



Strongly minimal Steiner systems II: coordinatization and quasigroups

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Abstract. Each strongly minimal Steiner k -system (M, R) (where R is a ternary collinearity relation) can be ‘coordinatized’ in the sense of (Ganter–Werner 1975) by a quasigroup if k is a prime-power. We show this coordinatization is never definable in (M, R) and the strongly minimal Steiner k -systems constructed in (Baldwin–Paolini 2020) never interpret a quasigroup. Nevertheless, by refining the construction, if k is a prime power, in each $(2, k)$ -variety of quasigroups (Definition 3.10) there is a strongly minimal quasigroup that interprets a Steiner k -system.

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1. Introduction

Steiner Triple Systems are in a 1-1-correspondence with the so-called squags (Steiner quasigroups: groupoids satisfying the identities $xx = x$, $xy = yx$, $x(xy) = y$). With the help of this correspondence, many combinatorial properties of Steiner Triple Systems can be described in an algebraic language, and algebraic methods have successfully been applied.

Ganter and Werner [16, opening paragraph]

A *linear space* is collection of points and lines that satisfy a minimal condition to call a structure a geometry: two points determine a line. For us, a *Steiner k -system* is a linear space such that every line (block) has cardinality $k > 2$. A quasigroup is a structure with a single binary operation whose multiplication table is a Latin square (each row or column is a permutation of the universe).

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The main contribution of [16] is to generalize the correspondence they describe to Steiner q -systems when $q = p^n$ for some n and show these are the only Steiner k -systems in our sense that can be coordinatized. (They also classify the coordinatization of Steiner systems where a block is determined by k elements with $k > 2$.)

We combine methods developed to study \aleph_1 -categorical first order theories to construct new families of quasigroups with the methods of universal algebra [16] to study interpretability between (coordinatization of) Steiner systems and quasigroups. Section 2 lays out model theoretic background on the general Hrushovski method (Section 2.1), the special case that yields Steiner systems (Section 2.2), and the notion of interpretation (Section 2.3). In Section 3, we distinguish interpretation from coordinatization of Steiner systems (e.g. [16]) and show the strongly minimal Steiner systems are interpretable in incomplete theories of quasigroups but not conversely. In Section 4, we extend the Hrushovski technology to construct strongly minimal quasigroups. We conclude with universal algebraic questions which arise from this construction.

Steiner k -systems are generally considered in a 2-sorted vocabulary with sorts for points and lines; for reasons discussed in Remark 2.19, we use the bi-interpretable [3] setting of a single sorted structure (M, R) with one ternary ‘collinearity’ relation for both linear spaces and Steiner systems. Such mathematicians as Steiner, Bose, Skolem, and Bruck have established deep connections between the existence of a Steiner system with v points and blocks of size k and divisibility relations among k and v . This interaction with number theory is reflected in a line of work from the 1950–1980’s including [33, 18, 16, 14]. It culminates with the proof that Steiner k -systems are ‘coordinatized’ by varieties of quasigroups if and only if k is a prime power, q . We consider Steiner systems of every infinite cardinality with two contrasting results. Building on Fact 1.1, we prove Theorem 1.2 as Theorem 3.15 of this paper. Below, we label as ‘facts’ theorems from our earlier papers that are used here.

Fact 1.1. [3] *For each $k \geq 3$ there are uncountably many μ and an associated theory T_μ such that T_μ is the theory of a strongly minimal Steiner k -system (M, R) . μ is a function into the natural numbers counting the realizations of good pairs (Definition 2.6).*

The argument for the Theorem 1.2 (given in Section 3 is heavily based on Theorem 3.15 [16], where ‘coordinatized’ is a particular form of ‘interpreted’. Mikado varieties are described in Definition 3.10. Item 2) is immediate since the line length given a quasigroup must be a prime power (Lemma 3.11). The stronger (and much harder) result that no quasi-group can be interpreted in any of the constructed strongly minimal Steiner k -systems is reported in Fact 3.18.

Theorem 1.2. *If T_μ is the theory of a strongly minimal Steiner k -system (M, R) constructed as in [3], then*

- (1) *If k is a prime power q , for each Mikado $(2, q)$ -variety V , there is a quasigroup in V that interprets (M, R) with lines as 2-generated subalgebras and thus T_μ is interpreted in an incomplete theory $\tilde{T}_{\mu, V}$.*

- (2) If k is not a prime power the Steiner system does not have such an interpretation.
- (3) Unless $q = 3$, the interpreting quasigroup is not interpretable in (M, R) .

A complete first order theory is strongly minimal if every definable set in every model is finite or cofinite. Equivalently, in a strongly minimal theory T the model theoretic notion of algebraic closure determines a combinatorial geometry (matroid) with all bases automorphic. Model theorists say $a \in \text{acl}_M(B)$ if for some $\phi(x, \mathbf{b})$ with $\mathbf{b} \in B$, M satisfies both $\phi(a, \mathbf{b})$ and, for some k , $(\exists^{<k} x)\phi(x, \mathbf{b})$. We write bold face \mathbf{b} to indicate a finite sequence of elements, while b is a singleton. Zilber conjectured that these geometries were all disintegrated ($\text{acl}(A) = \bigcup_{a \in A} \text{acl}_M(a)$), locally modular (group-like), or field-like. The examples here modify Hrushovski's construction that refuted this conjecture [22]. A geometry is *flat* [22, Section 4.2] if the dimension of a closed subspace is determined from its own closed subspaces by the inclusion–exclusion principle [3, Definition 3.8]. Hrushovski's flat counterexamples have generally been regarded as an amorphous class of exotic structures. Indeed, a distinguishing characteristic is the inability to formally define an associative operation with infinite domain in any structure with a flat acl -geometry. Here, we show that non-associative does not mean uninteresting.

Although the Steiner systems in Theorem 1.2.(3) do not define a quasigroup, when q is a prime power we can find strongly minimal quasigroups that induce strongly minimal Steiner q -systems.

Theorem 1.3. *For each q and each of the T_μ in Theorem 1.2 with line length $k = q = p^n$ (for prime p) and each Mikado $(2, q)$ variety of quasigroups V , there is a strongly minimal theory of quasigroups, $T_{\mu', V}$ such that taking the 2-generated sub-quasigroups as lines yields a strongly minimal Steiner q -system.*

We explain in Section 4 how μ' is generated from μ and why the quasigroups satisfying $T_{\mu', V}$ are in V .

In [2], we investigate various combinatorial problems about the classes of quasigroups constructed here. In particular, we find strongly minimal Steiner triple systems (whose automorphism groups are two-transitive) of every infinite cardinality, and then easily deduce they have uniform cycle graphs [9], and further that are ∞ -sparse in the sense of [10]. We discuss in the introduction and Remark 5.27 of [3] and in [2] the connections of this work with, among others, Barbina–Casanovas, Conant–Kruckman, Horsley–Webb, and Hyttinen–Paolini [7, 11, 21, 23]. These works construct first order theories of Steiner systems or projective planes that are at the other end of the stability spectrum from those here. Evans [12] uses the Hrushovski construction to address combinatorial issues about Steiner systems.

This paper depends heavily on the results and notation of [3, 6]. Certain arguments will require consulting those papers.

2. Model theoretic preliminaries

In Section 2.1, we lay out the general pattern of a construction of a strongly minimal set by the ‘Hrushovski method’. Fraïssé and Jónsson generalized the Hausdorff notion of ‘universal’ linear orders to universally axiomatizable classes of structures. Hrushovski provided a ‘pre-processing’ for this technique that gives a general method for constructing theories of various model theoretic complexities. We specify here various refinements of his method that apply in universal algebra and combinatorics. We deal primarily with ‘ab initio’ constructions that begin with a collection of finite structures as opposed to expansions of structures (‘bad fields’) or fusions. The two page [3, Section 2.1] (arxiv) summarizes the role of strongly minimal sets in model theory and how strongly minimal Steiner systems arise.

