathcal{A}$ if and only if every subalgebra of $\mathcal{A}$ is simple and $\mathcal{V}(\mathcal{A})$, the variety generated by $\mathcal{A}$, is arithmetic. Moreover, if every function on $\mathcal{A}$ compatible with *Con*$\mathcal{A}$ is a term function of $\mathcal{A}$ (i.e. $\mathcal{A}$ is hemi-primal [14]) and if *Con*$\mathcal{A}$ is finite then $\mathcal{V}(\mathcal{A})$ is arithmetic [12, 14]. Although these results were milestones in the development of universal algebra, there still exist algebras satisfying weaker but interesting conditions, see e.g. E. Fried and A. F. Pixley [6]. Further, the universal algebra ideals, introduced by A. Ursini [20] and K. Fichtner [5], are well behaved in varieties which need not be arithmetic but only arithmetic at 0, see [4] for this concept and basic properties. Investigations of ideals of an algebra with 0 focus on the neighbourhood of the constant 0 whereby some conditions like compatibility of functions or primality could be "localized" at 0. The aim of this paper is to show how this "localization" works and what parts of Pixley's result can be generalized to this local case.

1. Basic concepts

Let $\mathcal{A} = (A; F)$ be an algebra of similarity type τ. We say that $\mathcal{A}$ is an *algebra with* 0 if 0 is a nullary term function on $\mathcal{A}$. A variety $\mathcal{V}$ is $with$ 0 if 0 is either a nullary basic operation of $\mathcal{V}$ or an equationally defined constant.

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Let $\mathcal{A}$ and $\mathcal{V}$ be an algebra and a variety with 0. Following [20], an $(n+m)$-ary term $p(x_1, \ldots, x_n, y_1, \ldots, y_m)$ of $\mathcal{A}$ (of $\mathcal{V}$) is called an *ideal term of* $\mathcal{A}$ (*of* $\mathcal{V}$) *in* $y_1, \ldots, y_m$ if

$$p(x_1, \ldots, x_n, 0, \ldots, 0) \approx 0$$

is an identity of $\mathcal{A}$ (of $\mathcal{V}$). Clearly, the nullary operation 0 is an ideal term in every algebra with 0. Denote by $\mathcal{IT}(\mathcal{A})$ and $\mathcal{IT}(\mathcal{V})$ the sets of all ideal terms of $\mathcal{A}$ and $\mathcal{V}$, respectively.

A nonvoid subset I of A is *closed under the ideal term* $p(x_1, \ldots, x_n, y_1, \ldots, y_m)$ of $\mathcal{A}$ *in* $y_1, \ldots, y_m$ whenever $p(a_1, \ldots, a_n, b_1, \ldots, b_m) \in I$ for all $a_1, \ldots, a_n \in A$ and $b_1, \ldots, b_m \in I$. Next I is an *ideal* of $\mathcal{A}$ (of $\mathcal{V}$) if I is closed under all ideal terms from $\mathcal{IT}(\mathcal{A})$ (from $\mathcal{IT}(\mathcal{V})$). A finite subset $\mathcal{B}$ of $\mathcal{IT}(\mathcal{A})$ (of $\mathcal{IT}(\mathcal{V})$) is a *finite ideal-term basis* of $\mathcal{A}$ (of $\mathcal{V}$) if every nonempty subset I of A closed under every $p \in \mathcal{B}$ is an ideal of $\mathcal{A}(\mathcal{V})$.

As usual, *Con*$\mathcal{A}$ denotes the congruence lattice of $\mathcal{A}$ and ω_A denotes the least element of *Con*$\mathcal{A}$. The *kernel* of a binary relation Θ on A is the set

$$[0]_\Theta = \{a \in A; \langle a, 0\rangle \in \Theta\}.$$

The kernel of a congruence on $\mathcal{A}$ is an ideal of $\mathcal{A}$, but, in general, the converse is not true [7].

An algebra $\mathcal{A}$ with 0 is *permutable at* 0 if for all $\Theta, \Phi \in Con\mathcal{A}$ the relations $\Theta \circ \Phi$ and $\Phi \circ \Theta$ have the same kernel (here, as usual, $\Theta \circ \Phi = \{\langle x, y\rangle; \langle x, z\rangle \in \Theta$ and $\langle z, y\rangle \in \Phi$ for some $z \in A\}$). Next $\mathcal{A}$ is *distributive at* 0 if for all $\Theta_1, \Theta_2, \Psi \in Con\mathcal{A}$ both congruences $(\Theta_1 \vee \Theta_2) \wedge \Psi$ and $(\Theta_1 \wedge \Psi) \vee (\Theta_2 \wedge \Psi)$ have the same kernel and $\mathcal{A}$ is *dually distributive at* 0 if for all $\Theta_1, \Theta_2, \Psi \in Con\mathcal{A}$ both congruences $(\Theta_1 \wedge \Theta_2) \vee \Psi$ and $(\Theta_1 \vee \Psi) \wedge (\Theta_2 \vee \Psi)$ have the same kernel. In general, distributivity at 0 does not imply dual distributivity at 0. It was shown by J. Duda [4] that distributivity at 0 and dual distributivity at 0 coincide for algebras that are permutable at 0.

