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Algebra Universalis

Admissible subsets and completions of ordered algebras

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Abstract. We consider ordered universal algebras and give a construction of a join-completion for them using so-called $\mathscr{D}$-ideals. We show that this construction has a universal property that induces a reflector from a certain category of ordered algebras to the category of sup-algebras. Our results generalize several earlier known results about different ordered structures.

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1. Introduction

There are several ways how to embed an ordered algebra into a complete ordered algebra of the same type. One such possibility is given in [15], where it is shown that certain injective hulls of ordered algebras have properties similar to those of Dedekind–MacNeille completions. In this paper we will follow a different approach—constructing completions with the help of admissible ideals.

In [5], Bruns and Lakser introduced admissible subsets and so-called D-ideals in semilattices. They proved that the set of all D-ideals (which is a complete lattice) is the injective hull of the semilattice. In [3], Bishop studied the completion by complete ideals of a lattice and proved a universal property for it. Krishnan [7] contstructed a completion for pomonoids which is compatible with joins. Rasouli [9] used a similar approach to construct a completion for

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S-posets where S is a pomonoid. In a recent paper [16], completions of marked quantales are considered. Such structures (semilattices, lattices, pomonoids, S-posets, posemigroups) can be considered as special cases of ordered universal algebras of different types. It is natural to ask if the results in the mentioned papers have a common generalization to the ordered algebras.

In this article we will give a construction that assigns the sup-algebra of $\mathscr{D}$-ideals (denoted by $\mathscr{D}(A)$) to each ordered algebra $\mathcal{A}$. We will prove that $\mathscr{D}(A)$ is a join-completion for $\mathcal{A}$ and prove a universal property of this construction. As a consequence, we will obtain a reflector functor $\mathscr{D}$ to the category of sup-algebras. The source category of this functor has ordered algebras as objects, but the morphisms are not all homomorphisms, but those which preserve admissible joins. We note that also in [12] different sup-algebra completions are considered, one of them being $\mathcal{D}(A)$. But the definitions of $\mathscr{D}(A)$ and $\mathcal{D}(A)$ differ a litt'le bit and the universal property is not considered in [12].

2. Preliminaries

We recall some definitions that will be needed in this paper.

Definition 2.1 ([1, Definition 4.16]). A subcategory $\mathcal{A}$ of a category $\mathcal{B}$ is called a *reflective subcategory* if for every $\mathcal{B}$-object B there is a $\mathcal{B}$-morphism $r : B \to A$ from B to an $\mathcal{A}$-object A with the following universal property: for any $\mathcal{B}$-morphism $f : B \to A'$ from B to an $\mathcal{A}$-object A', there exists a unique $\mathcal{A}$-morphism $f' : A \to A'$ such that $f'r = f$. In other words, $\mathcal{A}$ is a reflective subcategory of $\mathcal{B}$ if the inclusion functor $\mathcal{A} \to \mathcal{B}$ has a left adjoint functor $\mathcal{B} \to \mathcal{A}$ (see [8], page 91), which is usually called a *reflector*.

In this paper we will show how to construct a reflector from a certain category of ordered algebras to the category of sup-algebras of the same type.

Let Ω be a type. An *ordered* Ω*-algebra* is a triplet $\mathcal{A} = (A, \Omega_A, \leqslant_A)$ comprising a poset $(A, \leqslant_A)$ and a set Ω_A of operations on A (for every k-ary operation symbol $\omega \in \Omega_k$ there is a k-ary operation $\omega_A \in \Omega_A$ on A) such that all the operations ω_A are monotone mappings ([4]).

Let $\mathcal{A}$ and $\mathcal{B}$ be ordered Ω-algebras. We say that a monotone mapping $f : \mathcal{A} \to \mathcal{B}$ is a *lax morphism*, if

$$\omega_B(f(a_1), \ldots, f(a_n)) \leqslant f(\omega_A(a_1, \ldots, a_n)) \tag{2.1}$$

for every $n \in \mathbb{N}$, $\omega \in \Omega_n$, $a_1, \ldots, a_n \in A$, and

$$\omega_B \leqslant f(\omega_A) \tag{2.2}$$

for every $\omega \in \Omega_0$.