The basic ideas of the Hrushovski method are:

i) Modify the Fraïssé construction of countable homogeneous-universal structures by replacing the relation of substructure between finite structures by a relation of strong substructure (\leq) defined using a pre-dimension function ϵ with $\epsilon(A) \in \mathbb{N}$ for each finite A in a specified class \mathbf{L}_0 .

ii) Employ a function μ to bound the number 0-primitive extensions of each finite structure to obtain a class (\mathbf{L}_μ, \leq) . Then apply [15] to that class so that closure in the geometry on the generic model is algebraic closure. Sections 2.1 and 2.2 are a series of definitions and results needed to apply the Hrushovski construction as modified in [3].

2.1. The Hrushovski method

In this section, we first describe this method axiomatically while listing the five kinds of parameters that must be specified for any particular family of constructions. We slightly generalize Hrushovski’s approach by using work of Kueker and Laskowski [25] to weaken the requirement imposed by both Fraïssé and the original Hrushovski constructions that the collection of finite structures is closed under substructure.

Definition 2.1 [25].

- (1) A countable collection (\mathbf{L}_0, \leq) of finite structures, with a transitive relation (\leq : *strong substructure*) on \mathbf{L}_0 that refines substructure, is *smooth* if $B \leq C$ implies $B \leq C'$ if $B \subseteq C' \subseteq C$.
- (2) Given a class of finite structures L_0 , \hat{L}_0 denotes the collection of structures of direct limits of members of L_0 .

Theorem 2.2 [25]. *If a smooth class satisfies the amalgamation and joint embedding properties there is a countable generic model \mathcal{G} (see Definition 2.7).*

The extension in [25] to an abstract treatment of a smooth class (L, \leq) includes the Hrushovski construction of strongly minimal sets since both the definitions of the class and the strong extension relation are by universal sentences. To my knowledge, the construction here of strongly minimal quasi-groups (Section 4) is the first place where a *AE*-axiomatizable smooth class is used to study strongly minimal sets.

In [1], we listed three of the major variants of the Hrushovski method as of 2010 and that number has at least doubled in the ensuing decade. Those variants range through the stability hierarchy and some involve infinitary logics. The *fine structure of the original method* [22] has been studied only in our recent work.

We now describe the general framework for several different constructions that appear in this paper; we use $(\sigma, \mathbf{L}_0^*, \epsilon, \mathbf{L}_0, \mathbf{U})$ to make clear *this context holds throughout the paper*. We replace \mathbf{L}, ϵ by \mathbf{K}, δ for the explicit cases and when the possibility of confusion is even stronger add further labels. Identifying the parameters of the method in Context 2.3 clarifies the relations among the variants as those parameters are instantiated differently at several points in this paper as well as in [3, 6, 2].

Context 2.3. *A Hrushovski sm-class depends on the choice of a quintuple $(\sigma, \mathbf{L}_0^*, \epsilon, \mathbf{L}_0, \mathbf{U})$ of parameters.*

- (1) *The vocabulary σ contains only relation and constant symbols.*
- (2) *\mathbf{L}_0^* is a countable \forall -axiomatizable collection of finite σ -structures.*
- (3) *A pre-dimension ϵ is a function from \mathbf{L}_0^* to the integers \mathbb{Z} that, with $A \subseteq B$, writing $\epsilon(A/B)$ for $\epsilon(A) - \epsilon(B)$ satisfies:*
 - (a) *ϵ is submodular: That is, if $A, B, C \subseteq D \in \mathbf{L}_0^*$, with $A \cap C = B$, then:*

$$\epsilon(A/B) \geq \epsilon(A/C),$$

which an easy calculation shows is equivalent to submodularity:

$$\epsilon(A \cup C) \geq \epsilon(A) + \epsilon(C) - \epsilon(B).$$

- (4) *\mathbf{L}_0 is a subset of \mathbf{L}_0^* defined using ϵ . Here, \mathbf{L}_0 is those $A \in \mathbf{L}_0^*$ such that for any subset A' of A , $\epsilon(A') \geq 0$.*

\mathbf{U} requires some preparation.

Requirement (3) that ϵ maps into \mathbb{Z} slightly weakens the result in Baldwin and Shi [5] that well-ordering of the range suffices to get an ω -stable generic model with a geometry rather than just a dependence notion. They show that by allowing real coefficients one obtains a stable theory with the forking relation as dependence. From such an ϵ , one defines notions of strong extension (\leq), primitive extension, and good pair.

Definition 2.4 (*Strong extensions*).

- (1) In any $N \in \hat{\mathbf{L}}_0$,

$$d_N(A/B) = \min\{\epsilon(A'/B) : B \subseteq A \subseteq A' \subseteq N\}.$$

We often write $d_N(A - B/B)$ for $d_n(A/B)$.

$$d_N(A) = d_N(A/\emptyset).$$

- (2) For any $N \in \hat{\mathbf{L}}_0$ with $B \subseteq N$, we write $B \leq N$ (read N is a *strong extension* of B) when $B \subseteq A \subseteq N$ implies $d_N(A) \geq d_N(B)$.
- (3) We write $B < A$ to mean that $B \leq A$ and B is a proper subset of A .

The following definitions describe the pairs $B \subseteq A$ such that, in the generic model constructed from the class \mathbf{L}_0 , A will be contained in the algebraic closure of B . We write $(A - B/B)$ or (A/B) for the same pair depending on whether the superset or the annulus is emphasized.

Definition 2.5 (*Primitive and good*). Let $B, C \in \mathbf{L}_0$ with $B \cap C = \emptyset$ and $C \neq \emptyset$. Write A for $B \cup C$.

- (1) A is a k -primitive extension of/over B if $B \leq A$, $\epsilon(A/B) = k$, and there is no A_0 with $B \subsetneq A_0 \subsetneq A$ such that $B \leq A_0 \leq A$. We may just write *primitive* when $k = 0$. We stress that in this definition, while B may be empty, A cannot be.
- (2) We say that the 0-primitive extension A/B is *good* if there is no $B' \subsetneq B$ such that (A/B') is 0-primitive. (Hrushovski called this a minimal simply algebraic or m.s.a. extension.)
- (3) If A is 0-primitive over B and $B' \subseteq B$ is such that we have that A/B' is good, then we say that B' is a *base* for A
- (4) If A/B is good, then we also write (A/B) is a *good pair*.

Hrushovski gave one technical condition on the function μ counting the number of realizations of a good pair that ensured the theory is strongly minimal rather than ω -stable of rank ω . Fixing a class \mathbf{U} of functions μ satisfying that condition in the base case and other conditions for special purposes provides a way to index a rich group of distinct constructions. At various times in this paper \mathbf{U} is instantiated as $\mathcal{U}, \mathcal{U}', \mathcal{C}$, or \mathcal{T} .

Definition 2.6 (μ and U). We describe the functions that impose algebraicity.

- (1) Let \mathbf{U} be collection of functions μ assigning to every isomorphism type β of a good pair (C/B) in \mathbf{L}_0 a non-negative integer.
- (2) For any good pair (C/B) with $B \subseteq M$ and $M \in \mathbf{L}_0$, $\chi_M(C/B)$ denotes the number of disjoint copies of C over B in M . Of course, $\chi_M(C/B)$ may be 0.
- (3) For any $\mu \in \mathbf{U}$, \mathbf{L}_μ is the class of structures M in \mathbf{L}_0 such that if (C/B) is a good pair $\chi_M(C/B) \leq \mu(C/B)$.