Following [4], we say that $\mathcal{A}$ is *arithmetic at* 0 if it is both permutable at 0 and distributive at 0. A variety $\mathcal{V}$ with 0 is *permutable at* 0, *distributive at* 0, *and arithmetic at* 0 if each $\mathcal{A} \in \mathcal{V}$ has the corresponding property at 0. It was shown by H. -P. Gumm and A. Ursini [7] that in a variety $\mathcal{V}$ permutable at 0 every ideal is a congruence kernel. The following Mal'tsev conditions were found in [1, 4, 7]:

PROPOSITION 1.1. *Let* $\mathcal{V}$ *be a variety with* 0. *Then*

1. $\mathcal{V}$ *is permutable at* 0 *if and only if there exists a binary term s of* $\mathcal{V}$ *such that* $s(x, x) \approx 0$ *and* $s(x, 0) \approx x$.
2. $\mathcal{V}$ *is arithmetic at* 0 *if and only if there exists a binary term s of* $\mathcal{V}$ *such that* $s(x, x) \approx 0 \approx s(0, x)$ *and* $s(x, 0) \approx x$.
3. $\mathcal{V}$ *is distributive at* 0 *if and only if there exist* $n > 1$ *and binary terms* $d_0, \ldots, d_n$ *of* $\mathcal{V}$ *such that*

$$d_0(x, y) \approx 0,\ d_n(x, y) \approx x,$$
$$d_i(0, x) \approx 0 \text{ for } i = 1, \ldots, n-1,$$

$$d_i(x, 0) \approx d_{i+1}(x, 0) \text{ for } i \text{ even}, 0 \le i < n;$$
$$d_i(x, x) \approx d_{i+1}(x, x) \text{ for } i \text{ odd}, 0 \le i < n.$$

Examples.

(1) The variety of all pseudocomplemented $\wedge$-semilattices with 0 is arithmetic at 0. Indeed in 2. it suffices to set $s(x, y) \approx x \wedge y^*$.
(2) The variety of all $\wedge$-semilattices with 0 is distributive at 0. Indeed in 3. choose $n = 2, d_0(x, y) \approx 0, d_1(x, y) \approx x \wedge y, d_2(x, y) \approx x$.

A variety $\mathcal{V}$ with 0 distributive at 0 is *n-distributive at* 0 if n is the least integer for which the condition 3. of Proposition 1.1 holds.

2. The binary discriminator

Recall that the *ternary discriminator* t^A and the *ternary dual discriminator* d^A on a set A are defined by setting

$$t^A(x, x, z) \approx z, \; t^A(x, y, z) = x,$$
$$d^A(x, x, z) \approx x, \; d^A(x, y, z) = z$$

for all $x, y, z \in A, x \neq y$. As it was pointed out in [6],

$$d^A(x, y, z) \approx t^A(z, t^A(x, y, z), x)$$

but t^A is not a term operation of $(A; d^A)$.

For a fixed element $0 \in A$ the *binary discriminator* b_0^A and the *dual binary discriminator* h_0^A on A are the binary functions on A defined by

$$b_0^A(x, y) = \begin{cases} x & \text{if } y = 0, \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

$$h_0^A(x, y) = \begin{cases} 0 & \text{if } y = 0, \\ x & \text{otherwise.} \end{cases} \tag{2}$$

We abbreviate t^A, d^A, b_0^A, h_0^A by t, d, b, h whenever A and 0 are clear from the context. The following facts are evident:

$(F1)$ $b(x, y) \approx t(0, y, x)$,
$(F2)$ $h(x, y) \approx d(0, y, x)$,
$(F3)$ $h(x, y) \approx b(x, b(x, y))$.

An algebra $\mathcal{A}$ is called a *discriminator algebra*, a *dual discriminator algebra*, a *binary discriminator algebra*, and a *dual binary discriminator algebra*, if the corresponding function is a term function on $\mathcal{A}$. We exhibit (dual) binary discriminator algebras which are not (dual) discriminator algebras:

Examples.

(1) Let $(S, \leq)$ be a chain with a least element 0 and a greatest element 1 (where $0 \neq 1$) and let $\wedge$ be the meet in $(S, \leq)$. Set $0^* = 1$ and $a^* = 0$ otherwise. Then the term $b(x, y) = x \wedge y^*$ is the binary discriminator of the pseudocomplemented $\wedge$-semilattice $\mathcal{S} = (S; \wedge, *, 0)$. It is well known [13] that the ternary discriminator is not a term function on $\mathcal{S}$ whenever *card* $S > 2$.

(2) For the two-element chain $(\{0, 1\}, \leq)$, the operation $x \wedge y$ is the dual binary discriminator of the $\wedge$-semilattice $\mathcal{R} = (\{0, 1\}, \wedge)$. However, b is not a term operation of $\mathcal{R}$ because b is not even monotone. Consequently, neither t nor b are term operations of $\mathcal{R}$.

