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Varieties of ordered algebras as categories

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Abstract. A variety is a category of ordered (finitary) algebras presented by inequations between terms. We characterize categories enriched over the category of posets which are equivalent to a variety. This is quite analogous to Lawvere's classical characterization of varieties of ordinary algebras. We also study the relationship of varieties to discrete Lawvere theories, and varieties as concrete categories over **Pos**.

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1. Introduction

Classical varieties of (finitary) algebras were characterized in the pioneering dissertation of Lawvere [13] as precisely the categories of models of Lawvere theories, see Section 5. Lawvere further characterized varieties as the categories with effective equivalence relations which have an abstractly finite, regularly projective regular generator. In [1] we have simplified this: a category is equivalent to a variety iff it has

(1) reflexive coequalizers and kernel pairs, and
(2) an abstractly finite, effectively projective strong generator G.

Effective projectivity means that the hom-functor of G preserves reflexive coequalizers—we recall this in Section 2. Abstract finiteness is a condition much weaker than finite generation (see Example 3.20). We have also presented another characterization: varieties are precisely the free completions of duals of Lawvere theories under sifted colimits [4].

The aim of our paper is to present a categorical characterization of varieties of ordered algebras. These are classes of ordered Σ-algebras (for finitary

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signatures Σ) that are specified by inequations between terms. Example: ordered monoids, or ordered monoids whose neutral element is the least one. Our characterization of categories equivalent to varieties of ordered algebra turns out to be analogous to the conditions (1) and (2) above, but we have to work in the realm of *categories enriched over* **Pos**, the (cartesian closed) category of posets and monotone maps. That is, a category is always equipped with a partial order on every hom-set, and composition is monotone. The concepts used by Lawvere need to be modified accordingly. Whereas in ordinary category one works with regular epimorphisms (coequalizers of parallel pairs) we work with *subregular epimorphisms* $e\colon X \to Y$ which are the coinserters of parallel pairs (i.e. there exist $u_0, u_1\colon U \to X$ such that e is universal w.r.t. $eu_0 \leq eu_1$). This leads to a modification of effective projectivity: we call G subeffective projective if its hom-functor into **Pos** preserves reflexive coinserters. Our main result characterizes varieties of ordered algebras via strong generators that are abstractly finite subeffective projectives.

Power introduced discrete Lawvere theories [16] that, for categories enriched over **Pos**, we recall in Section 5. We prove that varieties of ordered algebras are precisely the enriched models of discrete Lawvere theories. From this Kurz and Velebil [11] derived that they are precisely the free completions of duals of discrete Lawvere theories under (enriched) sifted colimits; we present a simplified proof in Section 5.

We can also view a variety $\mathcal{V}$ of ordered algebras as a concrete category by considering its forgetful functor $U\colon \mathcal{V} \to$ **Pos**. Using the above results, we derive a characterization of varieties as concrete categories in Section 6.

Related Work. Varieties of (possibly infinitary) ordered algebras were studied already in the 1970's by Bloom [5], and they were characterized as concrete categories by Bloom and Wright [6]. In Section 6 we show that the characterization in the latter paper, when restricted to finitary signatures, is closely related to our main result. However, we have not found an easy way of deriving one of those results from the other one.

Kurz and Velebil published more recently a fundamental paper on this topic [11] in which the main subject is the exactness property for categories enriched over **Pos**. The definition in [11] is quite natural, but rather involved, based on the technical concept of congruence (Definition 3.8 in loc.cit.). For exact enriched categories the characterization of finitary varieties in their Theorem 5.9 is essentially the same as in our main result, Theorem 4.7 below. So the main message of our paper is that one does not need exactness to characterize varieties of ordered algebras as abstract categories. (The exactness is 'condensed' into properties of the given generator.) Kurz and Velebil also proved that finitary varieties of ordered algebras are precisely the monadic categories over **Pos** for strongly finitary monads; see [2] for a simplified proof.

2. Lawvere's characterization of varieties

We recall shortly two of the major results of Lawvere's famous dissertation: a characterization of categories equivalent to varieties in the classical sense (classes of Σ-algebras presented by equations) and algebraic theories as a syntax for varieties. Varieties are considered as categories with homomorphisms as morphisms.

Let Σ be a finitary signature. Given a variety $\mathcal{K}$ of Σ-algebras, every set generates a free algebra of $\mathcal{K}$. Denote by G the free algebra on one generator. Then G is

(a) *abstractly finite* which means that all copowers $M.G = \coprod_M G$ (M a set) exist and every morphism $f\colon G \to M.G$ factorizes through a finite subcopower $M'.G \hookrightarrow M.G$ ($M' \subseteq M$ finite),
(b) a *strong generator with copowers*, i.e., all copowers $M.G$ exist and every object X is an extremal quotient of a copower via the canonical map $c_X = [f]\colon \coprod_{f\colon G\to X} G \to X$, and
(c) a *regular projective*, i.e., $\mathcal{K}(G, -)$ preserves regular epimorphisms.

Theorem 2.1 (Lawvere [13], Theorem 2.1). *A category is equivalent to a variety of finitary algebras iff it has*

(1) *coequalizers and finite limits,*
(2) *effective equivalence relations (i.e. every equivalence relation is a kernel pair of its coequalizer), and*
(3) *a regular generator which is an abstractly finite regular projective.*

Remark 2.2. 'Coequalizers' in Condition (1) are missing in [13]. This seems to be just a typo: at the end of the proof of Theorem 2.1 Lawvere forms a coequalizer $\overline{\overline{r}}$ of a parallel pair without commenting why it exists.

