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Some further results on pointfree convex geometry

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Abstract. Inspired by locale theory, pointfree convex geometry was first proposed and studied by Yoshihiro Maruyama. In this paper, we shall continue to his work and investigate the related topics on pointfree convex spaces. Concretely, the following results are obtained: (1) A Hofmann–Lawson-like duality for pointfree convex spaces is established. (2) The $\mathcal{M}$-injective objects in the category of S_0-convex spaces are proved precisely to be sober convex spaces, where $\mathcal{M}$ is the class of strict maps of convex spaces; (3) A convex space X is sober iff there never exists a nontrivial identical embedding $i\colon X \hookrightarrow Y$ such that its dualization is an isomorphism, and a convex space X is S_D iff there never exists a nontrivial identical embedding $k\colon Y \hookrightarrow X$ such that its dualization is an isomorphism. (4) A dual adjunction between the category $\mathbf{CLat}_D$ of continuous lattices with continuous D-homomorphisms and the category $\mathbf{CS}_D$ of S_D-convex spaces with CP-maps is constructed, which can further induce a dual equivalence between $\mathbf{CS}_D$ and a subcategory of $\mathbf{CLat}_D$; (5) The relationship between the quotients of a continuous lattice L and the convex subspaces of $\mathbf{cpt}(L)$ is investigated and the collection $\mathbf{Alg}(\mathbf{Q}(L))$ of all algebraic quotients of L is proved to be an algebraic join-sub-complete lattice of $\mathbf{Q}(L)$ of all quotients of L, where $\mathbf{cpt}(L)$ denote the set of non-bottom compact elements of L. Furthermore, it is shown that $\mathbf{Alg}(\mathbf{Q}(L))$ is isomorphic to the collection $\mathbf{Sob}(\mathcal{P}(\mathbf{cpt}(L)))$ of all sober convex subspaces of $\mathbf{cpt}(L)$; (6) Several necessary and sufficient conditions for all convex subspaces of $\mathbf{cpt}(L)$ to be sober are presented.

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1. Introduction

Locale theory can be considered as an algebraic theory of topological structures which does not presuppose the notion of a point and is primarily based on that of a region, since locale theory studies the lattice structure of open sets in an algebraic form, i.e., a "space" in locale theory is in fact a frame (for locale theory, see [23,28,34,35]). Usually, localic versions of theorems in the ordinary topology does not need non-constructive principles such as the law of exclude middle or the axiom of choice, and so locale theory can also be seen as constructive topology (see [2,8,10,9]). The following results are fundamental for locale theory, which clarify the categorical relationships between pointfree spaces and pointsets spaces.

- There is a dual adjunction between the category of frames with frame homomorphisms and the category of topological spaces with continuous maps (see [14]);
- Isbell duality: there is a dual equivalence between the category of spatial frames with frame homomorphisms and the category of sober spaces with continuous maps (see [7]);
- Hofmann–Lawson duality: there is a dual equivalence between the category of distributive continuous lattices (or continuous frames) with frame homomorphisms and the category of locally compact sober spaces with continuous maps (see [15]);
- There is a dual equivalence between the category of m-spatial frames with m-homomorphisms and the category of T_1-spaces with continuous maps (see [27]);
- There is a dual adjunction between the category of frames with D-homomorphisms and the category of T_D-spaces with continuous maps (see [6]);
- There is a dual equivalence between the category of T_D-spatial frames with D-homomorphisms and the category of T_D-spaces with continuous maps (see [6]).

Inspired by locale theory, pointfree convex geometry was first proposed and studied by Yoshihiro Maruyama in [26]. Similar to the cases of locale theory, pointfree convex geometry can also be viewed as an algebraic theory of construes which does not presuppose the notion of a point and is primarily based on that of a region. As a matter of fact, the theory pointfree convex geometry studies the lattice structure of convex sets in an algebraic form, i.e., a "convex space" in pointfree convex geometry is a continuous lattice. In [26] and [30], a categorical equivalence between sober convex spaces and algebraic lattices was established by considering some kinds of meet-complete filters and non-bottom compact elements as points respectively. Besides, the relationship between pointfree convex geometry and Hilbert's philosophy [10, 12] was discussed via this dual adjunction. The following results for pointfree convex geometry theory are fundamental, which investigate the categorical relationships between pointfree spaces and pointsets spaces.

• There is a dual adjunction between the category of continuous lattices with continuous homomorphisms and the category of convex spaces with CP-maps (see [26,30]);

• There is a dual equivalence between the category of algebraic lattices with continuous homomorphisms and the category of sober spaces with CP-maps (see [26,30]);

• There is a dual equivalence between the category of m-spatial continuous lattices with m-homomorphisms and the category of m-sober spaces (also called S_1-convex spaces in [33]) with CP-maps (see [26]).

The very famous theorem that a T_0-space is injective iff it is a T_0-space of "Scott open sets" in a continuous lattice was first proved by Scott in his paper [29] on the mathematical models for the Church-Curry λ-calculus, which is one of the most important results in domain theory. Based on this fact, a categorical isomorphism between injective T_0-spaces and continuous lattices was established. After about 10 years later, Jankowski [18,22] showed that a closure space is an absolute extensor for the category of all closure spaces which satisfy the compact theorem iff a contraction of X is a closure space of all filters with the empty set in a frame. Note that a closure space which satisfies the compact theorem is also called a convex space in [33] and an absolute extensor is in fact an injective object in the sense of Scott's theorem. Subsequently, Jankowski gave a uniform approach to the problem of the characterization of absolute extensors for the categories of T_0-spaces and convex spaces. Some applications of the theory of closure spaces to logic can be found in [1,19,20,21]. As we know, a frame with the filter convex structure is a sober convex space, which indicates that every injective object in the category of S_0-convex spaces is sober. Then one may ask: for which class of morphisms $\mathcal{M}$ including all convex-homeomorphisms and is contained in the class of monomorphisms are the $\mathcal{M}$-injective objects in $\mathbf{CS}_0$ exactly sober convex spaces. In Section 3, we shall give a complete answer.

T_D-space was originally introduced by Aull and Thron in [4], which is strictly between T_0 and T_1. In [13], Drake gave a necessary and sufficient condition of the lattice-equivalence [32] of topological spaces, that is, a topological space X has the property that for any T_0 space Y the lattices $\Gamma(X)$ and $\Gamma(Y)$ of all closed sets of X and Y are isomorphic implies that X is homeomorphic Y iff X is sober and T_D. On this basis, a dual adjunction between the category of T_D-spaces and a suitable subcategory $\mathbf{Frm}_D$ of the category $\mathbf{Frm}$ of frames was established. Furthermore, this adjunction can induce a categorical equivalence between the category of T_D-spaces and the subcategory of $\mathbf{Frm}_D$ given by T_D-spatial frames. In [24], the properties of T_D-spaces and spatial sublocales were investigated. Motivated by the topological separation axiom T_D and its importance in classical and point-free spaces, S_D-convex space was firstly introduced and studied by Shen et al. in [30]. Just as in the case of topological spaces, it was surprisingly shown in [30] that a convex space X has the property that for any S_0-convex space Y, $\mathfrak{C}(X)$ and $\mathfrak{C}(Y)$ of all convex subsets of X and Y are isomorphic implies X and Y are homeomorphic iff X is sober and S_D. However, a lot of important results mentioned above about T_D-spaces are still unknown for the case of S_D-convex spaces. Also, it should be recognized that the importance of S_D is in the rank of that of sobriety. As we will see in this paper, these two properties are closely related. For example, in Theorem 4.9 of Section 4, we will provide a comparison between the S_D-axiom and sobriety in classic convex theory showing they may be viewed as mirror images of each other or even dual to each other in certain sense. And in Theorem 6.11 of Section 6, we will present the close connection between the S_D-axiom and sobriety in the case of pointfree convex theory. Then it is natural to consider whether there exists a similar categorical framework for S_D-convex spaces as the classical duality between convex spaces and continuous lattices in [26] and [30]. In other words, whether the Banaschewski–Pultr-like duality in convex theory exists? If so, we will obtain that the "S_D-convex space" in pointfree convex geometry is also a continuous lattice and we can then consider the S_D-axiom as a bridge connecting the related properties between classical or pointfree convex theory and domain theory. All in all, comparing with the fundamental dualities for locale theory above, three results for pointfree convex geometry are still unknown and we will devote to studying them in Section 3 and Section 5.

As was seen in [17, 16], kernel operators and quotients also play key roles in pointfree convex geometry just as the roles of nuclei and sublocales in locale theory. The relationship between subspaces and sublocales was investigated in [28, 31], by exploiting the categorical equivalence between spatial frames and sober spaces. Concretely, for a frame L, it was proved that the collection $\mathbf{sp}(\mathbf{S}(L))$ of all spatial sublocales of L is a sublocale of the coframe $\mathbf{S}(L)$ consisting of all sublocales of L, and $\mathbf{sp}(\mathbf{S}(L))$ is isomorphic to the collection of all the sober subspaces of the sober space $\mathbf{pt}(L)$. Furthermore, the questions that when are all sublocales of L spatial and when are all the subspaces of $\mathbf{pt}(L)$ sober were explored and related necessary and sufficient conditions were given. In [16], the relationship between continuous kernel operators and quotients of continuous lattices was investigated. Concretely, it was shown that the continuous kernel operators and quotients on a continuous lattice are one-to-one corresponding and the question that for which a continuous lattice L is the complete lattice $Ker(L)$ of continuous kernel operators on L itself a continuous lattice was solved. Also, several necessary and sufficient conditions for all quotients of a continuous lattice L to be algebraic were given in [16]. However, the question that when are all convex subspace of $\mathbf{cpt}(L)$ sober is still unknown. In Section 6, we shall give a positive answer for this question and further show that the collection $\mathbf{Alg}(\mathbf{Q}(L))$ of all algebraic quotients of L is an algebraic join-sub-complete lattice of $\mathbf{Q}(L)$.

2. Preliminaries

In this section, we shall recall some basic facts about order theory and convex spaces. The readers can refer to [3] for category theory.

For a subset Y of a poset $(X, \leq)$, we write $\downarrow Y = \{x \in X : \exists y \in Y\ s.t.\ x \leq y\}$. A subset Y is a *lower set* of X if $Y = \downarrow Y$. When $Y = \{y\}$, we write $\downarrow \{y\}$

simply as $\downarrow y$ and we call it a *principal ideal* of X. A subset Y of a poset $(X, \leq)$ is *Scott closed* if it satisfies (1) $X = \downarrow X$ and (2) for any directed subset D, $D \subseteq X$ implies $\bigvee D \in X$ whenever $\bigvee D$ exists. The complements of Scott closed sets are called *Scott open sets*, such subsets of a poset X form a topology on X, which is the well-known Scott topology. A map $f\colon X \longrightarrow Y$ between posets is called *Scott continuous*, if it is continuous with respect to the Scott topologies. One can check that a map $f\colon X \longrightarrow Y$ between posets is Scott continuous iff it preserves existing suprema of directed sets. For a complete lattice L, the bottom element and the top element of L are denoted $\bot_L$ and $\top_L$, respectively. We say that x is *way below* y in L, in symbol $x \ll y$, iff for all directed subsets D of L, the relation $y \leq \sup D$ always implies $x \in \downarrow D$. An element satisfying $x \ll x$ is said to be *compact*. Let $K(L)$ denote the set of all compact elements of L. A complete lattice L is called *continuous* if for every $x \in L$, $x = \bigvee \Downarrow x$, where $\Downarrow x = \{y \in L \mid y \ll x\}$. A complete lattice L is called *algebraic* if for every $x \in L$, $x = \bigvee(\downarrow x \cap K(L))$. We say that x is *wedge below* y in L, in symbol $x \lhd y$, iff for all subsets A of L, the relation $y \leq \sup A$ always implies $x \in \downarrow A$. An element satisfying $x \lhd x$ is said to be *supercompact*. A subset P of a poset X is called *join-dense*, if for any $x \in X$ there exists a subset B of P such that $x = \bigvee B$. For a poset X, we always write $F \subseteq_\omega X$ to denote F is a non-empty finite subset of X. A *frame* is a complete lattice satisfying the distributive law of binary meets over arbitrary joins.