Definition 2.7. A countable structure \mathcal{G}_μ is *generic* for \mathbf{L}_μ if

- (1) it is a countable union of structures in \mathbf{L}_μ ;
- (2) and it is \leq -homogenous: if isomorphic finite A, B are each strong in \mathcal{G}_μ , they are automorphic in \mathcal{G}_μ .

Theorem 2.8. *If (\mathbf{L}_μ, \leq) is a smooth class with the amalgamation property then it has a countable generic model \mathcal{G}_μ .*

Smoothness is immediate for Hrushovski and for [3] as L_μ and \leq are given by universal sentences; Section 4 requires more care (Theorem 4.4) because the class of finite structures is $\forall\exists$ axiomatizable. Proofs that the theory of \mathcal{G}_μ is strongly minimal depend slightly on the particular instance of the schema described in this section; several such instantiations are explained in Sections 2.2 and 4.

Notation 2.9. The theory of the generic structure, \mathcal{G}_μ , is the desired strongly minimal theory T_μ .

2.2. Generic linear spaces

The construction of strongly minimal Steiner systems [3] takes place with the instantiation in Definition 2.12 for linear spaces of the pattern described in Context 2.3.

Formalizing the initial description,

Definition 2.10. A linear space is a structure (M, R) with a single ternary relation that is set-like (holds only of distinct tuples and in any order or none) and one further axiom: two points determine a line.

Definition 2.11. We say a *maximal* R -clique in a linear space M is a line (block) and sometimes write (partial) line for a clique that is not maximal. Note that if $B \subset A \subseteq M$ then a maximal clique in B may not be maximal in A .

- (1) Two unrelated points in a linear space (M, R) are regarded as being on a trivial line. A non-trivial line is any R -clique of at least 3-points. In a k -Steiner system, every line has $k > 2$ points and so is non-trivial.
- (2) For a line (maximal clique) $\ell \subseteq B$, we denote the cardinality of a line ℓ by $|\ell|$, and, for $A \subseteq B$, we denote by $|\ell|_A$ the cardinality of $\ell \cap A$.
- (3) We say that a non-trivial line ℓ contained in B is *based in* $A \subseteq B$ if $|\ell \cap A| \geq 2$, in this case we write $\ell \in L(B)$.
- (4) The *nullity of a line* ℓ contained in a linear space A is:

$$\mathbf{n}_A(\ell) = |\ell| - 2.$$

We deduce the notions of d_N, \leq , primitive, and good, exactly as in Section 2.1 from the following specification of $\sigma, \mathbf{L}_0^*, \delta$.

- Definition 2.12.**
- (1) $\sigma \rightarrow \tau$: τ has a single ternary relation, R .
 - (2) $\mathbf{L}_0^* \rightarrow \mathbf{K}_0^*$: \mathbf{K}_0^* is the class of finite linear spaces. In particular, R can hold only of three distinct elements and then in any order (i.e., is a 3-hypergraph).
 - (3) $\epsilon \rightarrow \delta$: $\delta(A) = |A| - \sum_{\ell \in L(A)} \mathbf{n}_A(\ell)$ where \mathbf{n}_A is defined in Definition 2.6.
 - (4) $\mathbf{L}_0 \rightarrow \mathbf{K}_0$: $\mathbf{K}_0 = \{A \in \mathbf{K}_0^* \text{ such that for any } A' \subseteq A, \delta(A') \geq 0\}$.
 - (5) $\mathbf{U} \rightarrow \mathcal{U}$:
 - (a) We write α for the isomorphism type of a pair of sets $(\{b_1, b_2\}, \{a\})$ with $R(b_1, b_2, a)$. (That is, rather than repeating the elements of the base by writing $(\{a, b_1, b_2\}/\{b_1, b_2\})$, we simply separate the two pieces of the diagram of the larger set.) $(\{b_1, b_2\}, \{a\})$ will be a good pair in each example considered.
 - (b) Let \mathcal{U} be the collection of functions μ assigning to every isomorphism type β of a good pair C/B in \mathbf{L}_0 :
 - (i) a natural number $\mu(\beta) = \mu(C/B) \geq \epsilon(B)$, if $|C - B| \geq 2$;
 - (ii) a natural number $\mu(\beta) \geq 1$, if $\beta = \alpha$.

The special treatment of α is to allow the consideration of Steiner 3-systems. Note that in Definition 2.12, the class \mathbf{K}_0^* is \forall -axiomatizable; in Section 4, we will need a $\forall\exists$ class for the relevant instantiation $\tilde{\mathbf{K}}_q$ of \mathbf{L}_0 .

[3, Lemma 3.10.3] demonstrates the class \mathbf{K}_0 satisfies amalgamation with the following construction of the canonical amalgam.

Definition 2.13 [3, Definition 3.7]. Let $A \cap B = C$ with $A, B, C \in \mathbf{K}_0$. We define $D := A \oplus_C B$ as follows:

- (1) the domain of D is $A \cup B$;
- (2) a pair of points $a \in A - C$ and $b \in B - C$ are on a non-trivial line ℓ' in D if and only if there is line ℓ based in C such that $a \in \ell$ (in A) and $b \in \ell$ (in B). Thus $\ell' = \ell$ (in D).

Baldwin and Paolini [3] demonstrate the class (\mathbf{K}_0, δ) satisfies the basic properties (including flatness) of a δ function in a Hrushovski construction and of the associated algebraic closure geometry. For $M \models T_\mu$, $a \in \text{acl}(B)$ if and only if $d_M(a/B) = 0$. The flatness implies that no model of T_μ (Fact 1.1) admits a definable binary associative function with infinite domain [22, Lemma 14].

The following lemma singles out the effect of the fact that our δ (Definition 2.12.(3)) depends on line length rather than the number of number of tuples realizing R .

Fact 2.14 (Line length). *By Lemma 5.18 of [3], lines in models of T_μ have length k if and only if $\mu(\alpha) = k - 2$.*

2.3. Interpretations

We carefully define the concept of interpretation as given in [20]. To bridge the several fields considered here we write these definitions in the notation of this paper. In Section 3 the Ganter–Werner notion of ‘coordinatizing’ Steiner systems by quasigroups is seen as a specific kind of interpretation.

By a *vocabulary* (alias: similarity type, language, signature) we mean a list of function and relation symbols. A boldfaced variable represents a finite sequence of variables. A formula is *unnested* if any atomic subformula of it is either a single occurrence of a relation symbol or an equality of terms containing only variables and at most one function symbol [20, p 58].

Definition 2.15 (*Interpretations*). Fix vocabularies τ and σ . Fix also a τ -structure B , a σ -structure A and a positive integer n . An *n-dimensional* interpretation Γ of B into A is:

- (1) (a) a σ -formula $\partial_\Gamma(x_0 \dots x_{n-1})$ (the domain of the interpreted model).
- (b) For each unnested atomic τ -formula $\phi(y_0, \dots, y_{m-1})$, a σ -formula $\phi_\Gamma(\mathbf{x})$ where \mathbf{x} is an m -tuple of n -tuples.
- (c) there is a surjective map $f_\Gamma: \partial_\Gamma(A^n) \rightarrow B$ such that for each unnested atomic τ -formula $\phi(\mathbf{y})$ and any $\mathbf{a}_i \in \partial_\Gamma(A^n)$

$$B \models \phi(f_\Gamma(\mathbf{a}_0), \dots, f_\Gamma(\mathbf{a}_{m-1})) \leftrightarrow A \models \phi(\mathbf{a}_0, \dots, \mathbf{a}_{m-1})$$

- (2) Note that 1a) and 1b) have established a function from the vocabulary of τ to formulas of σ ; this extends by inductions on formulas to arbitrary formulas (see [20, 5.3.2, Remark 1]). Such a map is called an interpretation of τ into σ .