There is another important property of G related to *reflexive coequalizers.* A pair $u, v\colon X \to Y$ is called reflexive if it consists of split epimorphisms with a joint splitting $d\colon Y \to X$ ($u \cdot d = v \cdot d = \mathrm{id}$). A reflexive coequalizer is a coequalizer of a reflexive pair. Whereas in varieties coequalizers are not **Set**-based in general, reflexive coequalizers are, see [4]. It then follows that G has the following property introduced by Pedicchio and Wood [15]:

Definition 2.3. An object is called an *effective projective* if its hom-functor preserves reflexive coequalizers.

Effective equivalence relations can be deleted from the above theorem, provided that in Condition (3) we replace regular projective by effective projective. This was observed by Pedicchio and Wood [15]. In [1] a full proof of the following modified theorem is presented:

Theorem 2.4. *A category is equivalent to a variety of finitary algebras iff it has*

(1) *reflexive coequalizers and kernel pairs, and*
(2) *a strong generator which is an abstractly finite effective projective.*

A pioneering result of Lawvere's thesis was a characterization of varieties as categories of models of algebraic theories. Let us recall this.

Notation 2.5. Denote by

$$\mathcal{N}$$

the full subcategory of **Set** on all natural numbers $n = \{0, \dots, n-1\}$. We have a canonical strict structure of finite coproducts in $\mathcal{N}$: given natural natural numbers n and k we equip $n+k$ with the injections $n \to n+k$ given by $i \mapsto i$, and $k \mapsto n+k$ given by $j \mapsto n+j$.

Definition 2.6. An *algebraic theory* is a small category $\mathcal{T}$ whose objects are natural numbers and having finite products together with a functor $I\colon \mathcal{N}^{\mathrm{op}} \to \mathcal{T}$ which is identity on objects and strictly preserves finite products.

The category $\mathbf{Mod}\mathcal{T}$ of *models* has as objects functors $A\colon \mathcal{T} \to \mathbf{Set}$ preserving finite products, and as morphisms natural transformations.

Theorem 2.7 (Lawvere). *Varieties are, up to equivalence, precisely the categories of models of algebraic theories.*

Whereas limits in a variety $\mathcal{V}$ are computed on the level of **Set** (indeed, they are preserved by the forgetful functor $U\colon \mathcal{V} \to \mathbf{Set}$), colimits in general are not. However, U preserves directed colimits and (as remarked above) reflexive coequalizers. These two types of colimits are generalized as follows:

Definition 2.8 ([4]). A small category $\mathcal{D}$ is called *sifted* if for all diagrams $D\colon \mathcal{D} \to \mathbf{Set}$ in **Set** colimits commute with finite products.

A *sifted colimit* in a category is a colimit of a diagram whose domain is a sifted category.

Example 2.9. Both directed colimits and reflexive coequalizers are sifted colimits. And these two types are, in a way, exhaustive. For example, if $\mathcal{K}$ and $\mathcal{L}$ are categories with colimits, then a functor $F\colon \mathcal{K} \to \mathcal{L}$ preserves sifted colimits iff it preserves filtered colimits and reflexive coequalizers, see [4].

Notation 2.10. For every category $\mathcal{K}$ denote by $\mathbf{Sind}\mathcal{K}$ the free completion under sifted colimits: given a category $\mathcal{L}$ with sifted colimits, every functor $F\colon \mathcal{K} \to \mathcal{L}$ has an extension $F'\colon \mathbf{Sind}\mathcal{K} \to \mathcal{L}$ preserving sifted colimits, unique up to a natural isomorphism.

Theorem 2.11 ([4]). *Varieties are up to equivalence precisely the categories $\mathbf{Sind}\mathcal{T}^{\mathrm{op}}$ for algebraic theories $\mathcal{T}$.*

Remark 2.12. In [4] an object A of a category $\mathcal{K}$ is called *perfectly presentable* if $\mathcal{K}(A, -)$ preserves sifted colimits. If $\mathcal{K}$ is a variety, these are precisely the retracts of free finitely generated algebras of $\mathcal{K}$. Moreover, a full subcategory $\mathcal{T}$ of $\mathcal{K}$ representing all perfectly presentable objects (up to isomorphism) is an algebraic theory with $\mathbf{Mod}\mathcal{T}^{\mathrm{op}}$ equivalent to $\mathcal{K}$.

3. Generators

Assumption 3.1. From now on we work with categories enriched over the (cartesian closed) category **Pos**. Thus a category $\mathcal{K}$ is understood to have partially ordered hom-sets and composition is monotone. Also 'functor' means automatically an enriched functor, i.e., one monotone on hom-sets. Natural transformations in the enriched sense are just the ordinary ones.

Every set is considered as the (discretely ordered) poset.

Remark 3.2. Recall the concept of a *coinserter* of a parallel (ordered) pair of morphisms u_0, $u_1 : U \to X$: it is a morphism $f : X \to Y$ universal w.r.t. $fu_0 \leq fu_1$. That is, given $f' : X \to Y'$ with $f'u_0 \leq f'u_1$, then (a) there exists $g : Y \to Y'$ with $f' = gf$ and (b) for every $\bar{g} : Y \to Y'$ from $gf \leq \bar{g}f$ it follows that $g \leq \bar{g}$.

Definition 3.3. A morphism is called a *subregular epimorphism* if it is a coinserter of a parallel pair.

Example 3.4. (1) Every subregular epimorphism $f : A \to B$ is an epimorphism – indeed, it has the stronger property that for parallel pairs $u_1, u_2 : B \to C$ we have $u_1 \leq u_2$ iff $u_1 f \leq u_2 f$.

If a category has finite copowers, every regular epimorphism $f : A \to B$ is subregular: if f is the coequalizer of $u, v : C \to A$, then it is the coinserter of $[u, v]$, $[v, u] : C + C \to A$.

(2) In **Pos** subregular epimorphisms are precisely the epimorphisms, i.e., the surjective morphisms. See Proposition 4.4 for a more general statement.

(3) If If $e = qp$ is a subregular epimorphism, then so is q. Indeed, let u_0, u_1 be a parallel pair with coinserter e. It is easy to verify that q is a coinserter of pu_0 and pu_1.