In [26], the author introduced the notion of convexity algebra as a pointfree convex space. A *convexity algebra* is a poset L which satisfies the following conditions:

(1) L has arbitrary meets;
(2) if $\{x_i \in L \mid i \in I\}$ is totally ordered in L, then $\{x_i \in L \mid i \in I\}$ has a join in L;
(3) for any doubly indexed family $\{x_{i,j} \mid i \in I, j \in J_i\}$, if $\{x_{i,j} \mid j \in J_i\}$ is totally ordered for every $i \in I$ and if $\left\{\bigwedge_{i\in I} x_{i,f(i)} \mid f \in F\right\}$ is totally ordered, then

$$\bigwedge_{i\in I}\bigvee_{j\in J_i} x_{i,j} = \bigvee_{f\in F}\bigwedge_{i\in I} x_{i,f(i)},$$

where $F = \prod_{i\in I} J_i \left(= \left\{f\colon I \longrightarrow \bigcup_{i\in I} J_i \mid \forall i \in I, f(i) \in J_i\right\}\right)$.

As a matter of fact, one can see that the notions of continuous lattice and convexity algebra are the same concepts (see [25] or [26]).

Let $f\colon P \longrightarrow Q$ and $g\colon Q \longrightarrow P$ be two monotone maps between posets. Then f is called a *left adjoint* of g (g is an *right adjoint* of f) if $f(x) \leq y \Leftrightarrow x \leq g(y)$ holds for all $x \in P$ and $y \in Q$. Then (f, g) is an adjunction iff $f \circ g \leq id_Q$ and $g \circ f \geq id_P$. If P and Q are complete lattices, then f has a right adjoint iff f preserves arbitrary joins and f has a left adjoint iff f preserves arbitrary meets. (see [11,14])

Definition 2.1 ([11,14]). Let L be a complete lattice. A *kernel operator* on L is a map $k\colon L \longrightarrow L$ satisfying the following conditions:

(1) k is monotone, i.e. $x \leq y$ implies $k(x) \leq k(y)$ for any $x \in L$;

(2) k is a contraction, i.e. $k(x) \leq x$ for any $x \in L$;
(3) k is idempotent, i.e. $k \circ k = k$.

We call a kernel operator k continuous if k is also Scott continuous.

Definition 2.2 ([16]). A subset Q of a continuous lattice L is called a *quotient* of L, if it satisfies the following conditions:

(1) Q is closed under arbitrary joins, that is, for any $M \subseteq Q$, $\bigvee M \in Q$.
(2) $x \ll_Q y$ implies $x \ll_L y$ for any $x, y \in Q$.

Lemma 2.3 ([11,14]). *Let L be a continuous lattice and k be a kernel operator on L. Then the following conditions are equivalent:*

(1) *$k(L)$ is a continuous lattice and $x \ll_{k(L)} y$ iff $x \ll_L y$ for any $x, y \in k(L)$.*
(2) *k is continuous.*

Let $Ker(L)$ denote the set of all continuous kernel operators on L and $\mathbf{Q}(L)$ denote the set of all quotients of L. It can be checked that $K(Q) = K(L) \cap Q$ for any $Q \in \mathbf{Q}(L)$ and the image $k(L)$ of a kernel operator k is a complete lattice in its own right with $\sup_{k(L)} X = \sup_L X$ for any $X \subseteq k(L)$. Then by Lemma 2.3, we have that $k(L) \in \mathbf{Q}(L)$ if $k \in Ker(L)$. For any $Q \in \mathbf{Q}(L)$, one can check that the map $k\colon L \longrightarrow L$ defined by $k(x) = \Downarrow x \cap Q$ is a continuous kernel operator. Furthermore, it follows that $Ker(L)$ and $\mathbf{Q}(L)$ are one-to-one corresponding.

Definition 2.4 ([33]). Let X be a set. A subfamily $\mathfrak{C}$ of 2^X is called a *convex structure* on X, if it satisfies the following conditions:

(1) $\emptyset, X \in \mathfrak{C}$;
(2) For any $\{A_i\}_{i\in I} \subseteq \mathfrak{C}$, $\bigcap_{i\in I} A_i \in \mathfrak{C}$;
(3) For any directed family $\{D_i\}_{i\in I} \subseteq \mathfrak{C}$, $\bigcup_{i\in I} D_i \in \mathfrak{C}$.

We call the pair $(X, \mathfrak{C})$, or simply X, a *convex space*, and every element in $\mathfrak{C}$ a *convex set*. We shall always denote by $\mathfrak{C}(X)$ the set of all convex sets of X.

Let $(X, \mathfrak{C})$ be a convex space. For any subset A of X, the *hull* $co_X(A)$ of A is defined as

$$co_X(A) = \bigcap\{B \in \mathfrak{C}(X) \mid A \subseteq B\}$$

The operator co_X is called the hull on X. It is obvious that every subset A of X is non-empty iff $co_X(A)$ is non-empty. A non-empty convex set is called a *polytope* if it is the hull of a finite subset of X. The collection of all the polytopes of X is denoted by $\mathbf{Po}(X)$. For convenience, we always write $co_X(x)$ for $co_X(\{x\})$ for any $x \in X$. One can easily check that $co_X(A) = \bigcup_{F\subseteq_\omega A} co_X(F)$ for any $A \subseteq X$. For a subset Y of X, it can be verified that the family $\mathfrak{C}(Y) = \{C \cap Y \mid C \in \mathfrak{C}(X)\}$ is a convex structure on Y. Then $(Y, \mathfrak{C}(Y))$ is called a *convex subspace* of X. A convex space X is called S_0, if $co_X(x) = co_X(y)$ implies $x = y$ for any $x, y \in X$. A convex space X is called S_1, if all singlelons in X are convex.

Let $f\colon X \longrightarrow Y$ be a map between convex spaces. Then f is called *convexity-preserving* (CP for short), if for any $C \in \mathfrak{C}(Y)$, $f^{-1}(C) \in \mathfrak{C}(X)$ and

f is called *convex-to-convex* (CC for short), if for any $C \in \mathfrak{C}(X)$, $f(C) \in \mathfrak{C}(Y)$. Furthermore, f is called a *convex-embedding* if f is injective, CP and CC from X to the convex subspace $f(X)$ of Y. A convex-embedding f is called a *convex-homeomorphism* if f is also surjective. We say that two convex space X and Y are *convex-homeomorphic* if there exists a convex-homeomorphism between X and Y, denoted by $X \cong Y$. One can easily check that f is CP iff $f(co_X(A)) \subseteq co_Y(f(A)))$ for any $A \subseteq X$ iff $f(co_X(F)) \subseteq co_Y(f(F))$ for any $F \subseteq_\omega X$.

Definition 2.5 ([26,30]). A convex space X is called *sober*, if every polytope is the hull of a unique singleton.

Definition 2.6 ([30]). A *sobrification* of a convex space X is a sober convex space Y together with a CP map $\eta_X : X \longrightarrow Y$, such that for any CP map $f : X \longrightarrow Z$ into a sober convex space Z, there exists a unique CP map $\hat{f} : Y \longrightarrow Z$ satisfying $f = \hat{f} \circ \eta_X$.

Theorem 2.7 ([30]). *For a convex space X, the space $\mathbf{cpt}(\mathfrak{C}(X))$ with the map $\eta_X : X \longrightarrow \mathbf{cpt}(\mathfrak{C}(X))$ defined by $\eta_X(x) = co_X(x)$ is a sobrification of X.*

Lemma 2.8 ([30]). *The sobrification of a convex space is unique up to convex-homeomorphism.*

Let $\mathbf{CS}$, $\mathbf{CS}_0$ and $\mathbf{Sob}$ denote the categories of convex spaces, S_0-convex spaces and sober spaces with CP-maps, respectively.

For a complete lattice L, let $\mathbf{cpt}(L)$ be the set of all non-bottom compact elements of L and $\mathfrak{C}(\mathbf{cpt}(L)) = \{K_a \mid K_a = \downarrow a \cap \mathbf{cpt}(L), a \in L\}$. It was shown in [33] that $(\mathbf{cpt}(L), \mathfrak{C})$ is a convex space. It should be noted that the requirement "$\bot_L \notin \mathbf{cpt}(L)$" is needed, because only in this way can we guarantee that $\emptyset = K_{\bot_L}$ is convex in $\mathbf{cpt}(L)$.

Definition 2.9 ([30]). A convex space X is called S_D, if for every $x \in X$, $co_X(x) \backslash \{x\}$ is convex.

In $\mathbf{CS}$, the relationships among sober, S_0, S_1 and S_D are depicted as in Figure 1

The implications in Figure 1 except $S_D \nRightarrow S_1$ were given in [30] and we shall give an explanation as below.

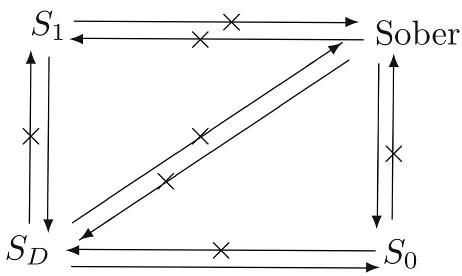

FIGURE 1. An illustration of the relationships

$S_D \not\Rightarrow S_1$: Let $X = \{x, y\}$ and $\mathfrak{C}(X) = \{\emptyset, \{x\}, X\}$. It is clear that $(X, \mathfrak{C})$ is a convex space. Since $co_X(x) = \{x\}$ and $co_X(y) = X$, it follows that X is S_D, but not S_1.

In the following, we gives two typical examples of S_D-convex spaces.

Example 2.10. (1) It is readily verified that every poset P endowed with the collection of all the lower subsets of P is an S_D-convex space, which is called the *Alexandroff convex space* on P.

(2) Every finite T_0-topological space is an S_D-convex space. Indeed, one can easily see that the finite convex spaces are exactly the finite closure spaces. Then the result follows by the fact in [4] that every finite T_0-topological space is T_D.

Let **CLat** denote the category of continuous lattices and continuous homomorphisms, and **AgLat** (**DAgLat**) denote the full subcategory of **CLat** consisting of algebraic lattices (distributive algebraic lattices). In [26] and [30], it was proved that the contravariant functor $\mathfrak{C}\colon \mathbf{CS} \longrightarrow \mathbf{CLat}$ is left adjoint to the contravariant functor $\mathbf{cpt}\colon \mathbf{CLat} \longrightarrow \mathbf{CS}$, where $\mathfrak{C}$ is defined by mapping every convex space X to $\mathfrak{C}(X)$ and every CP map $f\colon X \longrightarrow Y$ to $f^{-1}\colon \mathfrak{C}(Y) \longrightarrow \mathfrak{C}(X)$, and **cpt** is defined by mapping every continuous lattice L to $\mathbf{cpt}(L)$, and every continuous homomorphism $g\colon M \longrightarrow L$ to the restriction of the left adjoint of g to $\mathbf{cpt}(L) \longrightarrow \mathbf{cpt}(M)$. Furthermore, this adjunction can induce an dual equivalence between **Sob** and **AgLat**. More specifically, let us recall the following lemma (see [26,30]).