If $f : \mathcal{A} \to \mathcal{B}$ is monotone and operation-preserving, i.e., the inequalities in (2.1) and (2.2) turn out to be equalities, then f is a *homomorphism* of ordered algebras.

Throughout this text, a type Ω is fixed, all algebras that we consider will be Ω-algebras and all homomorphisms will be homomorphisms of Ω-algebras, even if Ω is not explicitly mentioned.

Linear functions on an ordered algebra $\mathcal{A}$ are defined as follows (see [14]).

(1) The identity mapping $A \to A, x \mapsto x$, is a linear function.
(2) If $n \in \mathbb{N}$, $\omega \in \Omega_n$, $i \in \{1, \ldots, n\}$, $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n \in A$ and $p : A \to A$ is a linear function, then the mapping

$$A \to A, \;\; x \mapsto \omega(a_1, \ldots, a_{i-1}, p(x), a_{i+1}, \ldots, a_n)$$

is a linear function.

Linear functions obtained by step (1) or by step (2) with $p = id_A$ are called *elementary translations* of $\mathcal{A}$. We write $L_{\mathcal{A}}$ for the set of all linear functions on an ordered algebra $\mathcal{A}$. Linear functions are the composites of elementary translations.

Definition 2.2. A subset M of an ordered algebra $\mathcal{A}$ is called *admissible*, if

- $\bigvee p(M)$ exists for all $p \in L_{\mathcal{A}}$ (in particular $\bigvee M = \bigvee id_A(M)$ exists),
- for all $p \in L_{\mathcal{A}}$,

$$p\left(\bigvee M\right) = \bigvee p(M).$$

Our definition generalizes the definition of an admissible subset of a semilattice which was introduced in [5]. Note that [12] also defines admissible subsets of an ordered algebra, but that definition differs from ours: instead of linear functions, unary polynomial functions are used.

We write $S{\downarrow} = \{a \in P \mid a \leqslant s \text{ for some } s \in S\}$ if S is a subset of a poset P, and we call S a *lower subset* of P if $S{\downarrow} = S$.

Definition 2.3. A lower subset S of an ordered algebra $\mathcal{A}$ is called a *$\mathscr{D}$-ideal* (cf. [5, p. 116] or [12, Definition3.6]) if for any admissible subset M of S, one has that $\bigvee M \in S$.

Let us denote by $\mathscr{D}(A)$ the set of all $\mathscr{D}$-ideals of A. Note that $a{\downarrow}$ is a $\mathscr{D}$-ideal for every $a \in A$ (see Remark 3.7 in [12]).

Definition 2.4. We say that an ordered algebra homomorphism $f : \mathcal{A} \to \mathcal{B}$ *preserves admissible joins* if, for any admissible subset M of A, the subset $f(M) \subseteq B$ is admissible and

$$f\left(\bigvee M\right) = \bigvee f(M).$$

All ordered algebras with admissible joins preserving homomorphisms constitute a category, which we denote by OAlg^*. This is a (non-full) subcategory of OAlg, the category of ordered algebras with all homomorphisms as morphisms.

3. Sup-algebras

An ordered Ω-algebra $\mathcal{Q} = (Q, \Omega_Q, \leqslant_Q)$ is called a *sup-algebra* if the poset $(Q, \leqslant_Q)$ is a complete lattice and all elementary translations preserve joins. Sup-algebras were introduced by Pedro Resende in [10] as a common generalization for many quantale-like structures. All sup-algebras with join-preserving homomorphisms form a category denoted by SupAlg.

Lemma 3.1. *Let $\mathcal{A}$ be an ordered algebra such that $(A, \leqslant)$ is a complete lattice. Then the following assertions are equivalent.*

(1) *$\mathcal{A}$ is a sup-algebra.*
(2) *All linear functions on $\mathcal{A}$ preserve joins.*
(3) *All subsets of $\mathcal{A}$ are admissible.*

Proof. (1) $\Leftrightarrow$ (2). This is true because linear functions are the composites of elementary translations.