Definition 3.5. By a *subkernel pair* of a morphism $f : X \to Y$ is meant a parallel pair u_0, $u_1 : U \to X$ universal w.r.t. $fu_0 \leq fu_1$. That is:

(1) every pair v_0, $v_1 : V \to X$ with $fv_0 \leq fv_1$ factorizes as $v_i = u_i \cdot k$ for some $k : V \to U$, and
(2) whenever $\bar{k} : V \to U$ fulfils $u_i k \leq u_i \bar{k}$ for $i = 0, 1$, then $k \leq \bar{k}$.

Example 3.6. (1) In **Pos** the subkernel pair of $f : X \to Y$ is the pair of projections of the subposet U of $X \times X$ on all (x_0, x_1) with $f(x_0) \leq f(x_1)$.

(2) Every subregular epimorphism $f : X \to Y$ is the coinserter of its subkernel pair. Indeed, let f be the coinserter of v_0, $v_1 : V \to X$, and let k be the factorization above. If $g : X \to Z$ fulfils $gu_0 \leq gu_1$, then $gv_0 = gu_0 k \leq gu_1 k = gv_1$, thus, g factorizes uniquely through f.

Definition 3.7. Let K be an object of an (enriched) category $\mathcal{K}$. A *tensor* $P \otimes K$ for a poset P is an object of $\mathcal{K}$ such that for every object X of $\mathcal{K}$ we have an isomorphism

$$\mathbf{Pos}\big(P, \mathcal{K}(K, X)\big) \simeq \mathcal{K}(P \otimes K, X) \tag{3.1}$$

in **Pos**, natural in $X \in \mathcal{K}$. We say that K *has tensors* if $P \otimes K$ exists for every poset P.

Example 3.8. (1) In **Pos**, $P \otimes K$ is just the categorical product $P \times K$.

(2) In a variety $\mathcal{K}$ of ordered algebras, let K be the free algebra on one generator. Then $P \otimes K$ is the free algebra on P. Indeed, for every algebra X in $\mathcal{V}$ the poset $\mathcal{K}(K, X)$ is precisely the underlying poset $|X|$ of X, thus the left hand side of (3.1) consist of all monotone functions from P to $|X|$. And they naturally correspond to homomorphism from $P \otimes K$ to X.

Notation 3.9. The isomorphism (3.1) is denoted by

$$\frac{P \xrightarrow{\ f\ } \mathcal{K}(K, X)}{P \otimes K \xrightarrow[\ \hat{f}\ ]{} X}$$

Example: given an object K, then for each object X the identity of $\mathcal{K}(K, X)$ yields the *canonical morphism*

$$c_X = \widehat{\mathrm{id}} \colon \mathcal{K}(K, X) \otimes K \to X\,.$$

The following is an 'inverse' example: we define

$$\eta_P \colon P \to \mathcal{K}(K, P \otimes K) \quad \text{by} \quad \widehat{\eta}_P = \mathrm{id}_{P \otimes K}\,.$$

Example 3.10. For a natural number n, considered as the discrete poset $\{0, \ldots, n-1\}$, we have a copower

$$n \otimes K = \coprod_n K\,.$$

We denote it by

$$n.K\,.$$

Given n morphisms $f_i \in \mathcal{K}(K, X)$, the corresponding map $f \colon n \to \mathcal{K}(K, X)$ yields

$$\hat{f} = \widehat{[f_i]} \colon n.K \to X\,.$$

Proposition 3.11 ([7], Propositions 6.5.5 and 6.5.6). *Let G be an object with tensors. Then the hom-functor*

$$\mathcal{K}(G, -) \colon \mathcal{K} \to \ \boldsymbol{Pos}$$

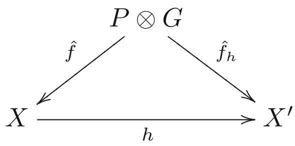
has the following enriched left adjoint

$$- \otimes G \colon \ \boldsymbol{Pos} \to \mathcal{K}\,.$$

It assigns to every monotone map $p \colon P \to Q$ the morphism $p \otimes G$ corresponding to $P \xrightarrow{p} Q \xrightarrow{\eta_Q} \mathcal{K}(X, Q \otimes X)$:

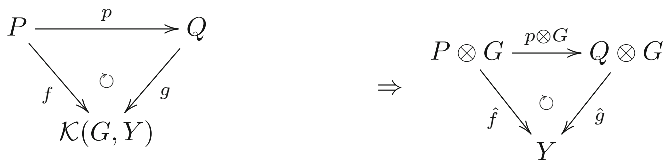
$$p \otimes G = \widehat{\eta_Q \cdot p} \colon P \otimes X \to Q \otimes X\,.$$

Remark 3.12. (1) The naturality of $f \mapsto \hat{f}$ in Notation 3.9 means that this map is monotone $\big(f \le g \colon P \to \mathcal{K}(G, X)$ implies $\hat{f} \le \hat{g}\big)$ and for every morphism $h \colon X \to X'$ we have a commutative triangle



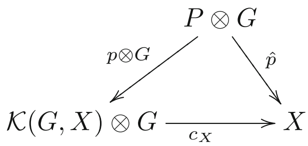
where $f_h \equiv P \xrightarrow{f} \mathcal{K}(G, X) \xrightarrow{h\cdot(-)} \mathcal{K}(G, X')$

(2) For every monotone map $p\colon P \to Q$ in **Pos** the following implication holds:



This follows from (1) since $p \otimes G = \widehat{\eta_Q \cdot p}$ and for $f = \eta_Q \cdot p$ we get $f_{\hat{g}} = g \cdot p = f$.