Lemma 2.11. (1) *Let L be a continuous lattice. Then $\mathbf{cpt}(L)$ is sober. Furthermore, L is algebraic iff the map $\delta_L\colon L \longrightarrow \mathfrak{C}(\mathbf{cpt}(L))$, defined by $\delta_L(a) = K_a$, is an order isomorphism.*

(2) *Let X be a convex space. Then $\mathfrak{C}(X)$ is an algebraic lattice. Furthermore, X is sober iff the map $\eta_X\colon X \longrightarrow \mathbf{cpt}(\mathfrak{C}(X))$, defined by $\eta_X(x) = co_X(x)$, is a convex-homeomorphism.*

3. Stable convex spaces and sober convex spaces

Let X be a convex space. The *specialization preorder* $\leq$ on X is defined by $x \leq y$ iff $x \in co_X(y)$, or alternatively $x \leq y$ iff $y \in C$ implies $x \in C$ for all $C \in \mathfrak{C}(X)$. The *specialization preorder* is a partial order iff X is an S_0-convex space. For an S_0-convex space X, one can easily verify that every convex set in $\mathfrak{C}(X)$ is always a lower subset of X with respect to the specialization order. For an S_1-convex space X, it is obvious that the specialization order reduces to the trivial partial order.

A convex space X is called *stable*, if for any $co_X(F), co_X(G) \in \mathbf{Po}(X)$ with $x \in co_X(F) \cap co_X(G)$, there exists a non-empty finite subset H of $\downarrow F \cap \downarrow G$ such that $x \in co_X(H)$, where $\downarrow F$ and $\downarrow G$ are lower subsets of X with respect to the specialization preorder.

Lemma 3.1. *Let X be a stable convex space. Then $\mathfrak{C}(X)$ is a distributive algebraic lattice under the inclusion order.*

Proof. It is obvious that $\mathfrak{C}(X)$ is an algebraic lattice. For the distributivity, we only need to show that $A \cap \bigvee_{i\in I} B_i \subseteq \bigvee_{i\in I}(A \cap B_i)$ for any $A \in \mathfrak{C}(X)$ and $\{B_i\}_{i\in I} \subseteq \mathfrak{C}(X)$, because the reverse inclusion is trivial. For any $x \in A \cap \bigvee_{i\in I} B_i$, there exist $F \subseteq_\omega A$ and $G \subseteq_\omega \bigcup_{i\in I} B_i$ such that $x \in co_X(F) \cap co_X(G)$ and so there exists a non-empty finite subset H of $\downarrow F \cap \downarrow G$ such that $x \in co_X(H)$. Since $H \subseteq_\omega \downarrow F \cap \downarrow G \subseteq \bigcup_{i\in I}(A \cap B_i)$ and $x \in co_X(H)$, we have that $x \in \bigvee_{i\in I}(A \cap B_i)$, which implies that $A \cap \bigvee_{i\in I} B_i \subseteq \bigvee_{i\in I}(A \cap B_i)$. □

The following theorem allows us to represent every distributive algebraic lattice in the form $\mathfrak{C}(X)$ for some stable sober convex space X.

Lemma 3.2. *Let L be a distributive algebraic lattice. Then* $\mathbf{cpt}(L)$ *is a stable sober convex space, and the map* $\delta_L : L \longrightarrow \mathfrak{C}(\mathbf{cpt}(L))$ *is an order isomorphism.*

Proof. By Lemma 2.11, we have that $\mathbf{cpt}(L)$ is sober and $\delta_L : L \longrightarrow \mathfrak{C}(\mathbf{cpt}(L))$ is an order isomorphism. Then we have that $\mathbf{Po}(\mathbf{cpt}(L)) = \{co_{\mathbf{cpt}(L)}(x) \mid x \in \mathbf{cpt}(L)\}$. It is clear that $co_{\mathbf{cpt}(L)}(x) = \downarrow_c x$ for any $x \in \mathbf{cpt}(L)$, where $\downarrow_c x$ denote the principle ideal in the subposet $\mathbf{cpt}(L)$, which means that the specialization order of the convex space $\mathbf{cpt}(L)$ and the partial order of the subposet $\mathbf{cpt}(L)$ coincide. Then one can easily check that X is stable. □

Proposition 3.3. *Let X be a sober convex space. Then X is stable iff $\mathfrak{C}(X)$ is a distributive algebraic lattice.*

Proof. The necessity follows by Lemma 3.1. Conversely, we let X be sober. Then Lemma 2.11 gives that X is convex-homeomorphic to $\mathbf{cpt}(\mathfrak{C}(X))$. Since $\mathfrak{C}(X)$ is a distributive algebraic lattice, $\mathbf{cpt}(\mathfrak{C}(X))$ is stable by Lemma 3.2 and hence X is stable. □

Theorem 3.4 ([30]). *The adjunction $\mathfrak{C} \dashv$* $\mathbf{cpt}$ *restricts to an equivalence between* **Sob** *and* **AgLat**.

Let **SSob** denote the full subcategory of **Sob** consisting of all stable sober convex spaces. Combining with Proposition 3.3 and Theorem 3.4, we can now obtain a Hofmann–Lawson-like duality for pointfree convex spaces.

Theorem 3.5. *The categories* **SSob** *and* **DAgLat** *are dually equivalent under the restrictions of the functors $\mathfrak{C}$ and* $\mathbf{cpt}$.

Now, we begin to consider the question mentioned in the introduction for which class of morphisms $\mathcal{M}$ such that the $\mathcal{M}$-injective objects in $\mathbf{CS}_0$ are exactly sober convex spaces. First, we introduce the notion of quasihomeomorphism on convex spaces as below.

Definition 3.6. A map $f : X \longrightarrow Y$ between convex spaces is called a *quasihomeomorphism* if the map $\mathfrak{C}(f) : \mathfrak{C}(Y) \longrightarrow \mathfrak{C}(X)$ is bijective and hence a lattice isomorphism. Furthermore, we call a map $e : X \longrightarrow Y$ of convex spaces a *strict embedding* if it is both a quasihomeomorphism and a convex-embedding.

Remark 3.7. The sobrification $(\eta_X, \mathbf{cpt}(\mathfrak{C}(X)))$ of a convex space X is a strict embedding if X is S_0, because $\mathfrak{C} \dashv \mathbf{cpt}$ is an adjunction.

Proposition 3.8. *Let $\phi\colon X \longrightarrow Y$ be a map between S_0-convex spaces. Then (ϕ, Y) is a sobrification of X iff Y is a sober convex space and ϕ is a strict embedding.*

Proof. The necessity follows by Lemma 2.8 and Remark 3.7. Conversely, we let $\phi\colon X \longrightarrow Y$ be a strict embedding. Then the map $\mathfrak{C}(\phi)\colon \mathfrak{C}(Y) \longrightarrow \mathfrak{C}(X)$ is a lattice isomorphism and hence $\mathbf{cpt} \circ \mathfrak{C}(\phi)\colon \mathbf{cpt}(\mathfrak{C}(X)) \longrightarrow \mathbf{cpt}(\mathfrak{C}(Y))$ is a convex-homeomorphism. Since Y is sober, Lemma 2.11 gives that the map $\eta_Y : Y \longrightarrow \mathbf{cpt}(\mathfrak{C}(Y))$ is a convex-homeomorphism. Furthermore, it is clear that $X \stackrel{\phi}{\longrightarrow} Y = X \stackrel{\eta_X}{\longrightarrow} \mathbf{cpt}(\mathfrak{C}(X)) \stackrel{\mathbf{cpt}\circ\mathfrak{C}(f)}{\longrightarrow} \mathbf{cpt}(\mathfrak{C}(Y)) \stackrel{\eta_Y^{-1}}{\longrightarrow} Y$. Since $\mathbf{cpt} \circ \mathfrak{C}(\phi)$, η_Y are convex-homeomorphisms and $(\eta_X, \mathbf{cpt}(\mathfrak{C}(X)))$ is a sobrification of X, it follows by Lemma 2.8 that (ϕ, Y) is a sobrification of X. □

Recall that a morphism $m\colon X \longrightarrow Y$ in a category $\mathcal{C}$ is called an *extremal monomorphism* if it is a monomorphism and if $m = f \circ e$ with e an epimorphism, then e must be an isomorphism. Similar to the cases in the category of topological spaces, it can be easily checked that the (monomorphisms) epimorphisms are precisely (injections) surjections in **CS**.

Let $\mathcal{C}$ be a category and let $\mathcal{M}$ be a class of morphisms in $\mathcal{C}$ including all convex-homeomorphisms and is contained in the class of monomorphisms. Recall that an object S in $\mathcal{C}$ is called $\mathcal{M}$-*injective* in $\mathcal{C}$ provided that for any morphism $h\colon A \longrightarrow B$ in $\mathcal{M}$ and for any morphism $f\colon A \longrightarrow S$ in $\mathcal{C}$ there exists a morphism $g\colon B \longrightarrow S$ in $\mathcal{C}$ such that $g \circ h = f$. In [18] and [22], the $\mathcal{N}$-injectives for the class $\mathcal{N}$ of all convex-embeddings are considered in $\mathbf{CS}_0$ and it was shown that the $\mathcal{N}$-injective objects in $\mathbf{CS}_0$ are exactly frames endowed with the filter convex structures. Similar to the case in the category of topological spaces, we find that the convex-embeddings in **CS** can be essentially viewed as the extremal monomorphisms.

Proposition 3.9. *In* **CS***, a CP map $m\colon X \longrightarrow Y$ is an extremal monomorphism iff it is a convex-embedding.*

Proof. Let $m\colon X \longrightarrow Y$ be a CP map, and $m = f \circ e$ with $e\colon X \longrightarrow Z$ an epimorphism. Then we have to check that e is a convex-homeomorphism. First, e is obviously an injection, because m is. Since e is an epimorphism, we have that e is a surjection and so a bijection. Since $f = m \circ e^{-1}$ is injective, we have that $f^{-1}(f(S)) = S$ for any $S \subseteq e(X)$ and so $f^{-1}(m(X)) = f^{-1}(f(e(X))) = e(X)$. Let C be an arbitrary convex set of X. Then $m(C) = D \cap m(X)$ for some $D \in \mathfrak{C}(Y)$. It follows that $e(C) = f^{-1}(f(e(C))) = f^{-1}(m(C)) = f^{-1}(D \cap m(X)) = f^{-1}(D) \cap f^{-1}(m(X)) = f^{-1}(D) \cap e(X) = f^{-1}(D)$, which implies that e is a CC map in Z. Thus, e is a convex-homeomorphism. Conversely, if $m\colon X \longrightarrow Y$ is an extremal monomorphism, then m is injective. Let $m = i \circ m|^{m(X)}$, where $i\colon m(X) \longrightarrow Y$ is the identical convex-embedding. Since $m|^{m(X)}$ is an epimorphism, it follows by the assumption that $m|^{m(X)}$ is a convex-homeomorphism. For any $C \in \mathfrak{C}(X)$, we have that $m(C) = m|^{m(X)}(C)$ is a convex set in the convex subspace $m(X)$. Thus, m is a convex-embedding. □

In the following, we shall choose the strict convex-embeddings as the members of $\mathcal{M}$ to study the injectives in $\mathbf{CS}_0$. Obviously, $\mathcal{M}$ is a class of morphisms including all convex-homeomorphisms and is contained in the class of monomorphisms.