(2) $\Leftrightarrow$ (3). This holds because all subsets of A have joins. □

Proposition 3.2. SupAlg *is a full subcategory of* OAlg^**.*

Proof. If $\mathcal{Q}$ and $\mathcal{R}$ are sup-algebras then all subsets of Q and R are admissible by Lemma 3.1. Hence a homomorphism $f : \mathcal{Q} \to \mathcal{R}$ preserves admissible joins if and only if it preserves all joins. □

If $\mathcal{Q}$ is a sup-algebra, then every elementary translation preserves joins, hence it has a right adjoint.

A closure operator j on a sup-algebra $\mathcal{Q}$ is a *nucleus* if it is a lax endomorphism of $\mathcal{Q}$. The subset $Q_j = \{q \in Q \mid j(q) = q\}$ can be made into a sup-algebra called a *quantic quotient* of $\mathcal{Q}$ (see [10, Theorem 2.2.7] or [13, Proposition 16]).

Lemma 3.3 [13, Proposition 15] *If $\mathcal{Q} = (Q, \Omega_Q, \leqslant_Q)$ is a sup-algebra and $S \subseteq Q$, then $S = Q_j$ for some nucleus j on $\mathcal{Q}$ if and only if S is closed under meets and under right adjoints of elementary translations. In this case, the nucleus j is defined by*

$$j(q) := \bigwedge \{s \in S \mid q \leqslant s\} \tag{3.1}$$

for $q \in Q$.

For an ordered algebra $\mathcal{A} = (A, \Omega_A, \leqslant_A)$, let $\mathscr{P}(A)$ be the set of all lower subsets of A. Then $\mathscr{P}(A)$ is a sup-algebra equipped with the inclusion as ordering and

$$\omega_{\mathscr{P}(A)}(D_1, \ldots, D_n) = \{\omega_A(d_1, \ldots, d_n) \mid d_i \in D_i, i = 1, \ldots, n\}{\downarrow}$$

as n-ary operations for every $\omega \in \Omega_n$, $n \in \mathbb{N}$. If $\omega \in \Omega_0$, then $\omega_{\mathscr{P}(A)} = \omega_A{\downarrow}$.

A nucleus j on $\mathscr{P}(A)$ is called *principal closed*, if $j(a{\downarrow}) = a{\downarrow}$ for all $a \in A$.

Proposition 3.4. *Let $\mathcal{A} = (A, \Omega_A, \leqslant_A)$ be an ordered algebra. Then $\mathscr{D}(A)$ is a quantic quotient of the sup-algebra $\mathscr{P}(A)$.*

Although our point of view is a little bit different from that of [12] (namely $\mathscr{P}(A)$ denotes the set of *all* subsets of A in [12]), Proposition 3.4 can still be proved precisely as Theorem 3.9 in [12].

Now, given an ordered algebra $\mathcal{A}$, Lemma 3.3 provides a nucleus $j : \mathscr{P}(A) \to \mathscr{P}(A)$, defined by (3.1), that is,

$$j(C) = \bigcap\{S \in \mathscr{D}(A) \mid C \subseteq S\} \tag{3.2}$$

for every $C \in \mathscr{P}(A)$, such that $\mathscr{D}(A) = \mathscr{P}(A)_j = \{C \in \mathscr{P}(A) \mid j(C) = C\}$. It can be shown that j is a principal closed nucleus. In the sup-algebra $\mathscr{D}(A)$ joins and meets are calculated as

$$\bigvee_{i\in I} S_i = j\Big(\bigcup_{i\in I} S_i\Big) = \bigcap\Big\{S \in \mathscr{D}(A) \,\Big|\, \bigcup_{i\in I} S_i \subseteq S\Big\},$$
$$\bigwedge_{i\in I} S_i = \bigcap_{i\in I} S_i,$$

and operations are

$$\begin{aligned} \omega_{\mathscr{D}(A)}(S_1,\ldots,S_n) &= j(\omega_{\mathscr{P}(A)}(S_1,\ldots,S_n)) \\ &= \bigcap\{S \in \mathscr{D}(A) \mid \omega_{\mathscr{P}(A)}(S_1,\ldots,S_n) \subseteq S\}, \end{aligned} \tag{3.3}$$

for every $n \in \mathbb{N}$, $\omega \in \Omega_n$. If $\omega \in \Omega_0$, then $\omega_{\mathscr{D}(A)} = j(\omega_{\mathscr{P}(A)}) = j(\omega_A{\downarrow})$. Nullary operations of $\mathscr{D}(A)$ can be described as follows.