Call an algebra $\mathcal{A}$ with 0 *ideal-free* if $\{0\}$ and A are the only ideals of $\mathcal{A}$. Let Z be a set and $f : Z \to A$. As usual,

$$\mathit{supp}\, f := \{z \in Z;\ f(z) \neq 0\}.$$

THEOREM 2.1. *Let $\mathcal{A}$ be an algebra with* 0 *and $\mathcal{V}$ the variety generated by $\mathcal{A}$. Then*

(1) *$\mathcal{A}$ is a binary discriminator algebra if and only if*

  (a) *$\mathcal{V}$ is permutable at* 0, *and*

  (b) *$\mathcal{A}$ is a dual binary discriminator algebra.*

(2) *If $\mathcal{A}$ is a dual binary discriminator algebra, then*

  (i) *$\mathcal{V}$ is* 2*-distributive at* 0, *and*

  (ii) *Every ideal I of any $\mathcal{B} \in \mathbb{SP}\mathcal{A}$ satisfies*

$$f \in I,\ g \in B,\ \mathit{supp}\ g \subseteq \mathit{supp}\ f \Longrightarrow g \in I. \tag{$*$}$$

In particular, every subalgebra of $\mathcal{A}$ is ideal-free.

Proof. (1) Let b be a term function of $\mathcal{A}$. In Proposition 1.1 (1) set $s := b$ to obtain that $\mathcal{V}$ is arithmetic at 0 and hence permutable at 0. By (F3) clearly h is a term function of $\mathcal{A}$. Conversely, let $\mathcal{V}$ be permutable at 0 and h a term function of $\mathcal{A}$. From Proposition 1.1 (1) we obtain that there exists a binary term s of $\mathcal{V}$ with $s(x, x) \approx 0$ and $s(x, 0) \approx x$. Set

$$p(x, y) :\approx s(x, h(x, y)).$$

Notice that $p(x,0) \approx s(x,h(x,0)) \approx s(x,0) \approx x$ while for $y \neq 0$ we have

$$p(x,y) = s(x,h(x,y)) = s(x,x) = 0.$$

This shows that $b = p$ is a term of $\mathcal{A}$.

(2) Let h be a term function of $\mathcal{A}$. To prove that $\mathcal{V}$ is 2-distributive at 0 set $d_0(x,y) \approx 0$, $d_1(x,y) \approx h(y,x)$ and $d_2(x,y) \approx x$. Notice that

$$\begin{aligned} d_1(0,x) &\approx h(x,0) \approx 0, \\ d_0(x,0) &\approx 0 \approx h(0,x) \approx d_1(x,0), \\ d_1(x,x) &\approx h(x,x) \approx x \approx d_2(x,x). \end{aligned}$$

By Proposition 1.1 (3) the variety $\mathcal{V}$ is 2-distributive at 0.

Now, let $I, \mathcal{B}, f$ and g be as in (ii) and let $B \subseteq A^Z$. Let $\mathcal{Y} := A^Z$. Denote by c_0 the constant map from Z into A with value 0. We show that $h^Y_{c_0}(x,y)(z) := h^A_0(x(z), y(z))$ is an ideal term of $\mathcal{Y}$ in its second variable. For every $x \in Y = A^Z$ clearly $d := h^Y_{c_0}(x, c_0)$ satisfies $d(z) = h^A_0(x(z), 0) = 0$ for every $z \in Z$ whence $d = c_0$ proving the claim.

Since $f \in I$, we obtain that $e := h^{\mathcal{Y}}_{c_0}(g, f) \in I$. For every $z \in Z \setminus supp\ f$ we get $e(z) = h^A_0(g(z), 0) = 0$ while for $z \in supp\ f$ clearly $e(z) = h^A_0(g(z), f(z)) = g(z)$. Together $e = g$ due to $supp\ g \subseteq supp\ f$.

For the last statement let $\mathcal{B}$ be a subalgebra of $\mathcal{A}$ and I an ideal of $\mathcal{B}$. If there exists $f \in I \setminus \{0\}$ then by what has been shown above (for $|Z| = 1$) every $g \in B$ belongs to I and hence $I = B$. □

COROLLARY 2.2. *If $\mathcal{A}$ is a binary discriminator algebra then $\mathcal{V}(\mathcal{A})$ is arithmetic at* 0 *and every subalgebra of $\mathcal{A}$ is ideal-free.*

Proof. It follows directly from Theorem 2.1 and (F3). □

3. 0-preserving functions

Let $\mathcal{A}$ be an algebra with 0. A function f on A is 0-*preserving* if $f(0,\dots,0) = 0$. All 0-preserving functions on A form a clone Z_A. It is a large clone; in fact a coatom of the lattice of all clones on A ordered by containment ([8] for A finite, [17] for A infinite). Clearly the set $Id\ \mathcal{A}$ of all ideals of $\mathcal{A}$ is solely determined by the clone of all 0-preserving term functions of $\mathcal{A}$. This raises the problem of studying $Id(C)$ for the various subclones C of Z_A. A particular subproblem is the determination of all ideal-free subclones C of Z_A (i.e., such that $(A; C)$ is ideal-free). The lattice $\mathcal{Z}_{\mathcal{A}}$ of all subclones of Z_A is explicitly known only for $|A| = 2$, see [16]; it is countable and essentially consists of four interlaced infinite descending chains. For $2 < |A| < \aleph_0$ the lattice $\mathcal{Z}_{\mathcal{A}}$ contains a copy of the direct power 2^N (of all 0–1 sequences ordered componentwise) and hence $|\mathcal{Z}_{\mathcal{A}}| = 2^{\aleph_0}$, [3]. For

A infinite we have $|\mathcal{Z}_{\mathcal{A}}| = 2^{(2^{|A|})}$; in fact a slight modification of an argument from [17, 18] shows that there are already that many essentially unary subclones of Z_A.