Proposition 3.10. *Let Z be a sober convex space. Then Z is $\mathcal{M}$-injective in $\mathbf{CS}_0$.*

Proof. Let $\eta\colon X \longrightarrow Y$ be a morphism in $\mathcal{M}$ and $(i_Y, \hat{Y})$ be a sobrification of Y. By Proposition 3.8, $i_Y \circ \eta \in \mathcal{M}$ and then $(i_Y \circ \eta, \hat{Y})$ is a sobrification of X. So for any CP map $f\colon X \longrightarrow Z$, there exists a unique CP map $\bar{f}\colon \hat{Y} \longrightarrow Z$ such that $\bar{f} \circ i_Y \circ \eta = f$. Let $g = \bar{f} \circ i_Y$. Then $g \circ \eta = f$ and g is a CP map, which implies that Z is $\mathcal{M}$-injective in $\mathbf{CS}_0$. □

Lemma 3.11. *In $\mathbf{CS}_0$, every retract of a sober convex space is sober.*

Proof. Let X be a sober convex space and Y be a retract of X in $\mathbf{CS}_0$. Then there exist two CP maps $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow X$ such that $f \circ g = id_Y$. For any non-empty finite subset F of Y, $g(F)$ is clearly a non-empty finite subset of X. Since X is sober, we have that $co_X(g(F)) = co_X(x)$ for a unique element x of X. Then it follows that $co_Y(F) = co_Y(f(g(F))) = co_Y(f(co_X(g(F)))) = co_Y(f(co_X(x))) = co_Y(f(x))$, as desired. □

Proposition 3.12. *Let X be an S_0-convex space. If X is $\mathcal{M}$-injective in $\mathbf{CS}_0$, then X is sober.*

Proof. Let X be an S_0-convex space and $(i_X, \hat{X})$ be a sobrification of X. Proposition 3.8 gives that $i_X \in \mathcal{M}$. Since X is $\mathcal{M}$-injective in $\mathbf{CS}_0$, there exists a CP map $f\colon \hat{X} \longrightarrow X$ such that $f \circ i_X = id_X$, that is, X is a retract of $\hat{X}$. Thus, it follows by Lemma 3.11 that X is sober. □

Combining with Proposition 3.10 and Proposition 3.12, we obtain the following theorem.

Theorem 3.13. *Let X be an S_0-convex space. Then X is $\mathcal{M}$-injective in $\mathbf{CS}_0$ iff X is a sober convex space.*

Lemma 3.14. *Let $f\colon X \longrightarrow Y$ be a CP map between S_0-convex spaces. Then f is a convex-embedding iff $f^{-1}\colon \mathfrak{C}(Y) \longrightarrow \mathfrak{C}(X)$ is surjective.*

Proof. Let f be a convex-embedding. Then it is clear that $f^{-1}(co_Y(f(C))) = C$ for any $C \in \mathfrak{C}(X)$ and hence f^{-1} is surjective. Conversely, since f^{-1} preserves arbitrary meets, f^{-1} has a left adjoint G and one can easily see that $G\colon \mathfrak{C}(X) \longrightarrow \mathfrak{C}(Y)$ is defined by $G(C) = co_Y(f(C))$ for any $C \in \mathfrak{C}(X)$. Since f^{-1} is surjective, it follows that $f^{-1}(co_Y(f(C))) = C$ for any $C \in \mathfrak{C}(X)$. Let $f(x_1) = f(x_2)$ with $x_1, x_2 \in X$. Then $co_X(x_1) = f^{-1}(co_Y(\{f(co_X(x_1))\})) = f^{-1}(co_Y(f(x_1))) = f^{-1}(co_Y(f(x_2))) = f^{-1}(co_Y(\{f(co_X(x_2))\})) = co_X(x_2)$ and hence $x_1 = x_2$, which implies that f is injective. Let $C \in \mathfrak{C}(X)$. It is routine to check that $f(C) = G(C) \cap f(X)$ and hence f is CC in $f(X)$. Thus, f is a convex-embedding. □

Lemma 3.15. *Let L be an algebraic lattice and $M \subseteq \mathbf{cpt}(L)$. Then the identical convex-embedding $i\colon M \longrightarrow \mathbf{cpt}(L)$ is a strict embedding iff M is join-dense in L.*

Proof. By Lemma 3.14, one can see that the identical convex-embedding $i\colon M \longrightarrow \mathbf{cpt}(L)$ is a strict embedding iff the map $i^{-1}\colon \mathfrak{C}(\mathbf{cpt}(L)) \longrightarrow \mathfrak{C}(M)$ is injective. If M is join-dense in L and $i^{-1}(K_x) = i^{-1}(K_y)$, then $K_x \cap M = K_y \cap M$ and hence $x = \bigvee(K_x \cap M) = \bigvee(K_y \cap M) = y$. This implies that i is a strict embedding. Conversely, it is clear that $i^{-1}(K_x) = K_x \cap M = K_{\bar{x}} \cap M = i^{-1}(K_{\bar{x}})$ and so $K_x = K_{\bar{x}}$, where $\bar{x} = \bigvee(K_x \cap M)$. Since $\mathbf{cpt}(L)$ is join-dense in L, it follows that $x = \bigvee(K_x \cap M)$, which implies that M is join-dense in L. □

Theorem 3.16. *For an S_0-convex space X, the following statements are equivalent:*

(1) *The sobrification of X is stable;*
(2) *X is convex-homeomorphic to a join-dense convex subspace of $\mathbf{cpt}(L)$ for some distributive algebraic lattice L;*
(3) *X allows a strict embedding into a stable sober convex space;*
(4) *X allows a strict embedding into a stable convex space;*
(5) *$\mathfrak{C}(X)$ is a distributive algebraic lattice.*

Proof. (1) ⇒ (2). Let $(\phi, \hat{X})$ be a sobrification of X. By Proposition 3.8, we have that ϕ is a strict embedding. By Theorem 2.7 and Lemma 2.8, we have that $\mathbf{cpt}(\mathfrak{C}(X))$ is a sobrification of X and the sobrification of X is unique up to isomorphism. Then there exists a convex-homeomorphism $j\colon \hat{X} \longrightarrow \mathbf{cpt}(\mathfrak{C}(X))$. Let $L = \mathfrak{C}(X)$. Since the map $j \circ \phi\colon X \longrightarrow \mathbf{cpt}(L)$ is a strict embedding, it follows by Lemma 3.1 and Lemma 3.15 that L is a distributive algebraic lattice and $j(\phi(X))$ is join-dense in L, respectively.

(2) ⇒ (3). It is immediate by Lemma 3.2 and Lemma 3.15.

(3) ⇒ (4). It is clear.

(4) ⇒ (5). It is immediate by Lemma 3.1 and Definition 3.6.

(5) ⇒ (1). Since $\mathfrak{C}(X)$ is a distributive algebraic lattice, Lemma 3.2 gives that the sobrification $\mathbf{cpt}(\mathfrak{C}(X))$ of X is stable. Then it follows by Lemma 2.8 that the sobrification of X is stable. □

4. Several characterizations of sober convex spaces and S_D-convex spaces

An element x of a complete lattice L is called *strongly irreducible* if for any $S \subseteq L$, $x = \bigvee S$ implies $x = y$ for some $y \in S$. It should be noted that every strongly irreducible element in a continuous lattice L must be a compact element. Indeed: let x be a strongly irreducible element of L. For every directed subset D of L with $x \leq \bigvee D$, we have that $x = \bigvee(x \wedge D)$. Then $x = x \wedge d$ for some $d \in D$ and so $x \leq d$. This implies that x is a compact element of L. Similarly, one can easily check that strongly irreducible elements and supercompact elements on a frame coincide. For a continuous lattice L, we call a nonempty Scott closed subset G of L *irreducible* if $a \wedge b \in G \Rightarrow a \in G$ or $b \in G$, and *completely irreducible* if for any nonempty subset A of L, $\bigwedge A \in G \Rightarrow A \cap G \neq \emptyset$. For a convex space $(X, \mathfrak{C})$, we shall use the notation $M(x) = \{C \in \mathfrak{C} \mid x \notin C\}$ for the *fixed Scott closed subset* of $\mathfrak{C}(X)$. One can easily check that every $M(x)$ is a completely irreducible Scott closed subset of $\mathfrak{C}(X)$. But the converse is not true generally.

Proposition 4.1. *Let L be a continuous lattice. Then the completely irreducible Scott closed subsets of L are completely determined by the compact elements in* $\mathbf{cpt}(L)$.

Proof. For any completely irreducible Scott closed subset G of L, we will verify that $k_G = \bigwedge\{a \in L \mid a \notin G\}$ is a compact element in $\mathbf{cpt}(L)$. First, $k_G \neq \perp_L$ as G is a nonempty Scott closed subset of L. Let $k_G \leq \bigvee D$ with D a directed subset of L. Suppose that $k_G \nleq d$ for any $d \in D$, then $d \in G$. So $D \subseteq G$ and hence $\bigvee D \in G$. Since G is a lower set, we have that $k_G \in G$ and so there exists an element $a \notin G$ such that $a \in G$, a contradiction. Thus, k_G is a compact element in $\mathbf{cpt}(L)$. Conversely, for any compact element k in $\mathbf{cpt}(L)$, we need to check that $G_k = \{a \in L \mid k \nleq a\}$ is a completely irreducible Scott closed subset of L. One can easily see that G_k is a nonempty Scott closed subset of L. If $\bigwedge A \in G_k$ for any $A \subseteq L$, then $k \nleq \bigwedge A$ and so $k \nleq a$ for some $a \in A$. Hence there exists an element $a \in A$ such that $a \in G_k$. Furthermore, we have that $k_{G_k} = \bigwedge\{x \in L \mid x \notin G_k\} = \bigwedge\{x \in L \mid k \leq x\} = k$. Finally, it remains to show that $G_{k_G} = G$, that is, $\{x \in L \mid \bigwedge\{a \in L \mid a \notin G\} \nleq x\} = G$. For any $x \in G$, if $\bigwedge\{a \in L \mid a \notin G\} \leq x$, then $\bigwedge\{a \in L \mid a \notin G\} \in G$ and so there exists an element $a \notin G$ such that $a \in G$, a contradiction. Thus, $x \in G_{k_G}$. For the reverse inclusion, if $x \notin G$, then $\bigwedge\{a \in L \mid a \notin G\} \leq x$ and so $x \notin G_{k_G}$. □

The proof of the following Proposition is in fact included in Theorem 4.21 in [30], we deduce it here for the completeness of this paper.

Proposition 4.2. *Let X be an S_0-convex space. Then X is an S_D-convex space iff every $co_X(x)$ of $\mathfrak{C}(X)$ is strongly irreducible.*

Proof. Let $co_X(x) = \bigvee_{i \in I} A_i$ for some family $\{A_i\}_{i \in I}$ of $\mathfrak{C}(X)$. Then $A_i \subseteq co_X(x)$ for any $i \in I$. Suppose that $x \notin A_i$ for any $i \in I$. Then $A_i \subseteq co_X(x) \backslash \{x\}$ for any $i \in I$. Since X is S_D, it follows that $co_X(x) \subseteq co_X(x) \backslash \{x\}$, a contradiction. Thus, $x \in A_{i_0}$ for some $i_0 \in I$, which implies that $co_X(x)$ is strongly irreducible. Conversely, assume that there is an element $x \in X$ such that $co_X(x) \backslash \{x\}$ is not convex, it follows that $co_X(x) \backslash \{x\} \subset co_X(co_X(x) \backslash \{x\}) \subseteq co_X(x)$. Then $co_X(co_X(x) \backslash \{x\}) = co_X(x)$ and so $co_X(x) = \bigvee\{co_X(y) \mid y \in co_X(x) \backslash \{x\}\}$. Since $co_X(x)$ is strongly irreducible, we have that $co_X(x) = co_X(y)$ for some $y \in co_X(x) \backslash \{x\}$, a contradiction, because X is S_0. □

Remark 4.3. (1) For a given convex space X, it is clear that every strongly irreducible convex set in $\mathfrak{C}(X)$ is of the form $co_X(x)$ for some $x \in X$. Thus, by Proposition 4.2, the strongly irreducible convex sets of an S_D-convex space X consists of all $co_X(x)$ for every $x \in X$.

(2) It should be noted that the associated compact elements of a completely irreducible Scott closed subset of L are not always strongly irreducible. For instance, for a non-S_D-convex space X, by Proposition 4.2, there exists a completely irreducible Scott subset $M(x)$ of $\mathfrak{C}(X)$, but its associated compact element $co_X(x)$ is not strongly irreducible.