(3) The canonical morphism makes for every monotone map $p\colon P \to \mathcal{K}(G, X)$ the following triangle commutative



This follows from (1): for $f = \eta_{\mathcal{K}(G,X)} \cdot p$ we get $\hat{f} = p \otimes G$ by Proposition 3.11. Since $\hat{c}_X = \mathrm{id}$, we have $f_{c_X} = p$.

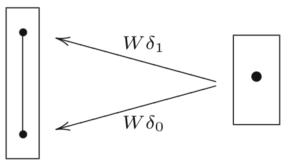
Remark 3.13. (1) Recall from [9] the concept of a *weighted colimit* in $\mathcal{K}$. Given a diagram $D\colon \mathcal{D} \to \mathcal{K}$ and a weight $W\colon \mathcal{D}^{\mathrm{op}} \to$ **Pos**, a weighted colimit is an object C of $\mathcal{K}$ with an isomorphism

$$\mathcal{K}(C, X) \cong [\mathcal{D}^{\mathrm{op}}, \mathcal{K}]\big(W, \mathcal{K}(D-, X)\big)$$

natural in $X \in \mathcal{K}$. The object C is also denoted by $\mathit{colim}_W D$.

Example: tensor $P \otimes K$ is a colimit of the diagram $D\colon 1 \to \mathcal{K}$ representing K weighted by $W\colon 1 \to$ **Pos** representing P.

(2) Another example: a coinserter of $u_0,\ u_1\colon U \to X$ is the colimit of the diagram where $\mathcal{D}$ is given by a parallel pair (δ_0, δ_1) to which D assigns (u_0, u_1), and W assigns the following monotone maps



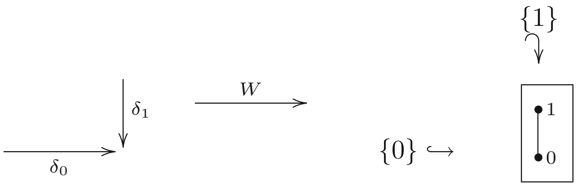
(3) Every left adjoint preserves weighted colimits ([7], Proposition 6.7.2).

(4) The dual concept is a *weighted limit*. Given a diagram $D\colon \mathcal{D} \to \mathcal{K}$ and a weight $W\colon \mathcal{D} \to$ **Pos**, a weighted limit is an object L with an isomorphism

$$\mathcal{K}(X, L) \simeq [\mathcal{D}, K]\big(W, \mathcal{K}(X, D-)\big)$$

natural in X.

(5) Example: the subkernel pair of $f\colon X \to Y$ (Definition 3.5) is the weighted limit where $\mathcal{D}$ is a cospan δ_0, δ_1 to which D assigns $D\delta_0 = D\delta_1 = f$ and $W\colon \mathcal{D} \to$ **Pos** assigns the embeddings of $\{0\}$ and $\{1\}$ to the chain $0 < 1$, respectively:



Example 3.14. (1) Every poset P is a *canonical coinserter* of the projections π_0, $\pi_1\colon P^{(2)} \to |P|$, where $|P|$ is a discrete poset underlying P and $P^{(2)}$ is the discrete poset of all comparable pairs in $P \times P$. More precisely, the following is a coinserter:

$$P^{(2)} \underset{\pi_0}{\overset{\pi_1}{\rightrightarrows}} |P| \xrightarrow{k_p} P$$

where k_p is carried by $\mathrm{id}_{|P|}$.

(2) Consequently, for every object G we have the corresponding coinserter:

$$P^{(2)} \otimes G \underset{\pi_0 \otimes G}{\overset{\pi_1 \otimes G}{\rightrightarrows}} |P| \otimes G \xrightarrow{k_p \otimes G} P \otimes G$$

Indeed, $- \otimes G$ preserves coinserters since they are weighted colimits, see 3.13(3).

(3) A *reflexive coinserter* is a coinserter of a reflexive pair u_0, u_1 (which means that u_0, u_1 are split epimorphisms with a joint splitting). Observe that the coinserters in (1) and (2) are reflexive: use the diagonal map $|P| \to P^{(2)}$.

Notation 3.15. (1) Given a full subcategory $\mathcal{A}$ of $\mathcal{K}$ (in the enriched sense: the ordering of hom-sets is inherited from $\mathcal{K}$), we denote by

$$E\colon \mathcal{K} \to \mathbf{Pos}^{\mathcal{A}^{\mathrm{op}}}$$

the functor assigning to an object K the restriction of $\mathcal{K}(-, K)\colon \mathcal{K}^{\mathrm{op}} \to$ **Pos** to $\mathcal{A}^{\mathrm{op}}$.

(2) In particular, if $\mathcal{A}$ consists of a single object G, then $\mathbf{Pos}^{\mathcal{A}^{\mathrm{op}}}$ is the category of posets with a (monotone) action of the ordered monoid $\mathcal{K}(G, G)^{\mathrm{op}}$. Morphisms are the monotone equivariant maps. Here the functor E assigns to K the poset $\mathcal{K}(G, K)$ with the action corresponding to $u\colon G \to G$ given by precomposition with u.

Definition 3.16. A morphism $f\colon X \to Y$ is called

(a) an *embedding* if given morphisms $u_0, u_1 \colon U \to X$ we have $u_0 \le u_1$ iff $fu_0 \le fu_1$

and

(b) an *extremal epimorphism* if whenever it factorizes through an embedding $m \colon Y_0 \to Y$, then m is invertible.

In ordinary categories an object G is called a strong generator provided that every monomorphism $m \colon X \to Y$ such that

$$m.(-) \colon \mathcal{K}(G, X) \to \mathcal{K}(G, Y)$$

jection is invertible. In case G has copowers, this is equivalent to each canonical map $c_K \colon \coprod_{f \colon G \to K} G \to K$ being an extremal epimorphism (one not factorizing through a proper subobject of K). Here is the enriched version:

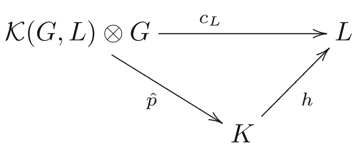
Definition 3.17 (Kelly [9]). An object G of K is a *strong generator* provided that the functor E of Notation 3.15(1) is conservative: a morphism m of $\mathcal{K}$ is invertible iff Em is.