- An interpretation Γ of a class \mathfrak{S} of τ -structures into a class V of σ -structures (or of the theories of such classes) is just an interpretation of τ into σ [20, (b) p 221]. This rather weak correspondence becomes useful when various properties in Definition 2.17 hold.

In this paper theory means first order theory; we will sometimes specify the type of theory with such terms as variety, complete, or strongly minimal.

The natural admissibility conditions (expressed by a set of first order formulas) assumed in Lemma 2.16.(2) are detailed at [20, page 214]; they allow for taking quotients in performing an interpretation and yield the following result [20, 5.3.2, 5.3.4]. We write \approx for isomorphism in the appropriate vocabulary.

Lemma 2.16 [20, Theorem 5.5.3]. *Suppose Γ is an n -dimensional interpretation of the τ -structure B into the σ -structure A .*

- (1) Definition 2.15.(1).(c) holds for all τ -formulas ϕ .
- (2) For every σ -structure A which satisfies the admissibility conditions there is a τ -structure B and a map $f = f_\Gamma$ with $f_\Gamma: \partial_\Gamma(A) \rightarrow B$ such that:
 - (a) If g and C are such that Γ with $g: \partial_\Gamma(A) \rightarrow C$ is also an interpretation of C into A then there is an isomorphism $i: B \rightarrow C$ such that $i(f\mathbf{a}) = g(\mathbf{a})$ for every $\mathbf{a} \in \partial_\Gamma(A^n)$.
 - (b) We write ΓA for the isomorphism class of $B \approx C$.

The interpretations constructed in this paper will be 1-dimensional, indeed on the same domain. But, Theorem 3.18 shows not even n -dimensional interpretations are possible in the other direction. Hodges introduced the following terminology to detail which additional properties of an interpretation were important for applications to decidability, consistency, model theoretic complexity, etc. In particular they will allow us in Section 3 to clarify the strength and weaknesses of coordinatization.

Definition 2.17 (*Properties of interpretations*). Let Γ be an interpretation of a class \mathfrak{S} of τ -structures into a class V of σ -structures.

- (1) Γ is *left total* if for every $A \in V$, $\Gamma A \in \mathfrak{S}$.
- (2) Γ is *right total* if for every $B \in \mathfrak{S}$, there is an $A \in V$ with $\Gamma A \approx B$.
- (3) Γ is *total* if it is both left and right total.

Definition 2.18 (*Relations of classes*).

- (1) Two classes of structures are mutually interpretable if each is interpreted in the other.
- (2) They are bi-interpretable if in addition the composition of the two interpretations (in either direction) is the identity.

Remark 2.19. (One sorted formalization) Since each sort is infinite, no theory in the two-sorted formulation of ‘linear space’ can be strongly minimal. However, we showed in Section 2 of [3] that there is a (2-dimensional) left and right total bi-interpretation between linear spaces in the two sorted formalization and linear spaces in a one-sorted logic with a single ternary ‘collinearity’ predicate. Fact 1.2.1 applied the Hrushovski method to linear spaces in the

one-sorted framework and using a geometrically motivated predimension function (Definition 2.12.(3)) produced strongly minimal Steiner k -systems that are model complete and satisfy the usual properties of counterexamples to Zilber's trichotomy conjecture. Their acl-geometries are flat, but not disintegrated nor locally modular.

3. Associating strongly minimal Steiner systems with quasigroups

We summarise and extend the substantial literature on coordinatization of k -Steiner systems to show T_μ is interpretable into a theory $\tilde{T}_{\mu,V}$ of quasigroups. We note in Lemma 3.6 that Steiner 3-systems are quasigroups. Then we give a short proof that the 'natural' coordinatizing quasigroup provided by [16] is not definable in the strongly minimal k -Steiner system (M, R) when $k > 3$. Thus, one can't 'invert' the coordinatization to obtain an interpretation of the quasigroup into the Steiner system.

Then we deduce from the argument of Baldwin–Verbovskiy [6] Theorem 3.18 that under a weak hypothesis (μ -triples, Definition 3.16) no binary function with domain M^2 is interpretable in any $(M, R) \models T_\mu$.

Notation 3.1. For each $k \in \omega$, \mathfrak{S}^k denotes the class of Steiner k -systems. If we write q , we mean a prime power.

[16] use the notation \mathfrak{S}_k^m ; for them m is block size and k is the number of points that determine a line. By restricting to linear spaces we have fixed that k as 2 and need only a single parameter, which we superscript.

Definition 3.2 [32].

- (1) A structure $(A, *)$ with one binary function $*$ is called a *groupoid* (or *magma*).
- (2) A *quasigroup* (Alias: multiplicative quasigroup [26], combinatorial quasigroup [32]) $(Q, *)$ is a groupoid $(Q, *)$ such that for $a, b \in Q$, there exist unique elements $x, y \in Q$ such that both

$$a * x = b \text{ and } y * a = b.$$

- (3) If every 2-generated subquasigroup has q -elements it is a *q -quasigroup*.

The general notion of a quasigroup is an *AE* Horn class in the vocabulary with function symbol $*$. So in general, a quotient or a subalgebra of a quasigroup $(Q, *)$ need not be a quasigroup. But Quackenbush provides a sufficient condition that every algebra in the variety generated by Q is a quasigroup.

Theorem 3.3 [30, Theorem 3]. *If $(Q, *)$ is a quasigroup with $V = (HSP(Q))$ and $F_2(V)$, the free V -algebra on 2 generators, is finite then every algebra in V is a quasigroup.*

Remark 3.4. Quackenbush's argument makes an interesting use of the finiteness of $F_2(V)$. By standard arguments $F_2(HSP(Q))$ is in $SP(Q)$ so it satisfies the cancellation laws. (Recall the operators H, S, P taking a class of algebras

\mathbf{K} to its homomorphic images, subalgebras, and direct products respectively.) Thus, left and right multiplication are injective. But since injective maps of finite sets are onto, each equation $ax = b$ has a solution and F_2 is a quasigroup. So we can treat the quasigroups that arise as structures with one binary operation and deal only with varieties. This allows us to apply directly the results of [3].

We discuss in detail three (families of) varieties of quasigroups corresponding to $k = 1, 2, p^n$.

Definition 3.5 [32].

- (1) A *Steiner quasigroup* is a groupoid which satisfies the equations: $x \circ x = x, x \circ y = y \circ x, x \circ (x \circ y) = y$.
- (2) A *Stein quasigroup* is a groupoid which satisfies the equations: $x * x = x, (x * y) * y = y * x, (y * x) * y = x$.
- (3) *block algebras*: [17] Let $q = p^n$ for some prime p and natural number n .
 - (a) A near-field is an algebraic structure in a vocabulary $(+, \times, 0, 1)$ satisfying the axioms for a division ring, except that it has only one of the two distributive laws.
 - (b) Given a near-field $(F, +, \cdot, -, 0, 1)$ of cardinality q and a primitive element $a \in F$, define a multiplication $*$ on F by $x * y = y + (x - y)a$. An algebra $(A, *)$ satisfying the 2-variable identities of $(F, *)$ is a *block algebra* [17] over $(F, *)$; $(F, *)$ is idempotent ($x * x = x$).