We call a Scott closed subset G of a continuous lattice L *radical* if it is completely irreducible and the associated compact element of G is strongly irreducible.

Lemma 4.4. *Let $(X, \mathfrak{C})$ be an S_0-convex space. Then each radical Scott closed subset of $\mathfrak{C}(X)$ is fixed.*

Proof. Let $\mathfrak{A}$ be a radical Scott closed subset of $\mathfrak{C}(X)$. Then the associated compact element $\bigcap\{C \in \mathfrak{C}(X) \mid C \notin \mathfrak{A}\}$ is strongly irreducible in $\mathfrak{C}(X)$ and so it follows by Remark 4.3 that $\bigcap\{C \in \mathfrak{C}(X) \mid C \notin \mathfrak{A}\} = co_X(x)$ for some $x \in X$. We next show that $\mathfrak{A} = M(x)$. Let $C \in \mathfrak{A}$. If $C \notin M(x)$, then $\bigcap\{C \in \mathfrak{C}(X) \mid C \notin \mathfrak{A}\} = co_X(x) \subseteq C$ and so $\bigcap\{C \in \mathfrak{C}(X) \mid C \notin \mathfrak{A}\} \in \mathfrak{A}$. Then there exists $C \notin \mathfrak{A}$ such that $C \in \mathfrak{A}$, a contradiction. This implies that $\mathfrak{A} \subseteq M(x)$. For the reverse inclusion, if $C \notin \mathfrak{A}$, then $co_X(x) \subseteq C$ and so $C \notin M(x)$. Thus, $\mathfrak{A} = M(x)$. □

In the following, instead of polytopes and the strongly irreducible convex subsets, we shall use completely irreducible Scott closed subsets and radical Scott closed subsets to give the corresponding characterizations of sober convex spaces and S_D-convex spaces, respectively.

Proposition 4.5. (1) *An S_0-convex space X is sober iff every completely irreducible Scott closed subset of $\mathfrak{C}(X)$ is fixed.*

(2) *An S_0-convex space X is S_D iff every fixed Scott closed subset of $\mathfrak{C}(X)$ is radical.*

Proof. (1) Let X be a sober convex space and let $\mathfrak{F}$ be an arbitrary completely irreducible Scott closed subset of $\mathfrak{C}(X)$. Then it follows by Proposition 4.1 that $\mathfrak{F} = \{C \in \mathfrak{C}(X) \mid co_X(F) \nsubseteq C\}$ for some finite subset F of X. Since X is sober, we have that $co_X(F) = co_X(x)$ for some $x \in X$ and then $\mathfrak{F} = \{C \in \mathfrak{C}(X)) \mid co_X(x) \nsubseteq C\} = M(x)$. Hence, $\mathfrak{F}$ is fixed. Conversely, we let $co_X(F)$ be an arbitrary polytope of X. Then again by Proposition 4.1, the strongly irreducible Scott closed subset $\{C \in \mathfrak{C}(X) \mid co_X(F) \nsubseteq C\} = M(x)$ for some $x \in X$. Then $co_X(F) = co_X(x)$. The uniqueness follows by the fact that X is S_0.

(2) Let X be S_D. Since $k_{M(x)} = \bigcap\{C \in \mathfrak{C} \mid C \notin M(x)\} = co_X(x)$ for any $x \in X$, it follows by Proposition 4.2 that $co_X(x)$ is strongly irreducible and so $M(x)$ is radical. Conversely, since the associated compact element of each $M(x)$ is $co_X(x)$, it follows by the assumption that every $co_X(x)$ is strongly irreducible. Then again by Proposition 4.2, we have that X is S_D. □

Remark 4.6. For a continuous lattice L, we let $\Upsilon(L)$ the collection of all completely irreducible Scott closed subsets of L. It is routine to check that the sets of the form $\triangle(x) = \{P \in \Upsilon(L) \mid x \notin P\}$ for $x \in L$ form a convex structure on $\Upsilon(L)$. Furthermore, one can find that this definition is equivalent to the cases in [26] and [30].

Lemma 4.7. *Let X be a convex space, $x \in X$ and i be the identical convex-embedding $X\backslash\{x\} \hookrightarrow X$. Then i is a strict embedding iff $co_X(x)\backslash\{x\}$ is not convex.*

Proof. By Lemma 3.14, $\mathfrak{C}(i)$ is clearly surjective. Then it suffices to show that $\mathfrak{C}(i)$ is injective iff $co_X(x)\backslash\{x\}$ is not convex. Let $A\backslash\{x\} = B\backslash\{x\}$ with $A, B \in \mathfrak{C}(X)$. Case 1: Both $A\backslash\{x\}$ and $B\backslash\{x\}$ are convex. If $co_X(x)\backslash\{x\}$ is not convex, then $co_X(x)\backslash\{x\} \subset co_X(co_X(x)\backslash\{x\}) \subseteq co_X(x)$ and hence $co_X(co_X(x)\backslash\{x\}) = co_X(x)$. If $x \in A$, then $co_X(x) = co_X(co_X(x)\backslash\{x\}) \subseteq A\backslash\{x\}$ and hence $x \in A\backslash\{x\}$, a contradiction. Then $x \notin A$. Similarly, $x \notin B$ and hence $A = B$. Case 2: $A\backslash\{x\}$ and $B\backslash\{x\}$ are both convex. Since A and B are convex, we can directly get that $x \in A \cap B$ or $x \notin A \cup B$ and both implies that $A = B$. Thus, $\mathfrak{C}(i)$ is injective. Conversely, we let the map $\mathfrak{C}(i)$ be injective. Suppose that $co_X(x)\backslash\{x\}$ is convex, then it follows that $\mathfrak{C}(i)(co_X(x)) = co_X(x)\backslash\{x\} = (co_X(x)\backslash\{x\})\backslash\{x\} = \mathfrak{C}(i)(co_X(x)\backslash\{x\})$ but $co_X(x) \neq co_X(x)\backslash\{x\}$, a contradiction. □

Theorem 4.8. *Let X be a convex space. Then the following statements are equivalent:*

(1) *X is S_D.*
(2) *For any CP map $f\colon Y \longrightarrow X$ with Y a convex space, f is surjective iff $f^{-1}\colon \mathfrak{C}(X) \longrightarrow \mathfrak{C}(Y)$ is injective.*

Proof. (1) ⇒ (2). Suppose f^{-1} is injective but f is not surjective, then $f^{-1}(\{x\}) = \emptyset$ for some $x \in X$. Thus $f^{-1}(co_X(x)\backslash\{x\}) = f^{-1}(co_X(x))$ and so $co_X(x)\backslash\{x\} = co_X(x)$, a contradiction. The converse is obvious.

(2) ⇒ (1). Suppose that X is not S_D, then there exists an element $x \in X$ such that $co_X(x)\backslash\{x\}$ is not convex. Take $Y = X\backslash\{x\}$ and $i\colon Y \longrightarrow X$ be the identical convex-embedding. Since i is not surjective, it follows that $i^{-1}\colon \mathfrak{C}(X) \longrightarrow \mathfrak{C}(Y)$ is not injective and so there exist convex subsets C_1 and C_2 of $\mathfrak{C}(X)$ such that $C_1\backslash\{x\} = i^{-1}(C_1) = i^{-1}(C_2) = C_2\backslash\{x\}$ but $C_1 \neq C_2$. It is obvious that neither $x \in C_1 \cap C_2$ nor $x \notin C_1 \cup C_2$ is possible. Then without loss of generality, we let $x \in C_1$ but $x \notin C_2$. Since $C_1\backslash\{x\} = C_2\backslash\{x\}$ and $x \notin C_2$, we have that $C_2 = C_1\backslash\{x\}$. Then it follows by $x \in C_1$ that $co_X(x) \cap C_2 = co_X(x) \cap (C_1\backslash\{x\}) = co_X(x)\backslash\{x\}$. This implies that $co_X(x)\backslash\{x\}$ is convex, a contradiction. □

The following theorem mirror each other in that sober convex spaces are maximal in the same sense in which S_D-convex spaces are minimal.

Theorem 4.9. (1) *A convex space X is sober iff there does not exist a nontrivial identical convex-embedding $i\colon X \hookrightarrow Y$ which is strict.*
(2) *A convex space X is S_D iff there does not exist a nontrivial identical convex-embedding $i\colon Y \hookrightarrow X$ which is strict.*

Proof. (1) If X is not sober, then there is a sobrification Y of X with $i\colon X \hookrightarrow Y$ the nontrivial identical convex-embedding. Furthermore, Proposition 3.8 gives that i is a strict embedding. Conversely, let X be sober. Suppose that $i\colon X \hookrightarrow Y$ is a nontrivial identical convex-embedding which is strict, then it follows by Lemma 2.11 that $X \cong \mathbf{cpt}(\mathfrak{C}(X)) \cong \mathbf{cpt}(\mathfrak{C}(Y))$. By Theorem 2.7, we have that $\mathbf{cpt}(\mathfrak{C}(Y))$ is a sobrification of Y and hence Y can be embedded into X, a contradiction.

(2) Suppose that X is not S_D, then there exists an element $x \in X$ such that $co_X(x)\backslash\{x\}$ is not convex. By Lemma 4.7, the nontrivial identical convex-embedding $i\colon X\backslash\{x\} \hookrightarrow X$ is strict. Conversely, let X be S_D. Suppose that there exists a nontrivial identical convex-embedding $i\colon Y \hookrightarrow X$ which is strict. Take an arbitrary element $x \in X\backslash Y$. Let $j\colon Y \hookrightarrow X\backslash\{x\}$ and $k\colon X\backslash\{x\} \hookrightarrow X$ be the nontrivial identical convex-embeddings. Then $i = k \circ j$ and hence $\mathfrak{C}(i) = \mathfrak{C}(j) \circ \mathfrak{C}(k)$. Since $\mathfrak{C}(i)$ is injective, we have that $\mathfrak{C}(k)$ is injective. By Lemma 3.14, we have that $\mathfrak{C}(k)$ is surjective. Then $\mathfrak{C}(k)$ is an isomorphism and so k is a strict embedding. It follows by Lemma 4.7 that $co_X(x) \setminus \{x\}$ is not convex, a contradiction. □

5. An adjunction between continuous lattices and S_D-convex spaces

A map $f\colon L \longrightarrow M$ between continuous lattices is called a *continuous homomorphism* if $f(\bot_L) = \bot_M$, and preserves arbitrary meets and directed joins. A continuous homomorphism f is called *a continuous D-homomorphism* if the left adjoint f^* of f preserves the strongly irreducible elements. A continuous lattice L is called *S_D-algebraic* if $L \cong \mathfrak{C}(X)$ for some S_D-convex space X. A continuous lattice L is *strongly S_D-algebraic* if it is algebraic and every compact element in $\mathbf{cpt}(L)$ is strongly irreducible. We denote by $\mathbf{CS}_D$ the category of S_D-convex spaces with CP-maps and by $\mathbf{CLat}_D$ the category of continuous lattices with continuous D-homomorphisms. Let $\mathbf{AgLat}_D$ denote the category of S_D-algebraic lattices with continuous D-homomorphisms.

In this section, we shall devote to constructing an adjunction between $\mathbf{CLat}_D$ and $\mathbf{CS}_D$. Furthermore, this adjunction can induce a dual equivalence between $\mathbf{AgLat}_D$ and $\mathbf{CS}_D$.

For a continuous lattice L, we let $\Phi(L)$ denote the subspace of $\mathbf{cpt}(L)$ of strongly irreducible elements, with $\Phi_a = K_a \cap \Phi(L) = \{x \in \Phi(L) \mid x \leq a\}, a \in L$, as the convex sets. In particular, we also have that

$$\Phi_{\bot_L} = \emptyset, \Phi_{\top_L} = \Phi(L), \Phi_{\bigwedge_{i\in I} a_i} = \bigcap_{i\in I} \Phi_{a_i}, \Phi_{\bigvee D} = \bigcup_{d\in D} \Phi_d$$

for any $\{a_i\}_{i\in I} \subseteq L$ and for any directed subset D of L.