An algebra $\mathcal{A}$ is 0-*primal* if Z_A is the clone of all term functions of $\mathcal{A}$. For $2 < |A| < \aleph_0$ D. Lau [10] found a general 0-primality criterion. It is based on the full list of all coatoms of $\mathcal{Z}_A$, i.e., of the clones covered by Z_A, called *maximal subclones* of Z_A. The criterion states that $(A; F)$ with $F \subseteq Z_A$ is 0-primal if and only if F is a subset of *no* maximal subclone of Z_A. It follows from the fact that each proper subclone of Z_A is included in a maximal subclone of Z_A (equivalently, Z_A is a compact element of $\mathcal{Z}_A$ or also Z_A is a finitely generated clone). To describe Lau's maximal subclones of Z_A, we recall the following concept due to Kuznetsov [9]. For a positive integer m an m-ary relation on A is subset ρ of A^m. An n-ary function on A *preserves* ρ if ρ is a subuniverse of $(A; f)^m$. Denote by *Pol* ρ the set of all functions on A preserving ρ. Here is Lau's list divided into 7 natural groups:

1. The clones $C = Z_A \cap Pol\ \rho$ where $m = 1$ and $\emptyset \neq \rho \subset A$ and $\rho \neq \{0\}$ (i.e. the clone of all 0-preserving functions f on A such that $f(r_1, \dots, r_n) \in \rho$ whenever $r_1, \dots, r_n \in \rho$).
2. The clones $C = Z_A \cap Pol\ \rho$ where $m = 2$ and ρ is a nontrivial equivalence relation on A having $\{0\}$ as its class (thus C is the set of all 0-preserving f such that $\rho \in Con(A; f)$).
3. The clones $C = Z_A \cap Pol\ (\leq)$ where $\leq$ is an order relation on A with the least element 0 and a greatest element. Here C is the set of all 0-preserving isotone functions (i.e. functions f on A such that $f(a_1, \dots, a_n) \leq f(b_1, \dots, b_n)$ whenever $a_1 \leq b_1, \dots, a_n \leq b_n$).

An m-ary relation ρ is *totally symmetric* if it is invariant under all coordinate exchanges; i.e., for every permutation π of $\{1, \dots, m\}$

$$\langle a_1, \dots, a_m \rangle \in \rho \Longrightarrow \langle a_{\pi(1)}, \dots, a_{\pi(m)} \rangle \in \rho.$$

The relation ρ is *totally reflexive* if $\langle a_1, \dots, a_m \rangle \in \rho$ whenever $a_1, \dots, a_m$ are not pairwise distinct; i.e., if $a_i = a_j$ for some $1 \leq i < j \leq m$. An element $c \in A$ is *central* for ρ if $\langle c, a_2, \dots, a_m \rangle \in \rho$ for all $a_2, \dots, a_m \in A$.

4. The clones $C = Z_A \cap Pol\ \rho$ where $1 < m < |A|$ and $\rho \subseteq A_m$ is totally symmetric and totally reflexive and 0 is a central element for ρ.
5. The clones $C = Pol\{\langle a, s(a) \rangle; a \in A\}$ where s is a permutation of A with a unique fixed point 0 and all cycles of the same prime length p.
6. The clones $C = Pol(\gamma \cup \{\langle 0, 0 \rangle\})$ where γ is a binary symmetric and areflexive relation on A with $\langle a, 0 \rangle \in \gamma$ and $\langle 0, a \rangle \in \gamma$ for all $a \in A \backslash \{0\}$ (i.e., a graph with a star at 0).
7. The clones $C = Pol\{\langle x, y, x + y \rangle; x, y \in A\}$ where $G = (A; +, 0)$ is an elementary abelian 2-group (i.e., the additive structure of vector space over the field GF(2)).

The clones listed in 1.–4. and 7. are meet reducible in the lattice $\mathcal{L}$ of clones while those listed in 5. and 6. are meet irreducible.

THEOREM 3.1. *If* $1 < |A| < \aleph_0$ *then the dual binary discriminator* h *belongs to the clones listed in* 1.–6.

Proof. Set $h = h_0^A$. We prove that h preserves all the relations from 1.–6. We proceed by contraposition.