While every group is a quasigroup, the Stein and Steiner quasigroups are rather special quasigroups since they are idempotent. Thus, a Stein or Steiner quasigroup $(Q, *)$ cannot be a group unless it has only one element. Further, it is routine to check from the defining equations that there is a *unique* (up to isomorphism) 2-generated Steiner (Stein) quasigroup and it has 3 (4) elements. So it is necessarily both simple and free. Block algebras are quasigroups but when $q > 4$ neither Stein nor Steiner quasigroups.

Lemma 3.6. *Each Steiner triple system is bi-interpretable with a Steiner quasigroup (Definition 3.5).*

Proof. Given the algebra, the lines are the 2-generated subalgebras, which are easily seen from the defining equations to have cardinality 3. Given a Steiner triple system, let $x \circ y$ be the third element of the line if $x \neq y$ and $x \circ x = x$. Since all lines are isomorphic to the unique 3 element Steiner quasigroup, the resulting algebra is a Steiner quasigroup. □

In general, [3] gave us a theory T_μ of Steiner q -systems for each prime power q ; when $\mu(\alpha) = 1$ we get a 3-Steiner system and hence also a quasigroup.

Corollary 3.7. *For each prime power q , there are 2^{\aleph_0} strongly minimal theories T_μ of Steiner q -systems and so, when $\mu(\alpha) = 1$, non-isomorphic (and even not elementarily equivalent) quasigroups of cardinality \aleph_0 .*

Proof. Lemma 3.6 provides an explicit 1-dimensional (the domain and range of the interpretation is the universe) bi-interpretation between Steiner triple systems and the Steiner 3-systems from Theorem 1.2.1 [3, Corollary 5.23]. □

We examine the notion of coordinatization from Ganter and Werner [16, 17] and describe the methods of their proof of the coordinatization of \mathfrak{S}^q by varieties of quasigroups. We then adapt these methods in the remainder of the paper to analyze the possible interpretations between strongly minimal Steiner systems and quasigroups. We begin with a precise definition of ‘coordinatization’.

Definition 3.8 (*Coordinatization*) [16, 17].

- (1) A Steiner q -system (M, R) is *coordinatized by a quasigroup* $(M, *)$ if the lines of (M, R) are the 2-generated subalgebras of $(M, *)$.
- (2) The class \mathfrak{S}^q of Steiner systems is *coordinatized by a variety* V if each $(M, R) \in \mathfrak{S}^q$ is coordinatized by an $(M, *) \in V$.

The proof of Lemma 3.9 motivates the generalization in Lemma 3.14.

Lemma 3.9 (Stein quasigroups and Steiner 4 systems) [17, page 5]. *Each Stein quasigroup induces a Steiner 4-system (M, R) . Moreover, each Steiner 4-system (M, R) is coordinatized by a Stein quasigroup, $(M, *)$.*

Proof. One direction is obvious; the lines are the 2-generated subalgebras of the quasigroup, which, as noted after Definition 3.5, all have cardinality 4. For the other direction, the universe of the algebra is M . We noted above just before Lemma 3.6, that all Stein 4-quasigroups are isomorphic and strictly 2-transitive. So, if we arbitrarily impose the structure of a Stein 4-quasigroup on each line the entire structure is a Stein quasigroup. It clearly satisfies the three equations of Definition 3.5.2 because they involve elements only within a single block and also the requirement that each equation $ax = b$ ($ya = b$) has a unique solution, as again the solution is within the block determined by a, b . \square

Coordinatization as in Definition 3.8 does not necessarily give an interpretation. It does, if the variety V satisfies the stronger properties of an (r, k) -variety which we now give. [16] used these conditions to extend the coordinatization phenomena to Steiner q -systems for prime power q .

Definition 3.10 [16, 27].

- (1) The variety V is an (r, k) -variety if every r -generated subalgebra of any $A \in V$ is isomorphic to $F_r(V)$, the free V -algebra on r generators, and $|F_r(V)| = k$.
- (2) A variety V is *binary* [14] if both all function symbols of V are binary and the defining equations involve only 2 variables.
- (3) A q -Mikado variety [16, p. 129] is a binary $(2, q)$ -variety.

Once the statements are understood, the proofs of the following equivalences are straightforward.

Lemma 3.11 [27]. *For a variety V of groupoids, the following are equivalent:*

- (1) V is a $(2, k)$ -variety;
- (2) Taking the 2-generated subalgebras of any $A \in V$ as lines yields a Steiner k -system;

- (3) *the automorphism group of any 2-generated algebra is strictly (i.e. sharply) 2-transitive.*

Abusing notation, we say that an algebra A satisfying Lemma 3.11.(3) is strictly 2-transitive. And, as [35] points out, applying the strict two transitivity to $F_2(V)$, an argument of Burnside [8], [31, Theorem 7.3.1] shows:

Corollary 3.12. *If V is a $(2, k)$ -variety then k is a prime power.*

The salient characteristic (crucial for e.g. Lemma 3.13) of the equational theories that arise in this paper is that each defining equation involves only two variables. In particular, none of the varieties are associative. We rely heavily on a ‘classical’ observation of Trevor Evans. It requires no written proof, but a little thought.

Lemma 3.13 [34, 13]. *If V is a binary variety of idempotent algebras and each line of a Steiner system (M, R) is expanded to an algebra from V then the resulting algebra is in V .*

Theorem 3.14. *For any Mikado variety V of q -quasigroups and any $(M, *) \in V$, the Definition 3.8 coordinatization of a Steiner system (M, R) is an interpretation of (M, R) into $(M, *)$ and thus yields a left and right total interpretation \mathfrak{S}^q into V .*

Proof. Left total: $\Gamma_V((M, *))$ is the τ -structure (M, R) with universe M where R is defined as follows.

Let $\theta_F(x, y, z)$ be the disjunction of the atomic formulas $z = f_i(x, y)$ where the $f_i(x, y)$ list the terms generating $F = F_2(V)$ from $\{x, y\}$. Then letting $R(x, y, z) \leftrightarrow \theta_F(x, y, z)$ defines a relation R such that (M, R) is a Steiner q -system consisting of the 2-generated subalgebras of $(M, *)$.

Right total: Conversely, let $(M, R) \in \mathfrak{S}^q$. By Lemma 3.13, expanding (M, R) by placing an arbitrary copy of $F_2(V)$ as multiplication on each line gives an algebra $(M, *)$ in V with $\Gamma_V(M, *) \approx (M, R)$. □

The argument so far, with different terminology, is in [16]. We now explore the restrictions on the interpreting quasigroup if the Steiner system is required to satisfy a T_μ . As Ganter and Werner [17, p 7] point out, this interpretation into quasigroups is not unique. They describe two different varieties of block algebras (one commutative and one not) over F_5 , depending on the choice of the primitive element a of F_5 (Definition 3.5). Either can be used to coordinatize a Steiner 5-system. Thus the theory of the Steiner system (M, R) does not even predict the equational theory of the coordinatizing algebra and certainly does not control the first order theory. That is why we label the interpreting theory in Theorem 3.15: $\hat{T}_{\mu, V}$. Theorem 3.15.(2) proves the coordinatizing multiplication is not defined in the Steiner System; so, we cannot invert *this* interpretation.

In Lemma 3.18, we show the stronger result that there is no interpretation of any sort of a quasigroup in a model of a T_μ .