Lemma 3.5. *If $\mathcal{A}$ is an ordered-algebra and $\omega \in \Omega_0$, then $\omega_{\mathscr{D}(A)} = \omega_A{\downarrow}$.*

Proof. Clearly,

$$\omega_A \in \bigcap\{S \in \mathscr{D}(A) \mid \omega_A \in S\} = \bigcap\{S \in \mathscr{D}(A) \mid \omega_A{\downarrow} \subseteq S\} = \omega_{\mathscr{D}(A)}$$

and hence $\omega_A{\downarrow} \subseteq \omega_{\mathscr{D}(A)}$. Take arbitrary $a \in \omega_{\mathscr{D}(A)}$. Then $a \in S$ for all $S \in \mathscr{D}(A)$ containing ω_A. In particular, $a \in \omega_A{\downarrow}$, and thus $\omega_{\mathscr{D}(A)} \subseteq \omega_A{\downarrow}$. □

Proposition 3.6 [15]. *Let $\mathcal{A}$ be an ordered algebra and let j be a principal closed nucleus on $\mathscr{P}(\mathcal{A})$. Then the mapping $\eta : \mathcal{A} \to \mathscr{P}(\mathcal{A})_j, a \mapsto a{\downarrow}$, has the following properties:*

(1) *η is a homomorphism of ordered algebras which is an order-embedding,*
(2) *$\eta(A)$ is join-dense in the lattice $\mathscr{P}(\mathcal{A})_j$,*
(3) *η preserves all existing meets in $\mathcal{A}$.*

4. Completions by $\mathscr{D}$-ideals

Generalizing the notion of join-completion of a poset (cf. [2], [11], or [6]) we say that a *join-completion* of an ordered algebra $\mathcal{A}$ is a pair $(\eta, \mathcal{R}(A))$, where

(1) $\mathcal{R}(A)$ is a sup-algebra,
(2) $\eta : \mathcal{A} \to \mathcal{R}(A)$ is a homomorphism of ordered algebras which is an order-embedding,
(3) the set $\eta(A)$ is join-dense in $\mathcal{R}(A)$.

$\cdot$	a	b	c
a	a	a	a
b	a	b	b
c	a	b	c

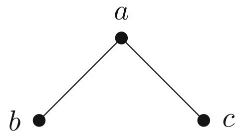


FIGURE 1. The multiplication and order of A

Meet-completions are defined dually.

It turns out that $\mathscr{D}(A)$, the sup-algebra of $\mathscr{D}$-ideals, is a join-completion of $\mathcal{A}$.

Proposition 4.1. *If $\mathcal{A}$ is an ordered algebra, then the mapping $r : A \to \mathscr{D}(A)$, $a \mapsto a\downarrow$, is a homomorphism of ordered algebras that is an order-embedding and preserves admissible joins. Moreover, $r(A)$ is join-dense in $\mathscr{D}(A)$ and r preserves all meets that exist in A.*

Proof. Let M be an admissible subset of A. Since $\mathscr{D}(A)$ is a sup-algebra, $p(\bigvee r(M)) = \bigvee p(r(M))$ for every $p \in L_{\mathscr{D}(A)}$ by Lemma 3.1. Thus $r(M)$ is an admissible subset of $\mathscr{D}(A)$.

It remains to prove that $r\left(\bigvee M\right) = \bigvee r(M)$ (all other claims follow from Proposition 3.4 and Proposition 3.6). Since $\mathscr{D}(A)$ is a sup-algebra, the join of $r(M)$ exists and, in fact,

$$\bigvee r(M) = \bigvee_{m\in M} r(m) = \bigcap \left\{ S \in \mathscr{D}(A) \;\middle|\; \bigcup_{m\in M} m\downarrow \subseteq S \right\}.$$

If S is a $\mathscr{D}$-ideal with $\bigcup_{m\in M} m\downarrow \subseteq S$, then $M \subseteq S$ and hence $\bigvee M \in S$ by the definition of a $\mathscr{D}$-ideal. This yields $\left(\bigvee M\right)\downarrow \subseteq S$, and thus $r\left(\bigvee M\right) \subseteq \bigvee r(M)$. The inclusion $\bigvee r(M) \subseteq r\left(\bigvee M\right)$ is clear. $\square$

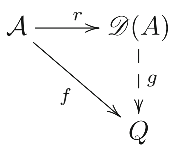
Example 4.2. In general, r need not preserve existing joins. Let $A = \{a, b, c\}$ be a commutative posemigroup with the multiplication and order given in Figure 1.