Lemma 5.1. *Let L be a continuous lattice. Then $\Phi(L)$ is an S_D-convex space.*

Proof. In order to show that $co_{\Phi(L)}(x)\backslash\{x\}$ is a convex set of $\Phi(L)$ for any $x \in \Phi(L)$, it suffices to check that $\{y \in \Phi(L) \mid y < x\} = \Phi_a$ with $a = \bigvee\{z \in \Phi(L) \mid z < x\}$. Let $y < x$ with $y \in \Phi(L)$. Then we have that $y \leq a$ and so

$y \in \Phi_a$. For the reverse inclusion, let $y \leq a$ with $y \in \Phi(L)$. Then $y \leq x$. If $y = x$, then $x = z_0$ for some $z_0 \in \Phi(L)$ with $z_0 < x$, a contradiction. So $y < x$ and hence $y \in \{y \in \Phi(L) \mid y < x\}$. Thus, $\Phi(L)$ is an S_D-convex space. □

Proposition 5.2. *Let L be a continuous lattice. Then the following statements are equivalent:*

(1) *L is S_D-algebraic;*
(2) *$a = \bigvee\{s \in \Phi(L) \mid s \leq a\}$ for any $a \in L$;*
(3) *For any $a, b \in L$, $a \not\leq b \Rightarrow s \leq a$ and $s \not\leq b$ for some $s \in \Phi(L)$;*
(4) *For any $a, b \in L$, $a \not\leq b \Rightarrow b \in G$ and $a \notin G$ for some radical Scott closed subset G of L.*

Proof. (1) $\Rightarrow$ (2). Without loss of generality, we let $L = \mathfrak{C}(X)$ with X an S_D-convex space. Then it is immediate by Proposition 4.2.

(2) $\Rightarrow$ (3). It is obvious.

(3) $\Rightarrow$ (4). One can easily check that $L\backslash \uparrow s$ is a radical Scott closed subset of L for any $s \in \Phi(L)$. Then if we let $G = L\backslash \uparrow s$, (4) is obvious.

(4) $\Rightarrow$ (1). We define a map $\lambda_L : L \longrightarrow \mathfrak{C}(\Phi(L))$ by $\lambda_L(a) = \Phi_a$. It is clear that λ_L is surjective and order-preserving. Let $\Phi_a = \Phi_b$ with $a \neq b$. Without loss of generality, we let $a \not\leq b$ and so there exists a radical Scott closed subset G of L such that $b \in G$ and $a \notin G$. Then the associated compact element k_G of G satisfies that $k_G \leq a$ and $k_G \not\leq b$. This is impossible, because $\Phi_a = \Phi_b$ and the associated compact element of a radical Scott closed subset is strongly irreducible. Then $a = b$ and so λ_L is injective. Thus, λ_L is an order isomorphism and it follows by Lemma 5.1 that L is S_D-algebraic. □

The correspondence $L \mapsto \Phi(L)$ clearly extend to a functor

$$\Phi : \mathbf{CLat}_D \longrightarrow \mathbf{CS}_D,$$

amounting to the restriction of the functor **cpt**. Then we have that $\Phi(h)(p) = h^*(p)$ and $(\Phi h)^{-1}(\Phi_a) = \Phi_{h(a)}$, where h^* is the left adjoint of h. This implies that Φh is continuous.

For an S_D-convex space X and a continuous lattice L, we define maps $\eta_X : X \longrightarrow \Phi(\mathfrak{C}(X))$ and $\delta_L : L \longrightarrow \mathfrak{C}(\Phi(L))$ by $\eta_X(x) = co_X(x)$ and $\delta_L(a) = \Phi_a$, respectively. By Proposition 4.2 and Lemma 5.1, we have that η_X and δ_L are well defined.

Lemma 5.3. *For an S_D-convex space X, η_X is a convex-homeomorphism.*

Proof. It is obvious that η_X is injective, because X is S_0. By Proposition 4.2 and Remark 4.3, we have that $\Phi(\mathfrak{C}(X)) = \{co_X(x) \mid x \in X\}$. Then η_X is surjective and $\Phi_C = \{co_X(x) \mid co_X(x) \subseteq C\}$ for any $C \in \mathfrak{C}(X)$. So $\eta_X^{-1}(\Phi_C) = \{x \in X \mid co_X(x) \subseteq C\} = \{x \in X \mid x \in C\} = C$, which implies that η_X is a CP map. For any $C \in \mathfrak{C}(X)$, we have that $\eta_X(C) = \{co_X(x) \mid x \in C\} = \{co_X(x) \mid co_X(x) \subseteq C\} = \Phi_C$, which implies that η_X is a CC map. Thus, η_X is a convex-homeomorphism. □

Lemma 5.4. *For a continuous lattice L, δ_L is a continuous D-homomorphism.*

Proof. It is clear that δ_L is a continuous homomorphism. It remains to show that the left adjoint $(\delta_L)^*$ of δ_L preserves the strongly irreducible elements. Since $\Phi(L)$ is S_D, it follows by Proposition 4.2 and Remark 4.3 that the strongly irreducible elements in $\mathfrak{C}(\Phi(L))$ are precisely $\{co_{\Phi(L)}(s) \in \mathfrak{C}(\Phi(L)) \mid s \in \Phi(L)\}$. For any $a \in L$ and $s \in \Phi(L)$, we have that $(\delta_L)^*(co_{\Phi(L)}(s)) \leq a$ iff $co_{\Phi(L)}(s) \subseteq \Phi_a$ iff $s \leq a$. Then $(\delta_L)^*(co_{\Phi(L)}(s)) = s$. Thus, δ_L is a continuous D-homomorphism. □

Lemma 5.5. *The maps η_X and δ_L constitute natural transformations $\eta\colon Id_{\mathbf{CS}_D} \longrightarrow \Phi \circ \mathfrak{C}$ and $\delta\colon Id_{\mathbf{CLat}_D} \longrightarrow \mathfrak{C} \circ \Phi$ respectively, where $Id_{\mathbf{CS}_D}$ and $Id_{\mathbf{CLat}_D}$ are the identical functors on $\mathbf{CS}_D$ and $\mathbf{CLat}_D$ respectively.*

Proof. Lemma 5.3 and Lemma 5.4 give that η_X is a CP map and δ_L is a continuous D-homomorphism, respectively. It remains to show the commutativity of the following diagrams.

$$\begin{array}{ccc} L & \xrightarrow{\delta_L} & \mathfrak{C}(\Phi(L)) \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle \mathfrak{C}(\Phi(\varphi))} \\ M & \xrightarrow[\delta_M]{} & \mathfrak{C}(\Phi(M)) \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{\eta_X} & \Phi(\mathfrak{C}(X)) \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \Phi(\mathfrak{C}(f))} \\ Y & \xrightarrow[\eta_Y]{} & \Phi(\mathfrak{C}(Y)) \end{array}$$

For any $a \in L$ and $x \in X$, we have that $\mathfrak{C}(\Phi(\varphi))(\delta_L(a)) = (\varphi^*)^{-1}(\Phi_a) = \{p \in \Phi(L) \mid \varphi^*(p) \in \Phi_a\} = \{p \in \Phi(L) \mid \varphi^*(p) \leq a\} = \{p \in \Phi(L) \mid p \leq \varphi(a)\} = \Phi_{\varphi(a)} = \delta_M(\varphi(a))$ and $\Phi(\mathfrak{C}(f))(\eta_X(x)) = (f^{-1})^*(co_X(x)) = \bigcap\{P \in \Phi(\mathfrak{C}(Y)) \mid co_X(x) \subseteq f^{-1}(P)\} = co_{\Phi(\mathfrak{C}(Y))}(f(x)) = \eta_Y(f(x))$, as desired. □

Proposition 5.6. *The contravariant functor Φ is the right adjoint to the contravariant functor $\mathfrak{C}$.*

Proof. In order to show the adjointness of the functors Φ and $\mathfrak{C}$, we need to show the commutativity of the following diagrams.

$$\begin{array}{ccc} \mathfrak{C}(X) & \xrightarrow{Id_{\mathfrak{C}(X)}} & \mathfrak{C}(X) \\ {\scriptstyle \delta_{\mathfrak{C}(X)}}\searrow & & \nearrow{\scriptstyle \mathfrak{C}(\eta_X)} \\ & \mathfrak{C}(\Phi\mathfrak{C}(X)) & \end{array} \qquad \begin{array}{ccc} \Phi(L) & \xrightarrow{Id_{\Phi(L)}} & \Phi(L) \\ {\scriptstyle \eta_{\Phi(L)}}\searrow & & \nearrow{\scriptstyle \Phi(\delta_L)} \\ & \Phi\mathfrak{C}(\Phi(L)) & \end{array}$$

For any $A \in \mathfrak{C}(X)$ and $p \in \Phi(L)$, we have that $\mathfrak{C}(\eta_X)(\delta_{\mathfrak{C}(X)}(A)) = \eta_X^{-1}(\Phi_A) = \{x \in X \mid co_X(x) \subseteq A\} = A$ and $\Phi(\delta_L)(\eta_{\Phi(L)}(p)) = \Phi(\delta_L)(co_{\Phi(L)}(p)) = (\delta_L)^*(co_{\Phi(L)}(p)) = p$, as desired. □

Combining with Proposition 5.2, Lemma 5.3, Lemma 5.4 and Proposition 5.6, we obtain the main result in this section as below.

Theorem 5.7. *The adjunction $\mathfrak{C} \dashv \Phi$ induces a dual equivalence between $\mathbf{CS}_D$ and $\mathbf{AgLat}_D$.*

6. The relationship between convex subspaces and quotients

In this section, we shall aim at investigating the sobrifications of the convex subspaces of a sober convex space X and the algebraizations of the quotients of a continuous lattice L, and further exploiting the categorical equivalence between sober convex spaces and algebraic lattices, which was established in [26] and [30], to show that the algebraic lattice of sober convex subspaces of $\mathbf{cpt}(L)$ and that of algebraic quotients of L are isomorphic. Furthermore, several necessary and sufficient conditions for all the convex subspace of X to be sober are given.

For a complete lattice L and a subset $Y \subseteq L$, we denote by $\mathcal{J}(Y)$ the collection of $\{\bigvee M \mid M \subseteq Y\}$.

Lemma 6.1 ([16]). *Let L be a continuous lattice. Then the following statements hold:*

(1) *The map $f \mapsto f(L)$ is an isomorphism of $Ker(L)$ onto $\mathbf{Q}(L)$.*
(2) *$Ker(L)$ is anti-isomorphic to the lattice $Cong^*(L)$ of all continuous lattice congruences of L (i.e. equivalence relations $R \subseteq L \times L$ which closed under arbitrary intersections and directed joins). The anti-isomorphism is given by $f \mapsto \{(x, y) \in L \times L \mid f(x) = f(y)\}$.*

Theorem 6.2 ([16]). *Let L be a continuous lattice. Then the following statements are equivalent:*

(1) *L is an algebraic lattice such that the set $K(L)$ does not contain any order dense chain;*
(2) *Every quotient of L is an algebraic lattice;*
(3) *$Ker(L)$ is a continuous lattice;*
(4) *$Ker(L)$ is an algebraic lattice.*

For a continuous lattice L, it was shown in [16] that $Ker(L)$ is a complete lattice and $k_1 \leq k_2 \Longleftrightarrow Q_{k_1} \subseteq Q_{k_2}$ for any $k_1, k_2 \in Ker(L)$. Then it follows by Lemma 6.1 that $\mathbf{Q}(L)$ partially ordered by inclusion is also a complete lattice with the lattice operations given by $\bigvee_{i\in I} Q_{k_i} = Q_{\bigvee_{i\in I} k_i}$ and $\bigwedge_{i\in I} Q_{k_i} = Q_{\bigwedge_{i\in I} k_i}$ for any $\{Q_{k_i}\}_{i\in I} \subseteq \mathbf{Q}(L)$, where $\bigvee_{i\in I} k_i$ is the pairwise join of the family of kernel operators $\{k_i\}_{i\in I}$ and $\bigwedge_{i\in I} k_i = \bigvee\{k \in ker(L) \mid \forall i \in I, k \leq k_i\}$.