Theorem 3.15. *If T_μ is the theory of a strongly minimal Steiner q -system (from Theorem 1.2.1) and V is a Mikado $(2, q)$ variety of quasigroups, then*

- (1) *There is a left and right total interpretation of T_μ , a complete theory of Steiner q -systems, into $\hat{T}_{\mu,V}$, an incomplete theory of quasigroups.*
- (2) *If $\mu(\alpha) = q - 2 > 1$, the multiplication given by the coordinatization is not definable in (M, R) .*

Proof. 1) We verify that the coordinatization of Theorem 3.14 is an interpretation. Let $\delta_F(x, y, f_1(x, y), \dots, f_k(x, y))$ denote the quantifier-free diagram of $F = F_2(V)$. The strict 2-transitivity of F guarantees the particular choice of the two elements x, y does not matter. Letting R/δ_F denote the substitution of δ_F for R , $\hat{T}_{\mu,V}$ is axiomatized by

$$Eq(V) \cup \{(\forall x, y)\delta_F(x, y, f_1(x, y), \dots, f_k(x, y))\} \cup \{\phi \upharpoonright (R/\delta_F) : \phi \in T_\mu\}.$$

Left and right total are as proved in Theorem 3.14.

2) Without loss of generality, let (M, R) be the countable generic for T_μ and suppose it is coordinatized by $(M, *)$. In a linear space A , two points a, b such that no $c \in A$ satisfies $R(a, b, c)$ satisfy $d_A(\{a, b\}) = 2$. Let $\{a, b\}$ be a strong substructure of (M, R) (i.e. $d_M(\{a, b\}) = 2$; see Definition 2.4.(1)) and let c_1, \dots, c_k fill out the line through a, b to a structure A . By genericity there is a strong embedding of A into M . Since V is a Mikado variety, all triples a, b, c_i realize the same quantifier free R -type and $A \leq M$ implies for any permutation ν of k fixing $0, 1$, for $2 \leq i < k$, there is an automorphism of (M, R) fixing a, b and taking c_i to $c_{\nu(i)}$. Thus, $a * b$ cannot be definable in (M, R) . \square

We have found an interpretation of Steiner q -systems in q -quasigroups. We showed Theorem 3.15.(2) that the coordinatization does not yield an interpretation in the other direction. But as we sketch in 3.16–3.21, with a minor (triples) hypothesis on μ , Definition 3.16, there is no interpretation of any dimension of any quasi-group into T_μ , even if we allow a first order definition of the image of $*$. In fact, the multiplication $*$ cannot be defined in T_μ ; as, there are no non-trivial definable binary functions on models of T_μ .

Definition 3.16 (*Triples*). Let \mathbf{K}_0^* be the class of finite linear spaces as in Definition 2.11. Define \mathcal{T} as the collection of functions μ from good pairs into ω such that $\mu \in \mathcal{U}$ (Definition 2.12.b.) and such that for every good pair (A/B) with $|A| > 1$ and $\delta(B) = 2$, $\mu(A/B) \geq 3$. We will write ‘ μ triples’, [6, Theorem 5.2].

We say ‘triples’ is a weak condition because it addresses only good pairs A/B where B has small dimension. If μ is in the class \mathcal{T} of triplable μ -functions, [6] ensures that there are no definable truly binary functions in the following precise sense; several equivalents are given by [6, Lemma 2.10]. The following notion generalizes Gratzner’s notion of function being distinguished by one of its variables [19, p 201].

Definition 3.17 (*Essentially unary functions*). Let T be a strongly minimal theory. An \emptyset -definable function $f(x_0 \dots x_{n-1})$ is called *essentially unary* if there is an \emptyset -definable function $g(u)$ such that for some i , for all but a finite number of $c \in M$, and all but a set of Morley rank $< n$ of tuples $\mathbf{b} \in M^n$, $f(b_0, \dots, b_{i-1}, c, b_{i+1}, \dots, b_{n-1}) = g(c)$.

With Verbovskiy, we introduced the notion of a decomposition of finite G -normal subsets [6, Section 2] of Hrushovski strongly minimal sets with respect to automorphism groups to prove:

Fact 3.18 [6, Theorem 5.6]. *For any strongly minimal Steiner system (M, R)*

- (1) *If $\mu \in \mathcal{T}$ (μ -triples), every definable function in a model of T_μ is essentially unary.*
- (2) *If $\mu \in \mathcal{U}$, T_μ does not have any commutative definable binary function.*

Fact 3.18 is proved for the basic Hrushovski construction and strongly minimal Steiner systems in [6]. The crucial distinction between (1) and (2) in Fact 3.18 is that in (2) there may be a definable ‘truly’ binary function [6, Section 4.2] but it cannot be commutative.

We now show the versatility of the method of construction by finding Steiner systems which both do and don’t admit first order definable unary functions. Of course, since we are dealing with idempotent quasigroups they can’t produce algebraic terms for unary functions. But both the quasigroups and the Steiner systems might define unary functions with more complicated definitions. We repeat a short argument from [3] to motivate the argument of Proposition 3.21.

Theorem 3.19. *If $\mu \in \mathcal{U}$ and $\mu(\alpha) \geq 2$, i.e. lines have length at least 4, then there is a 12-element linear A that is 0-primitive over a singleton $a \in A$. Moreover, if $\mu(A/\{a\}) = 1$, then T_μ has a non-identity definable unary function.*

Proof. Let A be the τ -structure in $\mathbf{K}*_0$ with 12 elements, $\{a, b, c\} \cup \{d_i : 1 \leq i \leq 9\}$. Let η be the isomorphism type of the pair $(\{a\}, \{b, c\} \cup \{d_i : 1 \leq i \leq 9\}) = (\{a\}, A - \{a\})$ where R holds of (a, b, c) , (a, d_{2i+1}, d_{2i+2}) (for $0 \leq i \leq 3$), (b, d_{2i+2}, d_{2i+3}) (for $0 \leq i \leq 2$), (b, d_8, d_1) , and finally each triple from $\{c, d_8, d_9, d_4\}$. There are 12 points, nine 3-point line segments and one with 4 points so $\delta(A) = 12 - (9 + 2) = 1$. By inspection, each proper substructure A' has $\delta(A') \geq 2$ so A is 1-primitive over $\{a\}$. But d_9 is the unique point that is in exactly one clique within A . Thus, if $\mu(\eta) = 1$, the formula $(\exists x_1, x_2, y_1, \dots, y_8)\Delta(x_0, x_1, x_2, y_1, \dots, y_8, y_9)$ (where Δ is the quantifier free diagram of A) defines d_9 over any a in any model of T_μ (a determines d_9). Since each element in the generic \mathcal{G}_μ is embeddable in a copy $A \subseteq \mathcal{G}_\mu$, each model of T_μ has a global definable unary function. □

While we have given only one example, one can extend the length of the cycle and get infinitely many examples. Note that the construction in Theorem 3.19 is iterable so the definable closure *may not be locally finite*.

Recall \mathcal{U} is the set of μ such that for any good pair B/A , $\mu(B/A) \geq \delta(B)$. While the construction with \mathcal{U} as the class \mathbf{U} of admissible μ does not imply trivial unary closure (for any X , closure of X under definable unary functions is X), we can obtain triviality by taking the class \mathbf{U} of admissible μ as a \mathcal{C} which we now define. Nothing changes from the construction in Section 2.2 except now $\mathbf{U} \rightarrow \mathcal{C}$.

Definition 3.20. Define \mathcal{C} by restricting \mathcal{U} by requiring for each good pair (A/B) that if $|B| = 1$ and some point in A is determined by B (such as d_9 by a in η in Theorem 3.19), then $\mu(A/B) = 0$.