1. Let $\rho \subseteq A$ and $h(a,b) \notin \rho$ for some $a, b \in A$. If $b \neq 0$ then $a = h(a,b) \notin \rho$ and we are done. Thus let $b = 0$. Then $b = 0 = h(a,0) \notin \rho$.
2. Let ρ be an a nontrivial equivalence relation on A with a class $\{0\}$ and $\alpha := \langle h(a,b), h(c,d)\rangle \notin \rho$. If $b \neq 0 \neq d$ then $\langle a, c\rangle = \alpha \notin \rho$. Thus let $0 \in \{b, d\}$. By symmetry we may assume $b = 0$. From $\langle 0, h(c,d)\rangle = \alpha \notin \rho$ we obtain $d \neq 0$ and $\langle b, d\rangle = \langle 0, d\rangle \notin \rho$.
3. Let $\leq$ be an order on A with the least element 0 and let $h(a,b) \not\leq h(c,d)$. Clearly $h(a,b) \neq 0$ and so $a \neq 0 \neq b$. If $d \neq 0$ then $a = h(a,b) \not\leq h(c,d) = c$. Thus let $d = 0$. Then $b \not\leq 0 = d$. (In fact, we have proved that $h \in Pol \leq$ for every order $\leq$ with the least element 0).
4. Let ρ be a totally symmetric m-ary relation and let 0 be a central element of ρ. Let $\alpha := \langle h(a_1, b_1), \ldots, h(a_m, b_m)\rangle \notin \rho$. Then $h(a_i, b_i) \neq 0$ for all $i = 1, \ldots, m$ and so $h(a_i, b_i) = a_i$ for all $i = 1, \ldots, m$. Then $\langle a_1, \ldots, a_m\rangle \notin \rho$.
5. Let s be a permutation of A with a fixed point 0. Let $h(c,d) \neq s(h(a,b))$. Then $h(c,d) = 0 = h(a,b)$ does not hold and therefore $d = 0 = b$ is impossible. If $d \neq 0 \neq b$ then $c = h(c,d) \neq s(h(a,b)) = s(a)$ and we are done. If $d = 0 \neq b$ then $d = 0 \neq s(b)$ and finally if $d \neq 0 = b$ then $d \neq 0 = s(0) = s(b)$.
6. Let $(A; \gamma)$ be a graph on A with a star at 0. Suppose $\alpha = \langle h(a,b), h(c,d)\rangle \notin \gamma \cup \{\langle 0,0\rangle\}$. Then $h(a,b) \neq 0 \neq h(c,d)$ and so $\langle a,c\rangle = \alpha \notin \gamma \cup \{\langle 0,0\rangle\}$.

□

REMARK. Denote by M the intersection of all the maximal clones listed above in 1.–6. It can be proved that M is the clone $Clo(h)$ generated by h. In fact $Clo(h)$ is even the intersection of smaller sets of clones maximal in Z_A; e.g., of all $Pol\{a\}(a \in A)$ and all $Pol\ \rho$ where ρ is a bounded order on A with the least element 0. It can be shown that h does not belong to any of the maximal subclones of Z_A of type 7. Note that the clones of type 7 exist only for $|A| = p^m$, p prime.

Let the universe A be infinite. The clones listed in 1–7 are well defined but in general not maximal in Z_A (i.e., not covered by Z_A in the lattice of clones). Nevertheless Theorem 3.1 is also valid for A infinite.

THEOREM 3.2. *Every maximal clone in* Z_A *is ideal-free.*

Proof. For the clones listed in 1–6 this follows from Theorem 3.1 and Theorem 2.1 (2). Thus consider the clones from 7. It is well known that we can identify A with the set $\mathbf{2}^m$ (of all 0–1 m-dimensional vectors) and $+$ with the componentwise mod 2 addition on $\mathbf{2}^m$. Set $\lambda = \{\langle x, y, x+y\rangle; x, y \in \mathbf{2}^m\}$. Let I be an ideal of $(\mathbf{2}^m; Pol\lambda)$ distinct from $\{\underline{0}\}$, where $\underline{0} = \langle 0, \ldots, 0\rangle$, and let $i \in I\backslash\{\underline{0}\}$. Let A be an $m \times m$ $0-1$ matrix and $h(x) :\approx xA$ (where xA is the product of the $1 \times m$ matrix x and the matrix A over $GF(2)$). It is almost immediate that $h \in Pol\lambda$. As $h(\underline{0}) \approx \underline{0}$, clearly $h(y)$ is an ideal term in y and so $h(i) \in I$. Now varying A we can get that $I = \mathbf{2}^m$. Thus the clone is ideal-free as well. ☐

For infinite A we do not know any analogue of Lau's theorem.

COROLLARY 3.3. *If $\mathcal{A}$ is a* 0*-primal algebra then*

(i) $\{0\}$ *is the only proper subalgebra of $\mathcal{A}$;*
(ii) *$\mathcal{A}$ is simple;*
(iii) $Aut\mathcal{A} = \{id_A\}$, *and*
(iv) *$\mathcal{A}$ is ideal-free.*

Proof. (i)–(iii) are proved by constructing appropriate unary 0-preserving operations. (iv) follows directly by Theorem 3.2. ☐

LEMMA 3.4. *If $\mathcal{A}$ is a binary discriminator algebra then every ideal of $\mathcal{A}$ is a congruence kernel.*

Proof. Evident since, by Corollary 2.2, $\mathcal{A}$ is ideal-free, i.e. $\{0\}$ and A are the only ideals of $\mathcal{A}$. ☐

Let $\emptyset \neq B \subseteq A$. For an n-ary function f on A we denote by

$$f(B, \ldots, B) = \{f(b_1, \ldots, b_n); b_1, \ldots, b_n \in B\}.$$

REMARK. Let $\mathcal{A} = (A, F)$ be an algebra, $\Theta \in Con\mathcal{A}$ and $K = [0]_\Theta$. If f is an n-ary function on A such that $f(K, \ldots, K) \subseteq K$ for each $\Theta \in Con\mathcal{A}$, then f is 0-preserving since $\{0\} = Ker\,\omega_A$.