Proposition 3.18. *Let G be a generator with tensors. Then it is a strong generator iff all canonical maps*

$$c_K \colon \mathcal{K}(G, K) \otimes G \to K \qquad (K \in \mathcal{K})$$

are extremal epimorphisms.

Proof. (1) Sufficiency. We are to prove that given a morphism $h \colon K \to L$ with Eh invertible, then h is invertible. By assumption we have a monotone map $p \colon \mathcal{K}(G, L) \to \mathcal{K}(G, K)$ inverse to $h.(-)$. Since G is a generator, this implies that h is an embedding: given u_0, u_1 in $\mathcal{K}(G, K)$ then $hu_0 \le hu_1$ iff $u_0 \le u_1$. Thus all we need to verify is that c_L factorizes through h. By Remark 3.12(1) the composite of $\hat{p} \colon \mathcal{K}(G, L) \otimes G \to K$ and h is $\hat{p}_h$, where $p_h \colon \mathcal{K}(G, L) \to \mathcal{K}(G, L)$ sends $u \colon G \to L$ to $h\big(p(u)\big) = u$. Thus $p_h = \mathrm{id}_{\mathcal{K}(G,L)}$. By definition $c_L = \widehat{\mathrm{id}}_{\mathcal{K}(G,L)}$. We conclude that the triangle below commutes



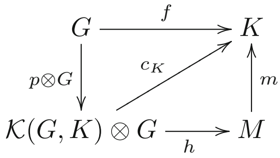
Therefore h is invertible.

(2) Necessity. If G is a strong generator, and if

$$c_K = mh \quad \text{for an embedding} \quad m \colon M \to K,$$

we are to prove that m is invertible. By assumtion, we just need to prove that $Em = m.(-) \colon \mathcal{K}(G, M) \to \mathcal{K}(G, K)$ is invertible. Every morphism $f \colon G \to K$ yields a unique (monotone) map $p \colon 1 \to \mathcal{K}(G, K)$ with $f = \hat{p}$ (using $1 \otimes G = G$).

The following diagram commutes:



due to Remark 3.12 (3). We thus obtain a mapping

$$d\colon \mathcal{K}(G,K) \to \mathcal{K}(G,M)\,, \quad d(f) = h(p \otimes G)\,.$$

The above diagram yields $Em \cdot d = \mathrm{id}$. Since m is an embedding in $\mathcal{K}$, Em is one in **Pos**. Thus $Em = d^{-1}$. □

For the following definition we recall the concept of a *slice category* $\mathcal{A}/K$. Let $\mathcal{A}$ be a subcategory of $\mathcal{K}$ and $K \in \mathcal{K}$. The objects of $\mathcal{A}/K$ are the morphisms $a\colon A \to K$ with $A \in \mathcal{A}$ in $\mathcal{K}$. Morphisms to another object $a'\colon A' \to K$ are the morphisms $f\colon A \to A'$ of $\mathcal{A}$ with $a = fa'$, and their ordering is inherited from $\mathcal{K}$. We have the forgetful functor $D\colon \mathcal{A}/K \to \mathcal{K}$ given by $D(A,a) = A$.

Definition 3.19. A full subcategory $\mathcal{A}$ of $\mathcal{K}$ is called *dense* if it satisfies one of the conditions below—they are equivalent by [9], Theorem 5.1:

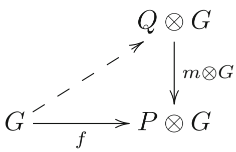
1. The functor $E\colon \mathcal{K} \to \mathbf{Pos}^{\mathcal{A}^{\mathrm{op}}}$ of Notation 3.15 is full and faithful ($f \le g$ iff $Ef \le Eg$ for parallel pairs f, g).

2. (a) For every object $K \in \mathcal{K}$ and every cocone of $D\colon \mathcal{A}/K \to \mathcal{K}$ with codomain L, there exists a morphism $k\colon K \to L$ such that the given cocone is $(k \cdot a)_{(A,a)\in\mathcal{A}/K}$, and

(b) given a morphism $\bar{k}\colon K \to L$ with $k \cdot a \le \bar{k} \cdot a$ for all (A,a), it follows that $k \le \bar{k}$.

The concept of an abstractly finite object (Section 2) has the following variant in enriched categories.

Definition 3.20. An object G is *abstractly finite* if it has tensors $P \otimes G$ (P a poset) and every morphism $f\colon G \to P \otimes G$ factorizes through a finite subtensor: we have a commutative triangle



for some finite subposet $m\colon Q \hookrightarrow P$.

Example 3.20. (1) A poset is abstractly finite in **Pos** (see the beginning of Section 2) iff it has only finitely many connected components. Thus there exist abstractly finite posets of an arbitrarily large cardinality.

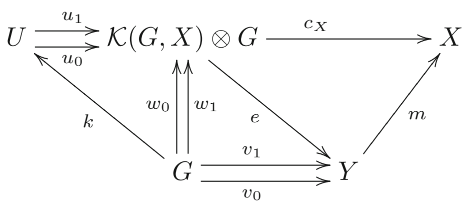
(2) A free algebra of a variety on an abstractly finite poset is abstractly finite.

(3) Every finitely generated object G in a category $\mathcal{K}$ is abstractly finite (but not conversely, as we have seen). Indeed, a copower $P.G$ with P infinite is a directed colimit of all subcopowers $Q.G$ with $Q \hookrightarrow P$ finite and nonempty. Given two such objects $Q.G$ and $Q'.G$, the connecting morphism, in case $Q \leq Q'$, is $i.G$ for the inclusion $i\colon Q \hookrightarrow Q'$. Since i is a split monomorphism in **Set**, $i.G$ splits in K. Thus $\mathcal{K}(G,-)$ preserves the above directed colimit, proving that G is abstractly finite.