We used the cycle graphs of [9] to prove in [3, 4.11] that there are 2^{\aleph_0} distinct strongly minimal Steiner theories T_μ ; this proof remains valid if μ is restricted to \mathcal{C} or even $\mathcal{C} \cap \mathcal{T}$ (Definition 3.16). The slight variant on the proof of amalgamation to show $\mathcal{C} \cap \mathcal{T}$ is non-empty follows that in [2, Lemma 5.1.2].

Recall that \mathbf{K}_μ is the instantiation for \mathbf{L}_μ with \mathbf{L}_0 taken as \mathbf{K}_0 .

Proposition 3.21. *If $\mu \in \mathcal{C} \cap \mathcal{T}$, and \mathcal{G}_μ is the generic for \mathbf{K}_μ , for any $a \in M$, $\text{dcl}(a) = \{a\}$.*

Proof. Clearly amalgamation can not introduce unary functions so we have a generic \mathcal{G}_μ with no unary functions. By completeness this holds for all model of T_μ . □

Question 3.22. We have found a multiplication $*$ with domain M for each model (M, R) of T_μ . The particular $*$ depends on M , the choice of V and free choices made in the construction (i.e. on an enumeration η of M).

- (1) Are all the $(M, R, *)$ (for the same V) elementarily equivalent? in the same equationally complete variety? Each is in a subvariety of V .
- (2) Do they represent continuum many distinct varieties? I.e, are the classes $HSP(\mathcal{G}_\mu)$ distinct for (sufficiently) distinct μ ?
- (3) What can be said about the model theoretic complexity of (completions of) the various $\hat{T}_{\mu,V}$?

4. Constructing strongly minimal quasigroups

We have shown that in general the strongly minimal Steiner k -systems in the vocabulary $\tau = \{R\}$ for $k > 3$ do not define quasigroups. More precisely by Corollary 3.12 and Lemma 3.11 if a k -quasigroup is definable in a model of T_μ , then $k = q = p^n$ for some n .

Working in a vocabulary $\tau' = \{R, *\}$ we will construct a generic structure that is both a Steiner system (M, R) and a quasigroup $(M, *)$. We require that the $*$ -algebra be in a given $(2, q)$ -variety V (Definition 3.10) that coordinatizes (M, R) . Then taking the reduct of $(M, R, *)$ to the vocabulary containing only $*$ we have a strongly minimal quasigroup with a flat acl-geometry.

Recall $\mu(\alpha) = q - 2$ implies the maximal cliques in \mathbf{K}_μ have at most q elements; in the generic model they all have q . We next ensure this maximality condition holds on each finite structure by restricting to a smaller class of τ -structures, $\hat{\mathbf{K}}_q$. There remain trivial lines (2 unrelated points which are thus in no clique.).

To construct a strongly minimal quasigroup, we modify the setting of Definition 2.12, where the basic parameters $\mathbf{L}_0^*, \epsilon, \mathbf{L}_\mu, U$ became $\mathbf{K}_0^*, \delta, \mathbf{K}_\mu, \mathcal{U}$ to construct the theory T_μ .

Definition 4.1. Fix $\mu \in \mathcal{U}$ with $\mu(\alpha) = q - 2$ and a $(2, q)$ -variety V of quasigroups.

- (1) Working with τ -structures:
 - (a) $\mathbf{L}_0^*, \epsilon, \mathbf{L}_0 \rightarrow \mathbf{K}_0^*, \delta$, and \mathbf{K}_0 are exactly as in Section 2.2.
 - (b) A new intermediate step. Let $\tilde{\mathbf{K}}_q \subseteq \mathbf{K}_\mu$ be the class of finite τ -structures (A, R) such that each maximal clique has q -elements. The restriction to ‘full lines’ is expressed by a single $\forall\exists$ τ -sentence.
- (2) $\sigma \rightarrow \tau'$: Expand $\tau = \{R\}$ to τ' by adding a ternary relation symbol H .
 - (a) $\mathbf{L}_0 \rightarrow \mathbf{K}'_\mu$: Let \mathbf{K}'_μ be the finite τ' -structures A' such that $A' \upharpoonright \tau \in \tilde{\mathbf{K}}_q$ and $A' \upharpoonright H$ is the graph of $F_2(V)$ on each line.
 - (b) $\epsilon \rightarrow \delta'$: For any $A' \in \mathbf{K}'_\mu$, let $\delta'(A') = \delta(A' \upharpoonright \tau)$. Define \leq' from δ' as usual. Note that each non-trivial line in A' has q elements.
 - (c) $\mathbf{U} \rightarrow \mathbf{U}'$: See Definition 4.3.(3).(b).

Construction 4.2. For $q > 3$ a prime power, fix μ with $\mu(\alpha) = q - 2$ and V as Definition 4.1. For clarity, we label the τ' -structures in this argument with primes.

We construct from $C \in \mathbf{K}_\mu$ a finite set of structures $C'_i \in \mathbf{K}'_\mu$ (for $i < r_C$). First, there is a canonical extension of C to $\tilde{C} \in \tilde{\mathbf{K}}_q$. Since there is a strong embedding of C into the generic \mathcal{G}_μ , there is a structure $\tilde{C} \subseteq \mathcal{G}_\mu$ whose universe consists of extensions of each clique of length at least 3 in C to have length q ; but with no new intersections. The extensions exist because each line \mathcal{G} has length q . And there are no intersections since $C \leq \mathcal{G}$. Thus \tilde{C} is a partial Steiner system in \mathbf{K}_μ .

Now we construct a finite family of τ' -expansions of \tilde{C} , $\langle C'_i : i < r_C \rangle$ by imposing on each non-trivial line $\ell \subset \tilde{C}$ a copy of $F_2(V)$ with graph $H \upharpoonright \ell$. Since C uniquely determines \tilde{C} , we denote these expansions by C'_i (rather than \tilde{C}'_i). There are finitely many non-isomorphic choices (depending on the interaction of H and R) to impose the $*$ -structure on \tilde{C} ; r_C is chosen to list all of them. By Lemma 3.13 they all are in \mathbf{K}'_0 (satisfy the condition on δ) and $C'_i \upharpoonright \tau = \tilde{C}$ for each i .

Since δ' ignores $*$, (A'/B') is a good pair for \mathbf{K}'_μ if the τ -reduct of (A'/B') is a good pair for \mathbf{K}_0 .

Definition 4.3 (*\mathbf{K}' good pairs and μ'*).

- (1) Let α' denote the τ' -isomorphism type $(a_1, a_2, b_1, \dots, b_{k-2})$ of a q -element line over two points; (\mathbf{b}/\mathbf{a}) is good with respect to \mathbf{K}'_0 .
- (2) For a fixed \mathbf{K}_0 -good pair (A/B) other than α , let $\langle \gamma'_i : i < r_{A/B} \rangle$ for some $r_{A/B} \in \omega$, be a list of the isomorphism types of \mathbf{K}'_μ -good pairs (A'/B') whose τ -reduct is (A/B) (with isomorphism type γ).
- (3) For any isomorphism type of a good pair γ' in τ'
 - (a) $\mu'(\alpha') = 1$
 - (b) $\mathbf{U} \rightarrow \mathbf{U}'$: Let \mathbf{U}' be the collection of μ' such that for any other γ'_i (with $\gamma = \gamma'_i \upharpoonright \tau$)

$$\mu'(\gamma_i) = \mu(\gamma).$$

- (c) $\mathbf{L}_\mu \rightarrow \mathbf{K}'_{\mu'}$: Let $\mathbf{K}'_{\mu'}$ be the class of structures D' in \mathbf{K}'_0 such that if (A'/B') is a good pair, then $\chi_{D'}(A'/B') \leq \mu'(A'/B')$.