We say that an n-ary function f on A is an *ideal-compatible function* of $\mathcal{A}$ if $f(I, \ldots, I) \subseteq I$ for each ideal I of $\mathcal{A}$.

COROLLARY 3.5. *An algebra $\mathcal{A}$ with* 0 *is* 0*-primal if and only if every ideal-compatible function of $\mathcal{A}$ is a term function of $\mathcal{A}$.*

Proof. It follows directly by Lemma 6, the foregoing Remark and the fact that every congruence kernel of $\Theta \in Con\mathcal{A}$ is an ideal of $\mathcal{A}$, see [7]. ☐

4. Main ideal term algebras

Although Corollary 2.2 provides a kind of a 0-version of Pixley's Theorem 3.1 from [13], we do not have its converse, i.e., an adaption of Theorem 5.1 from [12] to ideals, which is perhaps due to the complex structures of ideals in the general case. We have a converse if we impose a condition on the similarity type and a condition on the ideal terms.

From now on we assume that $\mathcal{A}$ is an algebra with 0 containing a binary term function $\circ$ and a unary term function denoted by $'$. Instead of $(x')'$ we write x''. Denote by $I(c)$ the ideal of $\mathcal{A}$ generated by the singleton $\{c\}$. Call $\mathcal{A}$ *a main ideal term algebra* if

$$(x \circ y'') \circ y' \approx 0 \quad \text{and} \quad x \circ 0' \approx x, \tag{$*$}$$

$$I(c) = \{a \circ c''; a \in A\} \quad \text{for every } c \in A\backslash\{0\}. \tag{$**$}$$

THEOREM 4.1. *Let $\mathcal{A}$ be a main ideal term algebra. Then $\mathcal{A}$ is a binary discriminator algebra if and only if $\mathcal{A}$ is ideal-free.*

Proof. ($\Leftarrow$) Let $\mathcal{A}$ be ideal-free. We show that $x \circ y'$ is the binary discriminator on A. First $x \circ 0' \approx x$ and so it remains to prove that $x \circ y' = 0$ for all $x \in A$ and all $y \in A\backslash\{0\}$. Since $\mathcal{A}$ has no proper ideals, $I(y) = A$ and thus by ($**$) we have $x = a \circ y''$ for some $a \in A$. Hence $x \circ y' = (a \circ y'') \circ y' = 0$ by ($*$). □

($\Rightarrow$): Corollary 2.2.

Although Theorem 4.1 provides a converse of Corollary 2.2 for main ideal term algebras, the verification of ($**$) may be far from obvious.

THEOREM 4.2. *Let $\mathcal{A}$ satisfy ($*$) and*

$$x \circ x'' \approx x,\ (x \circ y'')'' \approx (x \circ y'')'' \circ y'',\ (x \circ y) \circ z'' \approx x \circ (y \circ z'').$$

If $\{y'', x \circ y\}$ is a basis of ideal terms of $\mathcal{A}$ then $\mathcal{A}$ is a main ideal term algebra.

Proof. We must verify ($**$). Let $z \in A\backslash\{0\}$. Set $J = \{a \circ z''; a \in A\}$. By assumption, $z = z \circ z''$ and so $z \in J$. We prove that $J = I(z)$. Since $\{y'', x \circ y\}$ is a basis of $\mathcal{IT}(\mathcal{A})$, it suffices to show that J is closed under the ideal terms y'' and $x \circ y$. Let $c \in J$ be arbitrary. Then $c = a \circ z''$ for some $a \in A$ and $c'' = (a \circ z'')'' = (a \circ z'')'' \circ z'' = c'' \circ z'' \in J$. Let $x \in A$. Then $x \circ c = x \circ (a \circ z'') = (x \circ a) \circ z'' \in J$. Together J is an ideal of $\mathcal{A}$ containing z, i.e., $I(z) \subseteq J$. However, for every $a \in A$ we have $a \circ z'' \in I(z)$ thus also $J \subseteq I(z)$ proving ($**$). □

Example. Let $\mathcal{S} = (S; \wedge, ^*, 0)$ be a pseudocomplemented $\wedge$- semilattice, $x \circ y := x \wedge y$ and $x' := x^*$. A routine verification shows that $\mathcal{S}$ satisfies the conditions of Theorem 4.2. By Theorem 4.1, the binary discriminator on S is a term function on $\mathcal{S}$ if and only if $\mathcal{S}$ is ideal-free. It can be verified that this just happens when $\mathcal{S}$ is finite with exactly one atom.

REMARK. By Pixley's theorem, if the ternary discriminator is a term function on $\mathcal{A}$ then $\mathcal{A}$ is quasi-primal, i.e., every function on A preserving its subalgebras and inner isomorphisms is a term function of $\mathcal{A}$. This is not valid for the binary discriminator.