Definition 3.21. An object G is *subregularly projective* if its hom-functor $\mathcal{K}(G,-)\colon \mathcal{K} \to$ **Pos** preserves subregular epimorphisms. That is, given a subregular epimorphisms $e\colon X \to Y$, every morphism from G to Y factorizes through e.

Lemma 3.22. *Let $\mathcal{K}$ be a category with subkernel pairs and reflexive coinserters. For every subregularly projective strong generator G with copowers all canonical maps (Notation 3.9) are subregular epimorphisms.*

Proof. Let u_0, u_1 be a subkernel pair of c_X:



Since u_0, u_1 is obviously reflexive, we can form its coinserter e, and we get a unique $m\colon Y \to X$ with $c_X = me$. Our task is to prove that m is invertible. Since c_X is an extremal epimorphism (Proposition 3.18) we just need to prove that m is an embedding. As G is a generator, this amounts to showing that given v_0, $v_1\colon G \to Y$ with $mv_0 \leq mv_1$, it follows that $v_0 \leq v_1$. Since G is projective w.r.t. subregular epimorphisms, there exist morphisms $w_i\colon G \to \mathcal{K}(G,X) \otimes G$ with $v_i = e \cdot w_i$ $(i = 0,1)$. From $c_X u_0 \leq c_X u_1$ we conclude $c_X w_0 \leq c_X w_1$ (using that $c_X w_i = mew_i = mv_i$). Therefore, there exists $k\colon G \to U$ with $w_i = u_i k$. This proves $v_0 \leq v_1$ as desired: $v_i = ew_i = eu_i k$. □

Theorem 3.23. *Let $\mathcal{K}$ be a category with subkernel pairs and reflexive coinserters. If G is an abstractly finite subregularly projective strong generator, then the full subcategory of all finite copowers $n.G$, $n \in \mathbb{N}$, is dense.*

Proof. (1) Let $\mathcal{A}$ denote the full subcategory of $\mathcal{K}$ on all $n.G = \coprod_n G$, $n \in \mathbb{N}$. By Remark 3.19 (1), we are to prove that given objects K and L and a cocone of the canonical diagram $\mathcal{A}/K \to \mathcal{K}$ with codomain L, notation

$$\frac{n \otimes G \xrightarrow{\;f\;} K}{n \otimes G \xrightarrow[\;f'\;]{} L} \qquad (n \in \mathbb{N})$$

then (2a) in 3.19 holds: there exists a morphism

$$k\colon K \to L \quad \text{with} \quad f' = kf \quad \text{for all } f\colon n.G \to K.$$

Property (b) in 3.19 follows from (1). Given $\bar{k}$ with $k \cdot f \leq \bar{k} \cdot f$ for all $f\colon G \to K$ (we can restrict ourselves to $n = 1$), it follows that $k \cdot c_K \leq \bar{k} \cdot c_K$, use Remark 3.12(4). Thus $k \leq \bar{k}$, since c_K is a subregular epimorphism.

(2) We extend $(-)'$ to all finite tensors of G. Given a finite poset P and a morphism $f\colon P \otimes G \to K$ we define $f'\colon P \otimes G \to L$ as follows. We use the coinserter of Remark 3.14 (2):

$$\begin{array}{ccccc} P^{(2)}.G & \overset{\pi_1.G}{\underset{\pi_0.G}{\rightrightarrows}} & |P| \otimes G & \xrightarrow{k_p \otimes G} & P \otimes G \\ & & & \underset{f \cdot k_p \otimes G}{\searrow} & \downarrow f \\ & & & & K \end{array} \tag{a1}$$

(For simplifying our notation we assume that $\otimes$ binds stronger that composition. Thus $f \cdot (k_p \otimes G)$ is written as $f \cdot k_p \otimes G$.) For $f \cdot k_p \otimes G$ we already have, since $|P| \otimes G = |P|.G$ the corresponding morphism $(f \cdot k_p \otimes G)'\colon |P| \times G \to L$. We define f' by verifying the following inequality

$$(f \cdot k_p \otimes G)' \cdot \pi_0 \otimes G \leq (f \cdot k_p \otimes G)' \cdot \pi_1 \otimes G\,. \tag{a2}$$

Thus, $(f \cdot k_p \otimes G)'$ factorizes uniquely through the coinserter $k_p \otimes G$. Then $f'\colon P \otimes G \to L$ is defined as that factorization:

$$\begin{array}{ccc} P \otimes G & \xrightarrow{f'} & L \\ {\scriptstyle k_p \otimes G}\uparrow & \nearrow{\scriptstyle (f \cdot k_p \otimes G)'} & \\ |P| \otimes G & & \end{array} \tag{a3}$$

To verify (a2), use that since $(-)'$ is a cocone, it is monotone ($f \leq g$ implies $f' \leq g'$) and for every morphism $d\colon n.G \to m.G$ of $\mathcal{A}$ we have the following implication

$$\begin{array}{ccc} n.G & \xrightarrow{d} & m.G \\ {\scriptstyle f}\searrow & \leq & \swarrow{\scriptstyle g} \\ & K & \end{array} \quad \Rightarrow \quad \begin{array}{ccc} n.G & \xrightarrow{d} & m.G \\ {\scriptstyle f'}\searrow & \leq & \swarrow{\scriptstyle g'} \\ & L & \end{array} \tag{a4}$$