Note $\mu'(\alpha')$ was forced to be 1. As, when the τ -reduct is a line of length q over a two-point base, since two points determine a line and V is a $(2, q)$ variety, all quasigroups on the line are isomorphic. Since, except for α' , the various copies of each good pair have the same reduct to τ but may differ in their quasigroup structure, each good pair $\gamma = A/B$ in \mathbf{K}_μ has generated a finite number of distinct good pairs in $\mathbf{K}'_{\mu'}$. With this framework in hand we can complete the proof of Theorem 4.4. As the notation indicates, the variety V has not changed. But because \tilde{C} does not expand uniquely to a τ' -structure $\mu'(\gamma')$, in addition to having a different domain will have a larger value than μ of the engendering γ . We show how to apply the proofs of the crucial results 5.11 and 5.15 from [3] for this result.

Theorem 4.4. *For each q and each $\mu \in \mathcal{U}$, each of the T_μ in Theorem 1.2 with line length $k = q = p^n$ (for some n), and any Mikado $(2, q)$ variety (block algebras) of quasigroups V (i.e. block algebras), there is a strongly minimal theory of quasigroups $T_{\mu',V}$ that defines a class of strongly minimal Steiner q -systems. The associated quasigroups are not commutative.*

Proof. Choose $\mu \in \mathcal{U}'$ as in Definition 4.3. We can construct a generic, provided we prove amalgamation for $\mathbf{K}'_{\mu'}$. We now show that the amalgamation for the τ -class, as in Lemma 5.11 and Lemma 5.15 of [3] yields an amalgamation for τ' . Consider a triple D', E', F' in $\mathbf{K}'_{\mu'}$ with $D' \subseteq F'$ and with E' 0-primitive over D' . We put primes on the labels in [3] and omit the primes for the reducts to τ .

Apply Lemma 5.11 and Lemma 5.15 of [3] to obtain a τ -structure G that solves the amalgamation in \mathbf{K}_μ . *A priori* G might not be the domain of a structure in $\tilde{\mathbf{K}}_q$. However, since E is primitive over D , although there may be a line contained in the disjoint amalgam G with two points in each of D and $F - D$, each line that contains 2 points in $E - D$ can contain at most one from D . Thus the expansion \tilde{G} from Construction 4.2 is $\tilde{F} \oplus_D \tilde{E}$. (The tilde is intentionally omitted from D .) And the τ' -amalgam G' is obtained for \tilde{G} as in Construction 4.2. Since the definition of strong extension is by omitting specified configurations, \mathbf{K}'_μ has amalgamation and is smooth. So there is a generic by Theorem 2.2. Now the strong minimality of the generic $\mathcal{G}'_{\mu',V}$ follows exactly as in Lemmas 5.21 and 5.23 of [3] and, letting $T_{\mu',V} = \text{Th}(\mathcal{G}'_{\mu',V})$, we have proved Theorem 4.4. By Fact 3.18.(2), the quasigroup cannot be commutative. □

Necessarily in the construction given, a good pair (C/B) (other than α) of τ -structures in the reduct of a model of $T_{\mu',V}$ will have many (but finitely) more copies of C over B than $\mu(C/B)$. Thus, $T_{\mu',V} \upharpoonright \tau$ is not T_μ . But it is strongly minimal since there are fewer τ -definable than τ' -definable sets and each is finite or cofinite.

Remark 4.5. The reduct of $\mathcal{G}'_{\mu',V}$ to $\{H\}$ is a strongly minimal block algebra, in particular, a quasigroup. For ease of reading we give the definition R in the algebraic form of the reduct to H :

$$R(u, v, w) \leftrightarrow \bigvee_{\sigma \in F_2(V)} w = \sigma(u, v).$$

The theory of that reduct is essentially $T_{\mu',V \upharpoonright H}$.

Notation 4.6. Let $V' \subseteq V$ be the variety of quasigroups induced by $T_{\mu',V \upharpoonright H}$. For simplicity below, we will write $*$ for the multiplication with graph H on models of $T_{\mu',V}$.

A line of papers [4, 24, 28] study the model theoretic properties of free algebras that are saturated or in particular categorical in their cardinality. Although the strong minimality of the quasigroup constructed in Theorem 4.4 implies all these model theoretic conditions, we show it cannot be free. We rely on [2, Lemma 4.4.2], which we rephrase (and slightly correct from the 1st arxiv version) to use the notation here.

Fact 4.7. *If M is a strongly minimal set constructed by the methods of [3], $A \leq M$, and $|M - A|$ is infinite then there are infinitely many elements of M that are $*$ independent over A .*

Theorem 4.8. *The reduct $(M, *)$ of a model $(M, *, R)$ of $T_{\mu',V}$ is never V' -free on infinitely many generators.*

Proof. Suppose that $(M, *)$ is a free V' -algebra with an infinite basis Y . Then every permutation of Y extends to an automorphism of M , so Y is an infinite set of indiscernibles in M . Let X be an acl-basis of the strongly minimal set $(M, *)$. Then X is also an infinite set of indiscernibles. But it is immediate from strong minimality that there cannot be two distinct Ehrenfeucht–Mostowski types (for each k , the first order formulas that hold for each k tuple from the set of indiscernibles) over the empty set realized by such sequences. However, the basis Y cannot be algebraically independent in the model theoretic sense. If it were Y would realize the same k -types as X for each k and thus $Y \leq M$. But then by Fact 4.7, M has infinite V' dimension over Y , so Y is not a V' basis. □

Question 4.9. The use of the graph of the quasigroup in Construction 4.2 is similar to that in the study of model complete Steiner triple system of Barbina and Casanovas [7]. As noted in Remark 5.27 of [3], their generic structure M differs radically from ours: for them, $\text{acl}_M(X) = \text{dcl}_M(X)$, the theory is at the other end of the stability spectrum, and the generic model is atomic rather saturated.

Is it possible to develop a theory of q -block algebras for arbitrary prime powers similar to that for Steiner quasigroups with $q = 3$ in their paper? That is, to find a model completion for each of the various varieties of block algebras discussed in Definition 3.5.3?

[16, Theorem 4.6] asserts that every (m, q) -variety has the finite embedding property. Thus, the Fraïssé construction of a model completion should be immediate. *Where do the resulting theories lie in the stability classification?*

Question 4.10. We guaranteed that the quasigroup $\mathcal{G}_\mu \upharpoonright \{*\}$ is in V ; but it may satisfy more equations. Different varieties of quasigroups may have the same free algebra on two generators. Construction 4.2 depends on both the original \mathbf{K}_μ^q and $F_2(V)$. How many varieties can arise from the same $F_2(V)$? There are two variants on this question. One is, ‘how many varieties of quasigroup can have the same free algebra on two generators?’. The second asks about only varieties that arise from choosing a μ and a variety V as in Construction 4.2.

How do those varieties that the $\mathcal{G}_{\mu',V}$ generate behave? Immediately from known results each such variety satisfies the strong properties listed below.

Corollary 4.11. *For any model M of any $T_{\mu',V}$ the reduct of M to $*$ is in a variety that is congruence permutable, regular and uniform, [29, Theorem 3.1] or [16, Corollary 2.4].*

Question 4.12. Every finite algebra in a $(2, q)$ -variety has a finite decomposition into directly irreducible algebras [16, Corollary 2.4]. *Are there any similar results for infinite strongly minimal block algebras?*

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Declarations

Conflict of interest This paper is the work of the author. The author has no competing interests to declare that are relevant to the content of this article.

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