Example. Let $S = \{0, a, 1\}$ and $\mathcal{S} = (S; \wedge, ^*, 0)$ be the pseudocomplemented $\wedge$-semilattice with the induced order $0 < a < 1$. By the preceding example $x \wedge y^*$ is the binary discriminator on $\mathcal{A}$ and $\mathcal{A}$ is ideal-free. Consider the binary function $f(x, y)$ defined by setting $f(a, a) = 1$ and $f(x, y) = 0$ otherwise. As $\{0, 1\}$ is the only proper subalgebra of $\mathcal{S}$, clearly f preserves all the subalgebras of $\mathcal{S}$ and, trivially, also all inner isomorphisms of $\mathcal{S}$. We show that the relation

$$\rho := \{\langle 0, 0\rangle, \langle a, 1\rangle, \langle 1, 1\rangle\}$$

is a subuniverse of $\mathcal{S}^2$. It is easy to see that both 0 and x^* are compatible with ρ (use $0^* = 1$ and $a^* = 1^* = 0$). Finally ρ is a subuniverse of $(S; \wedge)^2$ due to $\langle 0, 0\rangle \wedge \langle a, 1\rangle = \langle 0, 0\rangle \wedge \langle 1, 1\rangle = \langle 0, 0\rangle$ and $\langle a, 1\rangle \wedge \langle 1, 1\rangle = \langle a, 1\rangle$ and so ρ is compatible with $\wedge$. However, ρ is not a subuniverse of $(S, f)^2$ because $\langle f(a, a), f(1, 1)\rangle = \langle 1, 0\rangle \notin \rho$. Thus f is not a term function of $\mathcal{S}$.

5. Binary discriminator varieties

A variety $\mathcal{V}$ with 0 of type τ is a *binary discriminator variety* if $\mathcal{V}$ is generated by a nontrivial class $\mathcal{K}$ of type τ with a binary term q such that q^A is the binary discriminator b_0^A for each $\mathcal{A} \in \mathcal{K}$.

For the ternary discriminator, the similarly defined *discriminator varieties* were studied by McKenzie [11]. We characterize the binary discriminator varieties:

THEOREM 5.1. *The following conditions are equivalent for a variety $\mathcal{V}$ with* 0*:*

(1) *$\mathcal{V}$ is a binary discriminator variety;*

(2) *there exists a binary term $b(x, y)$ satisfying*

  (i) $b(x, 0) \approx x, b(0, x) \approx 0 \approx b(x, x)$*;*

  (ii) $b(x, b(y, x)) \approx x$*;*

  (iii) *every n-ary operation f in the type of $\mathcal{V}$ satisfies*

$$b(f(x_1, \ldots, x_n), y) \approx b(f(b(x_1, y), \ldots, b(x_n, y)), y)$$

  (iv) *$\mathcal{V}$ is generated by a class $\mathcal{K}$ whose algebras are ideal-free;*

(3) *there exist a binary term $b(x, y)$ of $\mathcal{V}$ satisfying* (i)–(iii) *above and $\mathcal{V}$ is generated by a class $\mathcal{K}$ whose algebras have no proper congruence kernels.*

Proof. (1) $\Rightarrow$ (2): It is an easy exercise to verify that for each $\mathcal{A} \in \mathcal{K}$ the binary discriminator b_0^A satisfies (i)–(iii) and whence (i)–(iii) hold in $\mathcal{V}$. (iv) By Corollary 2.2 every $\mathcal{A} \in \mathcal{K}$ is ideal-free.

(2) $\Rightarrow$ (3): Immediate because every congruence kernel is an ideal.

(3) $\Rightarrow$ (1): Let $\mathcal{A} \in \mathcal{K}$. By (i) it remains to show that $z := b(x, y) = 0$ for all $x, y \in A \backslash \{0\}$. Suppose to the contrary that $z \neq 0$. Set

$$\gamma_z := \{\langle c, d\rangle \in A^2; b(c, z) = b(d, z)\}. \tag{+}$$

Clearly γ_z is an equivalence relation on A induced by the right translation $r_z(x) :\approx b(x, z)$ of b. Notice that by (i)

$$r_z(0) = b(0, z) = 0 = b(z, z) = r_z(z) \tag{$\times$}$$

and by (ii)

$$r_z(y) = b(y, z) = b(y, b(x, y)) = y \neq 0. \tag{$\times\times$}$$

Moreover, γ_z is a congruence on $\mathcal{A}$ because (iii) guarantees the substitution property. From $(\times)$ and $(\times\times)$ we see that $0, z \in [0]_{\gamma_z}$ while $y \notin [0]_{\gamma_z}$. Thus $\{0\} \neq [0]_{\gamma_z} \neq A$ and so γ_z has a proper congruence kernel. This contradiction shows $z = 0$ and then b is the binary discriminator. □

Example. Every variety is generated by all its subdirectly irreducible (SI) members. It is easy to show that in the variety $\mathcal{V}$ of pseudocomplemented $\wedge$-semilattices generated by the 3-element chain the only SI members are the 2- and 3-element chains. By the above examples in both of these SI members $x \wedge y^*$ is the binary discriminator. Hence $\mathcal{V}$ is a binary discriminator variety.