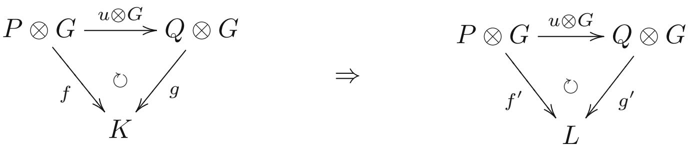
Applying this to $d = \pi_k.G$, for $k = 0, 1$, we get

$$\begin{aligned} (f \cdot k_p \otimes G)' \cdot (\pi_k.G) &= (f \cdot k_p \otimes G \cdot \pi_k \otimes G)' \\ &= (f \cdot [k_p \cdot \pi_k] \otimes G)'\,. \end{aligned}$$

Since $(-)'$ is monotone, and $k_p \cdot \pi_0 \le k_p \cdot \pi_1$ implies $[k_p \cdot \pi_0] \otimes G \le [k_p \cdot \pi_1] \otimes G$ (because $- \otimes G$ is locally monotone by Proposition 3.11), we get

$$\begin{aligned}(f \cdot k_p \otimes G)' \cdot \pi_0 \otimes G &= (f \cdot [k_p \cdot \pi_0] \otimes G)' \\ &\le (f \cdot [k_p \cdot \pi_1] \otimes G)' \\ &= (f \cdot k_p \otimes G)' \cdot \pi_1 \otimes G\,.\end{aligned}$$

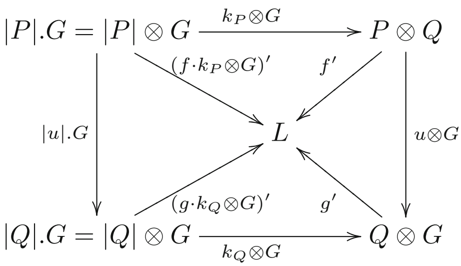
(3) Given a monotone map $u\colon P \to Q$ we prove the following implication



Since k_P and k_Q are carried by identity maps, we have

$$u \cdot k_P = k_Q \cdot |u|\colon |P| \otimes Q \to Q \otimes G\,.$$

Consider the following diagram

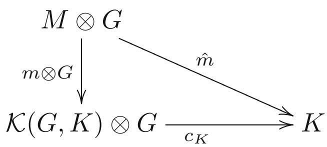


The square commutes because k_P and k_Q are carried by identity maps, and the upper and lower triangles commute by (a3). The left-hand triangle commutes since (a4) yields for $d = |u| \otimes G$ the following equality

$$\begin{aligned}\big(g \cdot k_Q \otimes G\big)' \cdot \big(u.G\big) &= \big(g \cdot k_Q \otimes G \cdot |u| \otimes G\big)' \\ &= \big(g \cdot u \otimes G \cdot k_P \otimes G\big)' \\ &= \big(f \cdot k_P \otimes G\big)'\,.\end{aligned}$$

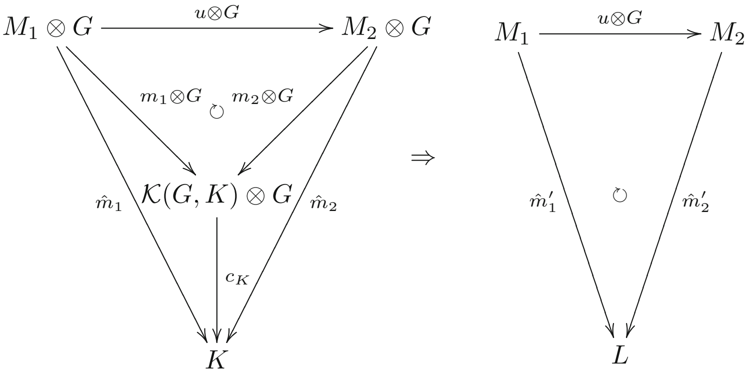
Thus the right-hand triangle commutes when precomposed by $k_P \otimes G$. Since $k_P \otimes G$ is an epimorphism (see Proposition 3.11), $f' = g' \cdot u \otimes G'$.

(4) It follows from Remark 3.12(3) that the canonical morphism c_K fulfils, for every finite subposet $m\colon M \hookrightarrow \mathcal{K}(G, K)$, that the following triangle commutes:



(a5)

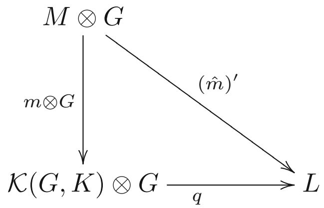
Since $\mathcal{K}(G,K)$ is a directed colimit of its finite subposets in **Pos**, we conclude (from Proposition 3.11 again) that all $m \otimes G$ form a colimit cocone in $\mathcal{K}$. The morphisms $\hat{m}'\colon M \otimes G \to L$ form a cocone of that diagram. Indeed, given $M_1 \subseteq M_2 \subseteq \mathcal{K}(G,K)$ we denote by $u\colon M_1 \to M_2$ the inclusion map and derive $\hat{m}_1' = \hat{m}_2' \cdot u \otimes G$ from (2):



Thus there exists a unique morphism

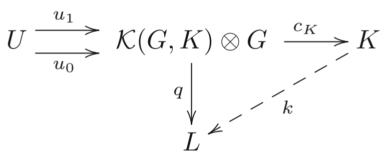
$$q\colon \mathcal{K}(G,K) \otimes G \to L$$

making the following triangles commutative



(a6)

for all finite subposets $m\colon M \hookrightarrow \mathcal{K}(G,K)$. By Lemma 3.22 we can express c_K as a coinserter of a parallel pair u_0, u_1:



In the next point we prove that $qu_0 \leq qu_1$. Thus q factorizes as $k \cdot c_K$. Then k is the desired morphism: we prove

$$f' = k \cdot f \quad \text{for all} \quad f\colon n \otimes G \to K\,.$$

It is sufficient to verify this for $n = 1$ since for general $f = [f_0, \ldots, f_n]$ we have $f' = [f_0', \ldots f_n']$ (apply (a4) to the coproduct injections $u\colon G \to n \otimes G$) and thus $f_i' = k \cdot f_i$ imply $f' = k \cdot f$.