Its quantale of $\mathscr{D}$-ideals is $\mathscr{D}(A) = \{a\downarrow, b\downarrow, c\downarrow, \{b, c\}, \emptyset\}$. We see that $r(\bigvee\{b, c\}) = r(a) = \{a, b, c\}$, but

$$\bigvee r(\{b, c\}) = \{b\} \vee \{c\} = \bigcap\{S \in \mathscr{D}(A) \mid \{b, c\} \subseteq S\} = \{b, c\}.$$

Our main result is the following. It generalizes, for example, [3, Theorem 3] about lattices and [9, Theorem 3.2] about S-posets.

Theorem 4.3. *Let $\mathcal{A}$ be an ordered algebra. Then, for every sup-algebra $\mathcal{Q}$ and every OAlg^*-morphism $f : \mathcal{A} \to \mathcal{Q}$, there exists a unique SupAlg-morphism $g : \mathscr{D}(A) \to \mathcal{Q}$ such that the diagram*



commutes.

Proof. Proposition 4.1 has established that r is an OAlg^*-morphism. Given a sup-algebra $\mathcal{Q}$ and a morphism $f : \mathcal{A} \to \mathcal{Q}$ in the category OAlg^*, define $g : \mathscr{D}(A) \to Q$ by

$$g(D) = \bigvee_{d \in D} f(d)$$

for every $\mathscr{D}$-ideal D of A. We need to prove that g is a join-preserving sup-algebra homomorphism. For every $\omega \in \Omega_0$, using Lemma 3.5, we have

$$g\left(\omega_{\mathscr{D}(A)}\right) = \bigvee_{d \in \omega_A \downarrow} f(d) = f(\omega_A) = \omega_Q.$$

We also need to prove the equality

$$\omega_Q(g(D_1), \ldots, g(D_n)) = g(\omega_{\mathscr{D}(A)}(D_1, \ldots, D_n))$$

for any $n \in \mathbb{N}, \omega \in \Omega_n$ and $\mathscr{D}$-ideals $D_1, \ldots, D_n$ of A. By the definition of g, we have

$$g(\omega_{\mathscr{D}(A)}(D_1, \ldots, D_n)) = \bigvee_{d \in \omega_{\mathscr{D}(A)}(D_1, \ldots, D_n)} f(d),$$

where, according to (3.3),

$$\omega_{\mathscr{D}(A)}(D_1, \ldots, D_n) = \bigcap \{S \in \mathscr{D}(A) \mid \omega_{\mathscr{P}(A)}(D_1, \ldots, D_n) \subseteq S\}. \tag{4.1}$$

We compute

$$\omega_Q(g(D_1), \ldots, g(D_n)) = \omega_Q\left(\bigvee_{d_1 \in D_1} f(d_1), \ldots, \bigvee_{d_n \in D_n} f(d_n)\right)$$

$$= \bigvee_{d_1 \in D_1, \ldots, d_n \in D_n} \omega_Q(f(d_1), \ldots, f(d_n)) \qquad (\mathcal{Q} \text{ is a sup-algebra})$$

$$= \bigvee_{d_1 \in D_1, \ldots, d_n \in D_n} f(\omega_A(d_1, \ldots, d_n)) \qquad (f \text{ is a homomorphism})$$

$$= \bigvee_{d \in \omega_{\mathscr{P}(A)}(D_1, \ldots, D_n)} f(d). \qquad (\text{operations in } \mathscr{P}(A))$$

By (4.1) we have $\omega_{\mathscr{P}(A)}(D_1, \ldots, D_n) \subseteq \omega_{\mathscr{D}(A)}(D_1, \ldots, D_n)$, so in $\mathcal{Q}$ we obtain the inequality $\omega_Q(g(D_1), \ldots, g(D_n)) \leqslant g(\omega_{\mathscr{D}(A)}(D_1, \ldots, D_n))$.