Proposition 6.3. *Let L be a continuous lattice. Then every principle ideal of L is a quotient of L.*

Proof. It is routine to check that the map $x \wedge -: L \longrightarrow L$ is a kernel operator for any $x \in L$. Since L is continuous, it follows that $x \wedge -$ is Scott continuous and then every $x \wedge -$ is a continuous kernel operator. Furthermore, one can easily check that $(x \wedge -)(L) = \downarrow x$ and so every principle ideal of L is a quotient of L. □

Lemma 6.4. *Let L be a continuous lattice. For any $Y \subseteq \mathbf{cpt}(L)$, we have that*

(1) *$Y \subseteq \mathbf{cpt}(\mathcal{J}(Y))$ and $\mathcal{J}(Y)$ is an algebraic quotient of L.*
(2) *If Y is a sober convex subspace of $\mathbf{cpt}(L)$, then $Y = \mathbf{cpt}(\mathcal{J}(Y))$.*

Proof. (1) It is clear that $\mathcal{J}(Y)$ is closed under arbitrary joins. Let $M_1, M_2 \subseteq Y$ with $\bigvee M_1 \ll \bigvee M_2$ in $\mathcal{J}(Y)$. If $\bigvee M_2 \leq \bigvee D$ for some directed subset D of L, then $M_2 \subseteq \downarrow D$ and hence $\bigvee M_2 \leq \bigvee(\downarrow D \cap \mathcal{J}(Y))$. Since $\downarrow D \cap \mathcal{J}(Y)$ is directed in $\mathcal{J}(Y)$, we have that $\bigvee M_1 \leq x$ for some $x \in \downarrow D \cap \mathcal{J}(Y)$ and hence $\bigvee M_1 \leq d$ for some $d \in D$, which implies that $\bigvee M_1 \ll \bigvee M_2$ in L. Then $\mathcal{J}(Y)$ is a quotient of L. So $K(\mathcal{J}(Y)) = K(L) \cap \mathcal{J}(Y)$ and hence $Y \subseteq \mathbf{cpt}(\mathcal{J}(Y))$. Furthermore, we have that $\mathcal{J}(Y)$ is algebraic, because all elements in $\mathcal{J}(Y)$ are joins of compact elements in $\mathcal{J}(Y)$.

(2)Let Y be a sober convex subspace of $\mathbf{cpt}(L)$. Since $\mathbf{cpt}(\mathcal{J}(Y)) = \mathcal{J}(Y) \cap \mathbf{cpt}(L)$, it follows that $\bigvee F \in \mathbf{cpt}(\mathcal{J}(Y))$ for any $F \subseteq_\omega Y$. Furthermore, we have that $\mathbf{cpt}(\mathcal{J}(Y)) = \{\bigvee F \mid F \subseteq_\omega Y\}$, because $\bigvee M = \bigvee\{\bigvee F \mid F \subseteq_\omega M\}$ for any $M \subseteq Y$ and $\{\bigvee F \mid F \subseteq_\omega M\}$ is directed in $\mathcal{J}(Y)$. For any $F \subseteq_\omega M$, since Y is sober, we have that $co_Y(F) = co_Y(y)$ for a unique $y \in Y$ and so $\bigvee F = y \in Y$, which implies that $\mathbf{cpt}(\mathcal{J}(Y)) \subseteq Y$. □

Proposition 6.5. *For any continuous lattice L, the algebraization quotient of L is $\mathcal{J}(\mathbf{cpt}(L))$.*

Proof. It is routine to check that the universal property of the algebraization quotient of L can be translated into a condition that the algebraization quotient of L is the largest quotient of L which is algebraic. Then it suffices to show that $\mathcal{J}(\mathbf{cpt}(L))$ is the largest quotient of L which is algebraic. Lemma 6.4 gives that $\mathcal{J}(\mathbf{cpt}(L))$ is an algebraic quotient of L. For an arbitrary algebraic quotient Q of L, we have that $\mathbf{cpt}(Q) = \mathbf{cpt}(L) \cap Q$ and so $\mathbf{cpt}(Q) \subseteq \mathbf{cpt}(L)$. Then it follows by the facts that $\mathcal{J}$ is monotone and Q is an algebraic quotient of L that $Q = \mathcal{J}(\mathbf{cpt}(Q)) \subseteq \mathcal{J}(\mathbf{cpt}(L))$, as desired. □

Proposition 6.6. *Let L be a continuous lattice and $Y \subseteq \mathbf{cpt}(L)$. Then the pair $(i_Y, \mathbf{cpt}(\mathcal{J}(Y)))$ is a sobrification of Y, where the sobrification map i_Y is the identical convex-embedding $Y \subseteq \mathbf{cpt}(\mathcal{J}(Y)))$.*

Proof. By Lemma 6.4, $\mathcal{J}(Y)$ is an algebraic quotient of L. Then Lemma 2.11 gives that $\mathbf{cpt}(\mathcal{J}(Y))$ is sober. Then by the proof of Lemma 6.4, we have that $\mathbf{cpt}(\mathcal{J}(Y)) = \{\bigvee F \mid F \subseteq_\omega Y\}$. By Lemma 6.4, $Y \subseteq \mathbf{cpt}(\mathcal{J}(Y))$ and we let $i_Y : Y \longrightarrow \mathbf{cpt}(\mathcal{J}(Y))$ be the identical convex-embedding. Then i_Y is of course a CP map. Suppose that $f : Y \longrightarrow Z$ is a CP map with Z a sober convex space. For any finite subset F of Y, $f(F)$ is a finite subset of Z and hence there exists a unique point $z \in Z$ such that $co_Z(f(F)) = co_Z(z)$. We define a map $\bar{f} : \mathbf{cpt}(\mathcal{J}(Y)) \longrightarrow Z$ by $\bar{f}(\bigvee F) = z$.

(1) Obviously, $\bar{f}$ is well defined and $\bar{f} \circ i_Y = f$.

(2) $\bar{f}$ is CP. Let $C \in \mathfrak{C}(Z)$ and $F \subseteq_\omega Y$. Then $f^{-1}(C) \in \mathfrak{C}(Y)$ and we let $f^{-1}(C) = K_a \cap Y$ for some $a \in L$. It follows that $\bigvee F \in \bar{f}^{-1}(C)$ iff $co_Z(f(F)) \subseteq C$ iff $F \subseteq f^{-1}(C)$ iff $F \subseteq K_a \cap Y$ iff $\bigvee F \in K_a \cap \mathbf{cpt}(\mathcal{J}(Y))$. Thus, $\bar{f}^{-1}(C) = K_a \cap \mathbf{cpt}(\mathcal{J}(Y))$, which is a convex set of $\mathbf{cpt}(\mathcal{J}(Y))$. Thus, $\bar{f}$ is a CP map.

(3) $\bar{f}$ is unique. Let $g : \mathbf{cpt}(\mathcal{J}(Y)) \longrightarrow Z$ be an arbitrary CP map such that $g \circ i_Y = f$. We need to show that $g = \bar{f}$. For any $F \subseteq_\omega Y$, we have that $co_Z(g(\bigvee F)) = co_Z(g(co_{\mathbf{cpt}(\mathcal{J}(Y))}(\bigvee F))) = co_Z(g(co_{\mathbf{cpt}(\mathcal{J}(Y))}(F))) =$

$co_Z(g(F)) = co_Z(g(i_Y(F))) = co_Z(f(F)) = co_Z(\bar{f}(\bigvee F))$. So $co_Z(g(\bigvee F)) = co_Z(\bar{f}(\bigvee F))$ and hence $g(\bigvee F) = \bar{f}(\bigvee F)$ as Z is S_0. Thus, $g = \bar{f}$. □

The following proposition will indicate that there exists a symmetry between the algebraization of quotients and the sobrification of convex subspaces.

Proposition 6.7. *Let L be a continuous lattice. Then $(\mathcal{J}, \mathbf{cpt})$ is an adjunction between the posets $\mathcal{P}(\mathbf{cpt}(L))$ and $\mathbf{Q}(L)$. Furthermore, the fixpoints of $\mathcal{J} \circ \mathbf{cpt}$ are exactly the algebraic quotients and the fixpoints of $\mathbf{cpt} \circ \mathcal{J}$ are exactly the sober convex subspaces.*

Proof. It is clear that the map $\mathbf{cpt}$ is well defined and Lemma 6.4 gives that $\mathcal{J}$ is well defined and monotone. In order to show that $(\mathcal{J}, \mathbf{cpt})$ is an adjunction, it suffices to show that $\mathcal{J} \circ \mathbf{cpt} \leq id_{\mathcal{P}(\mathbf{cpt}(L))}$ and $\mathbf{cpt} \circ \mathcal{J} \leq id_{\mathbf{Q}(L)}$. $\mathcal{J} \circ \mathbf{cpt} \leq id_{\mathcal{P}(\mathbf{cpt}(L))}$ is obvious. By Lemma 6.4, we have that $Y \subseteq \mathbf{cpt}(\mathcal{J}(Y))$ for any $Y \subseteq \mathbf{cpt}(L)$, which means that $\mathbf{cpt} \circ \mathcal{J} \leq id_{\mathbf{Q}(L)}$. Furthermore, Proposition 6.5 give that the fixpoints of $\mathcal{J} \circ \mathbf{cpt}$ are exactly the algebraic quotients and Proposition 6.6 gives that the fixpoints of $\mathbf{cpt} \circ \mathcal{J}$ are exactly the sober convex subspaces. □

We let $\mathbf{Alg}(\mathbf{Q}(L))$ denote the collection of all algebraic quotients of L. By Theorem 6.1 and Theorem 6.2, one can see that $\mathbf{Q}(L)$ may not be continuous, not to say algebraic. However, when consider the subposet $\mathbf{Alg}(\mathbf{Q}(L))$ of $\mathbf{Q}(L)$, it will be shown to be an algebraic lattice. For a compact element c of a continuous lattice L, we always write $Q_c = \{0, c\}$. Obviously, every Q_c is included in $\mathbf{Q}(L)$, even in $\mathbf{Alg}(\mathbf{Q}(L))$.