REMARK. There are essential differences between the (ternary) discriminator varieties and the binary discriminator varieties. For example, every discriminator variety is regular and congruence uniform. This does not hold for the binary discriminator varieties as one can observe in the previous example. Indeed, the 3-element chain $0 < a < 1$ has two congruences with the same 0-class, namely ω and $\Theta(a, 1)$ (where $\Theta(c, d)$ denotes the least congruence of $\mathcal{A}$ containing $\langle c, d\rangle$). Further, $\Theta(a, 1)$ has two blocks of unequal size. Moreover, if $\mathcal{V}$ is a discriminator variety, for $\mathcal{A} \in \mathcal{V}$ and $a, b, c \in A$ there exists $d \in A$ with $\Theta(a, b) = \Theta(c, d)$. It is an easy exercise to show that

$$d = t(t(a, b, c), t(a, b, t(a, c, b)), t(a, c, b)).$$

However, for the above mentioned 3-element chain (considered as a pseudocomplemented $\wedge$-semilattice) $\Theta(a, 1) \neq \Theta(0, x)$ for each $x \in A$.

We define a *dual binary discriminator variety* as the variety with 0 generated by a class $\mathcal{K}$ of nontrivial similar algebras with a binary term q such that q^A is the dual binary discriminator h_0^A for each $\mathcal{A} \in \mathcal{K}$.

We characterize the dual binary discriminator varieties:

THEOREM 5.2. *Let $\mathcal{V}$ be a variety with* 0*. Then $\mathcal{V}$ is a dual binary discriminator variety if and only if there exists a binary term h of $\mathcal{V}$ such that*

(i) $h(x, x) \approx x$, $h(x, 0) \approx 0 \approx h(0, x)$*;*
(ii) $h(h(x, y), y) \approx h(x, y)$*;*
(iii) *$\mathcal{V}$ is generated by a class $\mathcal{K}$ such that for each $\mathcal{A} \in \mathcal{K}$ and every $y \in A\backslash\{0\}$ the right translation $r_y(x) \approx h(x, y)$ is injective.*

Proof. ($\Rightarrow$): Let $\mathcal{V}$ be a dual binary discriminator variety and $\mathcal{K}$ its generating class. Then for each $\mathcal{A} \in \mathcal{K}$ the term operation $h^{\mathcal{A}}$ is the dual binary discriminator h_0^A. It is easy to check that it satisfies (i) and (ii); consequently (i) and (ii) hold in $\mathcal{V}$. To prove (iii) let $\mathcal{A} \in \mathcal{K}$ and $y \in A\backslash\{0\}$. Then $r_y(x) \approx h_0^A(x, y) \approx x$ and so $r_y(x)$ is injective.

($\Leftarrow$): Let $\mathcal{V}$ satisfy the conditions (i)–(iii). Suppose to the contrary that there exist $\mathcal{A} \in \mathcal{K}$ and $x, y \in A\backslash\{0\}$ such that $z := h^{\mathcal{A}}(x, y) \neq x$. Consider the right translation $r_y(x) :\approx h^{\mathcal{A}}(x, y)$. By (ii) $r_y(z) = h^{\mathcal{A}}(z, y) = h^{\mathcal{A}}(h^{\mathcal{A}}(x, y), y) = h^{\mathcal{A}}(x, y) = z = r_y(x)$ in contradiction to (iii). Thus $h^{\mathcal{A}}(x, y) = x$ for all $x, y \in A\backslash\{0\}$. Moreover from (i) also $h^{\mathcal{A}}(x, 0) \approx 0 \approx h^{\mathcal{A}}(0, x)$ and so $h^{\mathcal{A}} = h_0^{\mathcal{A}}$. □

THEOREM 5.3. *Let $\mathcal{V}$ be a dual binary discriminator variety, h the corresponding term, $\mathcal{A} \in \mathcal{V}$ and $y \in A$. Then $\gamma_y := \{\langle x, z\rangle \in A^2; h^{\mathcal{A}}(x, y) = h^{\mathcal{A}}(z, y)\}$ satisfies*

$$\gamma_y \circ \Theta(0, y) = A^2.$$

Proof. By (i) of Theorem 5.2 clearly $\gamma_0 = A^2$ and so the assertion holds for $y = 0$. Thus let $y \neq 0$ and $x \in A$. By (ii) of Theorem 5.2 we have

$$\langle x, h^{\mathcal{A}}(x, y)\rangle \in \gamma_y.$$

Observe that by (i)

$$\begin{aligned} \langle h^{\mathcal{A}}(x, y), 0\rangle &= \langle h^{\mathcal{A}}(x, y), h^{\mathcal{A}}(x, 0)\rangle \in \Theta(0, y) \\ \langle 0, y\rangle &= \langle h^{\mathcal{A}}(y, 0), h^{\mathcal{A}}(y, y)\rangle \in \Theta(0, y) \end{aligned}$$

and so $\langle h^{\mathcal{A}}(x, y), y\rangle \in \Theta(0, y)$. Hence $\langle x, y\rangle \in \gamma_y \circ \Theta(0, y)$ proving $\gamma_y \circ \Theta(0, y) = A^2$. □

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