To prove the opposite inequality, we first show that, for every $q_0 \in Q$, the subset

$$\mathscr{K} = \{a \in A \mid f(a) \leqslant q_0\}$$

of A is a $\mathscr{D}$-ideal. Since f is monotone, $\mathscr{K}$ is a lower subset of $\mathcal{A}$. Assume that $M \subseteq \mathscr{K}$ is an admissible subset. By the definition of $\mathscr{K}$, $f(m) \leqslant q_0$ for every $m \in M$. As f preserves admissible joins, we obtain

$$f\Big(\bigvee M\Big) = \bigvee_{m \in M} f(m) \leqslant q_0,$$

which means that $\bigvee M \in \mathscr{K}$. Thus $\mathscr{K}$ is a $\mathscr{D}$-ideal.

Now consider $\mathscr{K}$ corresponding to the element

$$q_0 := \bigvee_{d \in \omega_{\mathscr{P}(A)}(D_1,\ldots,D_n)} f(d) = \omega_Q(g(D_1),\ldots,g(D_n)).$$

Then we have $\omega_{\mathscr{P}(A)}(D_1,\ldots,D_n) \subseteq \mathscr{K}$. Since $\mathscr{K}$ is a $\mathscr{D}$-ideal of $\mathcal{A}$, it belongs to the set $\{S \in \mathscr{D}(A) \mid \omega_{\mathscr{P}(A)}(S_1,\ldots,S_n) \subseteq S\}$ and hence (4.1) implies that $\omega_{\mathscr{D}(A)}(D_1,\ldots,D_n) \subseteq \mathscr{K}$. We conclude that

$$\begin{aligned} g(\omega_{\mathscr{D}(A)}(D_1,\ldots,D_n)) = & \bigvee_{d \in \omega_{\mathscr{D}(A)}(D_1,\ldots,D_n)} f(d) \leqslant \bigvee_{d \in \mathscr{K}} f(d) \\ & \leqslant q_0 = \omega_Q(g(D_1),\ldots,g(D_n)). \end{aligned}$$

The equality $g(\omega_{\mathscr{D}(A)}(D_1,\ldots,D_n)) = \omega_Q(g(D_1),\ldots,g(D_n))$ follows.

Next we verify that g preserves joins. Assume that $\{D_i \mid i \in I\}$ is a set of $\mathscr{D}$-ideals of $\mathcal{A}$. Write $\widetilde{D} = \bigvee_{i \in I} D_i$ for the join in $\mathscr{D}(A)$. Then

$$\widetilde{D} = \bigcap \Big\{ S \in \mathscr{D}(A) \ \Big|\ D \subseteq S \Big\},$$

where $D = \bigcup_{i \in I} D_i$. The inequality $\bigvee_{i \in I} g(D_i) \leqslant g\Big(\bigvee_{i \in I} D_i\Big)$ is clear. We put $q_0 := \bigvee_{d \in D} f(d)$ and consider the set

$$\widetilde{\mathscr{K}} = \Big\{ a \in A \ \Big|\ f(a) \leqslant q_0 \Big\}.$$

Then $\widetilde{\mathscr{K}}$ is also a $\mathscr{D}$-ideal of $\mathcal{A}$ by the argument that we used for $\mathscr{K}$ above. Since $\widetilde{\mathscr{K}}$ is a $\mathscr{D}$-ideal and $D \subseteq \widetilde{\mathscr{K}}$, we have $\widetilde{D} \subseteq \widetilde{\mathscr{K}}$. So $\bigvee_{d \in \widetilde{D}} f(d) \leqslant \bigvee_{d \in \widetilde{\mathscr{K}}} f(d)$. For every $a \in \widetilde{\mathscr{K}}$, the inequality $f(a) \leqslant q_0$ holds, therefore $\bigvee_{d \in \widetilde{D}} f(d) \leqslant q_0$. Hence we have

$$\begin{aligned} g\Big(\bigvee_{i \in I} D_i\Big) = g\left(\widetilde{D}\right) = & \bigvee_{d \in \widetilde{D}} f(d) \\ & \leqslant q_0 = \bigvee_{i \in I} \bigvee_{d \in D_i} f(d) \\ & = \bigvee_{i \in I} g(D_i) \leqslant g\Big(\bigvee_{i \in I} D_i\Big) \end{aligned}$$

yielding $\bigvee_{i\in I} g(D_i) = g\Big(\bigvee_{i\in I} D_i\Big)$.