Lemma 6.8. *Let L be a continuous lattice. Then every $Q_c = \{0, c\}$ is compact in $\mathbf{Q}(L)$ for any $c \in \mathbf{cpt}(L)$.*

Proof. Let $\{Q_{k_i}\}_{i\in I} \subseteq \mathbf{Q}(L)$ be a directed family with $\{k_i\}_{i\in I} \subseteq Ker(L)$ and $Q_c \subseteq \bigvee_{i\in I} Q_{k_i}$. Then $\{k_i\}_{i\in I}$ is directed in $Ker(L)$ and $Q_c \subseteq Q_{\bigvee_{i\in I} k_i}$. So $c = (\bigvee_{i\in I} k_i)(c) = \bigvee_{i\in I}(k_i(c))$. It follows that $c \leq k_{i_1}(c)$ for some $k_{i_1} \in Ker(L)$ and so $c = k_{i_1}(c)$. Then $c \in Q_{k_{i_1}}$ and hence $Q_c \subseteq Q_{k_{i_1}}$. Thus, Q_c is compact in $\mathbf{Q}(L)$. □

Proposition 6.9. *Let L be a continuous lattice. Then $\mathbf{Alg}(\mathbf{Q}(L))$ is an algebraic join-sub-complete lattice of $\mathbf{Q}(L)$.*

Proof. By Proposition 6.7, one can check that the algebraization map $\mathcal{J} \circ \mathbf{cpt}\colon \mathbf{Q}(L) \longrightarrow \mathbf{Q}(L)$ is a kernel operator and it follows that the fixpoints of $\mathcal{J} \circ \mathbf{cpt}$ is closed under arbitrary joins in $\mathbf{Q}(L)$. Then $\mathbf{Alg}(\mathbf{Q}(L))$ is closed under arbitrary joins in $\mathbf{Q}(L)$, which implies that $\mathbf{Alg}(\mathbf{Q}(L))$ is a join-sub-complete lattice of $\mathbf{Q}(L)$. Then it follows that Q_c is compact in $\mathbf{Alg}(\mathbf{Q}(L))$ for any $c \in \mathbf{cpt}(L)$. So it suffices to show that $Q = \bigvee\{Q_c \mid c \in \mathbf{cpt}(Q)\}$ for any $Q \in \mathbf{Alg}(\mathbf{Q}(L))$. Clearly, $Q_c \subseteq Q$ for any $c \in \mathbf{cpt}(Q)$. Let $Q' \in \mathbf{Alg}(\mathbf{Q}(L))$ and such that $Q_c \subseteq Q'$ for any $c \in \mathbf{cpt}(Q)$. Then $\mathbf{cpt}(Q) \subseteq Q'$. For any $x \in Q$, we have that $x = \bigvee(\downarrow x \cap \mathbf{cpt}(Q))$ and it follows that $x \in Q'$. This implies that $Q \subseteq Q'$ and so $Q = \bigvee\{Q_c \mid c \in \mathbf{cpt}(Q)\}$. Thus, $\mathbf{Alg}(\mathbf{Q}(L))$ is an algebraic join-sub-complete lattice of $\mathbf{Q}(L)$. □

Proposition 6.10. *Let L be a continuous lattice. Then the map*

$$\mathbf{cpt}\colon \mathbf{Alg}(\mathbf{Q}(\mathbf{L})) \longrightarrow \mathbf{Sob}(\mathcal{P}(\mathbf{cpt}(\mathbf{L})))$$

is an order isomorphism.

Proof. By Proposition 6.7, we have that $\mathcal{J}\colon \mathcal{P}(\mathbf{cpt}(L)) \longrightarrow \mathbf{Q}(L)$ is the left adjoint of $\mathbf{cpt}\colon \mathbf{Q}(L) \longrightarrow \mathcal{P}(\mathbf{cpt}(L))$. Then this adjunction can induce an order isomorphism $\mathbf{cpt}$ which maps the fixpoints of $\mathcal{J}\circ\mathbf{cpt}$ onto that of $\mathbf{cpt}\circ\mathcal{J}$. Again by Proposition 6.7, these fixpoints are algebraic quotients of L and sober convex subspaces of $\mathbf{cpt}(L)$ and so the map $\mathbf{cpt}\colon \mathbf{Alg}(\mathbf{Q}(L)) \longrightarrow \mathbf{Sob}(\mathcal{P}(\mathbf{cpt}(L)))$ is an order isomorphism. □

We call an element x of a continuous lattice L *moderately irreducible* if for any $P \subseteq \mathbf{cpt}(L)$, $x = \bigvee P$ implies $x = p$ for some $p \in P$. It is clear that every strongly irreducible element of L is moderately irreducible. For an algebraic lattice L, one can easily check that a compact element in $\mathbf{cpt}(L)$ is moderately irreducible iff it is irreducible. For the question that when are all the quotients of L algebraic, Hofmann and Mislove had given some nice characterizations in [16]. Then a question naturally arises, when are all convex subspaces of $\mathbf{cpt}(L)$ sober? The following theorem will give several necessary and sufficient conditions for this question.

Theorem 6.11. *Let L be a continuous lattice. Then the following are equivalent:*

(1) *All the compact elements in $\mathbf{cpt}(\mathbf{L})$ is moderately irreducible;*
(2) *All the convex subspaces of $\mathbf{cpt}(\mathbf{L})$ are sober;*
(3) *The restriction of the map $\mathbf{cpt}\colon \mathbf{Q}(L) \longrightarrow \mathcal{P}(\mathbf{cpt}(\mathbf{L}))$ to $\mathbf{Alg}(\mathbf{Q}(L))$ is an order isomorphism;*
(4) *The space $\mathbf{cpt}(\mathbf{L})$ is S_D.*

Proof. (1) ⇒ (2). By Lemma 6.4, we have that $Y \subseteq \mathbf{cpt}(\mathcal{J}(Y))$ for any $Y \subseteq \mathbf{cpt}(L)$. For any $\bigvee P \in \mathbf{cpt}(\mathcal{J}(Y))$ with $P \subseteq Y$, it follows by the assumption that $\bigvee P = p$ for some $p \in P$ and hence $\bigvee P \in Y$. Then $Y = \mathbf{cpt}(\mathcal{J}(Y))$. By Proposition 6.6, it follows that every convex subspace of $\mathbf{cpt}(L)$ is sober.

(2) ⇒ (3). By Proposition 6.10.

(3) ⇒ (1). Suppose that there exists a subset $P \subseteq \mathbf{cpt}(L)$ and a compact element $c \in \mathbf{cpt}(L)$ with $c \notin P$ and such that $c = \bigvee P$. Then by the assumption, $P = \mathbf{cpt}(Q)$ for some $Q \in \mathbf{Alg}(\mathbf{Q}(L))$. Since $K(Q) = K(L) \cap Q$, we have that $K(Q) = P \cup \{\bot_L\}$ is directed in L. Then it follows by the fact $\bigvee P = \bigvee K(Q)$ that there exists an element $p \in K(Q)$ such that $c \leq p$. If $p = \bot_L$, then $\bot_L = c \in \mathbf{cpt}(L)$, a contradiction. If $p \neq \bot_L$, then $p \in P$ and so $c = p$ as $p \leq c$ is obvious, which implies that $c \in P$, a contradiction. Thus, every compact element in L is moderately irreducible.

(4) ⇒ (1). Suppose that there exists a subset $P \subseteq \mathbf{cpt}(L)$ and a compact element $p \in \mathbf{cpt}(L)$ with $p \notin P$ and such that $p = \bigvee P$. Then $p = \bigvee\{q \in \mathbf{cpt}(L) \mid q < p\}$. For any $a \in L$, if $co_{\mathbf{cpt}(L)}(p)\backslash\{p\} \subseteq K_a$, then $q < p$ implies $q \leq a$ for any $q \in \mathbf{cpt}(L)$ and it follows that $p \leq a$, which implies that $p \in K_a$. Hence $p \in co_X(co_X(p)\backslash\{p\})$, because $co_X(co_X(p)\backslash\{p\}) = \bigcap\{K_a \mid$

$co_X(p)\backslash\{p\} \subseteq K_a\}$. Since $\mathbf{cpt}(L)$ is S_D, we have that $co_X(co_X(p)\backslash\{p\}) = co_X(p)\backslash\{p\}$ and then $p \in co_X(p)\backslash\{p\}$, a contradiction.

(1) ⇒ (4). In order to show that $\mathbf{cpt}(L)$ is S_D, we only need to check that $co_{\mathbf{cpt}(L)}(p)\backslash\{p\} = K_{a_p}$ for any $p \in \mathbf{cpt}(L)$, where $a_p = \bigvee\{r \in \mathbf{cpt}(L) \mid r < p\}$. It is obvious that $co_X(p)\backslash\{p\} \subseteq K_{a_p}$. For any $q \in K_{a_p}$, we have that $q \leq a_p$ and so $q \leq p$. If $p = q$, then $p = \bigvee\{r \in \mathbf{cpt}(L) \mid r < p\}$ and so it follows by the assumption that $p = r_1 < p$ for some $r_1 \in \mathbf{cpt}(L)$, a contradiction. So $q < p$ and hence $K_{a_p} \subseteq co_{\mathbf{cpt}(L)}(p)\backslash\{p\}$. □

Finally, we shall give several characterizations for a continuous lattice to be strongly S_D-algebraic.

Theorem 6.12. *Let L be a continuous lattice. Then the following are equivalent:*
(1) *L is strongly S_D-algebraic;*
(2) *L is algebraic and $\mathbf{cpt}(\mathbf{L})$ is an S_D-convex space X;*
(3) *$L \cong \mathfrak{C}(X)$ for some sober S_D-convex space;*
(4) *L is algebraic and $\mathbf{Alg}(\mathbf{Q}(\mathbf{L})) \cong \mathcal{P}(\mathbf{cpt}(\mathbf{L}))$.*

Proof. (1) ⇒ (2). By Lemma 5.1.

(2) ⇒ (3). This is obvious as $L \cong \mathfrak{C}(\mathbf{cpt}(L))$ and $\mathbf{cpt}(L)$ is always sober.

(3) ⇒ (4). Obviously, L is algebraic as $\mathfrak{C}(X)$ is. Without loss of generality, we let $L = \mathfrak{C}(X)$ for some sober S_D-convex space X. Since X is S_D, Proposition 4.2 gives that $\Phi(L) = \{co_X(x) \mid x \in X\}$ and so $\mathbf{cpt}(L) = \Phi(L)$ as X is sober. Thus, it follows by Proposition 4.2 and Theorem 6.11 that $\mathbf{Alg}(\mathbf{Q}(L)) \cong \mathcal{P}(\mathbf{cpt}(L))$.

(4) ⇒ (1). By Theorem 6.11 and the assumption, every compact element in $\mathbf{cpt}(L)$ is moderately irreducible. Let $c \in \mathbf{cpt}(L)$ and $c = \bigvee G$ for some $G \subseteq L$. Since L is algebraic, $c = \bigvee G = \bigvee \bigcup_{x\in G} \downarrow x \cap K(L)$ and so $c = c_1$ for some $c_1 \in \bigcup_{x\in G} \downarrow x \cap K(L)$. Then $c \leq x_1$ for some $x_1 \in G$ and hence $c = x_1$ as $x_1 \leq c$ is obvious, as desired. □

Remark 6.13. For an algebraic lattice L, all the conditions in Theorem 6.11 and Theorem 6.12 are equivalent.

7. Conclusions

In [26] and [30], a dual adjunction between **CS** and **CLat** was built, which restricts to a dual equivalence between **Sob** and **AgLat**. In this paper, a further restriction of this dual equivalence yields the Hofmann–Lawson-like duality for pointfree convex spaces. Meanwhile, we also built a dual adjunction between $\mathbf{CS}_D$ and $\mathbf{CLat}_D$ by introducing the notion of strongly irreducible elements and it further induces a dual equivalence between $\mathbf{CS}_D$ and $\mathbf{AgLat}_D$. In addition, we obtain several characterizations of sober convex spaces and S_D-convex spaces, which indicates that they are dual to each other in certain sense, and also present the close connection between the sobriety and S_D-axiom in the case of pointfree convex geometry.

In the future, we will consider the following two problems.

• It is well known that the duality between the category of bitopological spaces and that of biframes was firstly established by Banaschewski et al. in [5], which generalized the classic duality between the category of topological spaces and that of frames. So we will consider building this duality between the category of "biconvex spaces" and that of "bicontinuous lattices" and further building the duality between the category of "bi-S_D-convex spaces" and that of "bicontinuous lattices".

• In Section 6, by exploiting the duality between **CS** and **CLat**, we investigate the relationships between the quotients of a continuous lattice L and the convex subspaces of $\mathbf{cpt}(L)$. Then based on the duality between $\mathbf{CS}_D$ and $\mathbf{CLat}_D$ and the close connection between the sobriety and S_D-axiom, we shall investigate the relationships between the S_D-quotients of a continuous lattice L and the convex subspaces of $\Phi(L)$.

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Conflict of interest The author declares that he has no conflict of interest.

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