It is straightforward to check that $gr = f$. It remains to show that g is unique with that property. Suppose that $h : \mathscr{D}(A) \to \mathcal{Q}$ is a SupAlg-morphism such that $hr = f$. Then $h(a\!\downarrow) = f(a)$ for every $a \in A$. For any $D \in \mathscr{D}(A)$,

$$\begin{aligned}\bigvee_{d\in D}(d\!\downarrow) &= \bigcap\Big\{S \in \mathscr{D}(A) \;\Big|\; \bigcup_{d\in D} d\!\downarrow \subseteq S\Big\} \\ &= \bigcap\{S \in \mathscr{D}(A) \mid D \subseteq S\} = D,\end{aligned} \tag{4.2}$$

so

$$g(D) = \bigvee_{d\in D} f(d) = \bigvee_{d\in D} h(d\!\downarrow) = h\Bigg(\bigvee_{d\in D}(d\!\downarrow)\Bigg) = h(D).$$

This completes the proof. □

Example 4.4. In general, g is not an order-embedding when f is an order-embedding in Theorem 4.3.

Let $S = \{a, b, c\}$ be the posemigroup considered in Example 4.2. Then $Q = S^0$ with externally adjoined zero element 0 being the bottom element is also a posemigroup. If $f : S \longrightarrow Q$ is the inclusion mapping then

$$g(\{b, c\}) = f(b) \vee f(c) = f(a) = g(a\!\downarrow) = g(\{a, b, c\})$$

shows that g is not an order-embedding.

Proposition 4.5. SupAlg *is a reflective subcategory of* OAlg^* *with the reflector functor* $\mathscr{D} : \mathsf{OAlg}^* \to \mathsf{SupAlg}$ *defined by the assignment*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\;r_A\;} & \mathscr{D}(A) \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \mathscr{D}(f)} \\ \mathcal{B} & \xrightarrow{\;r_B\;} & \mathscr{D}(B) \end{array}$$

where $\mathscr{D}(f)(D) = j(f(D)\!\downarrow)$ *for every* $D \in \mathscr{D}(A)$ *and* $j : \mathscr{P}(B) \to \mathscr{P}(B)$ *is defined as in* (3.2).

Proof. By Theorem 4.3, we know that SupAlg is a reflective subcategory of OAlg^*. Hence the inclusion functor $\mathsf{SupAlg} \to \mathsf{OAlg}^*$ has a left adjoint functor $\mathscr{D} : \mathsf{OAlg}^* \to \mathsf{SupAlg}$ which, by [1, Proposition 4.22], can be described explicitly as follows. It maps an object $\mathcal{A}$ of the category OAlg^* to an object $\mathscr{D}(A)$ of SupAlg and a morphism $f : \mathcal{A} \to \mathcal{B}$ in OAlg^* to the unique morphism $\mathscr{D}(f) : \mathscr{D}(A) \to \mathscr{D}(B)$ in SupAlg such that the square

$$\begin{array}{ccc} A & \xrightarrow{\;r_A\;} & \mathscr{D}(A) \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \mathscr{D}(f)} \\ B & \xrightarrow{\;r_B\;} & \mathscr{D}(B) \end{array}$$

commutes. Using (4.2) we conclude that

$$\mathscr{D}(f)(D) = \mathscr{D}(f)\left(\bigvee_{d\in D} r_A(d)\right) = \bigvee_{d\in D} \mathscr{D}(f)\left(r_A(d)\right) = \bigvee_{d\in D} r_B\left(f(d)\right)$$
$$= \bigvee_{d\in D} f(d)\!\downarrow = j\left(\bigcup_{d\in D} f(d)\!\downarrow\right) = j(f(D)\!\downarrow),$$

for each $D \in \mathscr{D}(A)$. □

Data availability Data sharing not applicable to this article as datasets were neither generated nor analysed.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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