Given $f\colon G \to K$ we have the subposet $m\colon \{f\} \hookrightarrow \mathcal{K}(G,K)$ with $\hat{m} = f$, for which (a6) yields

$$q \cdot m \otimes G = f' .$$

Since $q = k \cdot c_K$ and $c_K \cdot m \otimes G = \hat{m} = f$ by (a5), we get

$$k \cdot f = f' .$$

(5) It remains to verify $q \cdot u_0 \le q \cdot u_1$. Since G is a generator, this is equivalent to

$$q \cdot u_0 \cdot r \le q \cdot u_1 \cdot r \quad \text{for all} \quad r\colon G \to U .$$

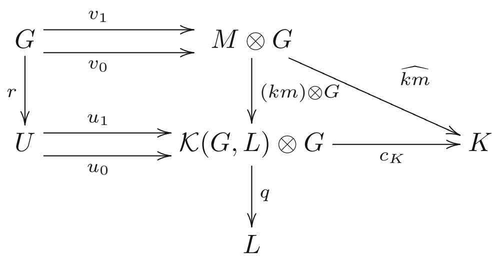
For the poset $P = \mathcal{K}(G,K)$ we have the morphism $k\colon |P| \to P$ carried by the identity map. It is a subregular epimorphism in **Pos** (Example 3.4), thus, due to Proposition 3.11 $k \otimes G\colon |P| \otimes G \to P \otimes G$ is a subregular epimorphism in $\mathcal{K}$. Since G is a subregular projective, we have factorizations

$$u_i r = (k \otimes G) u_i' \quad \text{for} \quad u_i'\colon G \to |P| \otimes G \qquad (i = 0,1) .$$

Moreover, G is abstractly finite, thus there exists a finite subset $m\colon M \to |P|$ with factorizations

$$u_i' = (m \otimes G).v_i \quad \text{for} \quad v_i\colon G \to M.G \qquad (i = 0,1) .$$

In other words: v_i is a factorization of $u_i r$ through $(km) \otimes G$:



We know that $c_k u_0 r \le c_k u_1 r$, and this by (a5) impllies

$$\widehat{km} v_0 \le \widehat{km} v_1 .$$

From (a4) we then obtain $\widehat{km}' v_0 \le \widehat{km}' v_1$, thus,

$$q u_0 r = \widehat{km}' v_0 \le \widehat{km}' v_1 = q u_1 r .$$

□

Remark 3.24. (1) A full subcategory $\mathcal{K}$ of $\mathcal{L}$ is called reflective if the embedding $\mathcal{K} \hookrightarrow \mathcal{L}$ has a left adjoint. Suppose that $\mathcal{L}$ has weighted limits and colimits, then so does every full reflective subcategory ([9], Section 3.5). This is in particular the case if $\mathcal{L}$ is the functor category $[\mathcal{A}^{\mathrm{op}}, \mathbf{Pos}]$ for $\mathcal{A}$ small ([9], Section 3.3).

(2) Let G be an object with copowers in a category with reflexive coinserters. Then every parallel pair $f,\ g\colon \coprod_A G \to \coprod_B G$ has a conical coequalizer. Indeed for $X = \coprod_A G + \coprod_A G + \coprod_B G$ consider the reflexive pair $[f,g,\mathrm{id}], [g,f,\mathrm{id}]$:
$X \to \coprod_B G$. Its coinserter is precisely a conical coequalizer of f and g.

Proposition 3.25. *Let G be an abstractly finite, subregularly projective strong generator in $\mathcal{K}$. If $\mathcal{K}$ has subkernel pairs and reflexive coinserters, then it has weighted limits and weighted colimits.*

Proof. (1) Let $\mathcal{A}$ be the full subcategory of finite copowers $n.G$ which we know is dense by Theorem 3.23. This implies that the functor E (Notation 3.15) is full and faithful. In fact, fullness is precisely (2a) in Definition 3.19. To see the faithfulness:

$$Ef_0 \leq Ef_1 \quad \text{implies} \quad f_0 \leq f_1\,,$$

use (2b) in that definition.

Thus, all we need to prove is that E has a left adjoint (see the last step of our proof): then we apply Remark 3.24.

(2) Since G, being abstractly finite, has tensors, all copowers $M.G$ exist and are conical, i.e., given u_0, $u_1\colon M.G \to X$ then $u_0 \leq u_1$ iff this holds when precomposed by every coproduct injection.

(3) For every object H of $[\mathcal{A}^{\mathrm{op}},\ \mathbf{Pos}]$ we construct a conical diagram $D_H\colon \mathcal{E}_K \to \mathcal{K}$ 'of elements' of H.

The objects of $\mathcal{E}_H$ are pairs (A, x) where $A \in \mathcal{A}$ and $x \in HA$. For a pair (A, x) and (B, y) of objects morphisms $f\colon (A, x) \to (B, y)$ are those morphisms $f\colon A \to B$ of $\mathcal{A}$ with $Hf(y) = x$. They are ordered as in $\mathcal{A}(A, B)$. And we define

$$D_H\colon \mathcal{E}_H \to \mathcal{K}\,, \quad (A, x) \mapsto A\,.$$

From the conical copowers of G it follows that D_H has a conical colimit. We use the standard construction of conical colimits via conical coproducts and conical coequalizers, completely analogous to the non-enriched case ([14], Thm. V.2.1): put

$$X = \coprod_{(A,x)} A \quad \text{and} \quad Y = \coprod_{f\colon (A,x)\to(B,y)} B$$

where X is a coproduct ranging over objects of $\mathcal{E}_H$ and Y is one ranging over morphisms of $\mathcal{E}_H$. Let us denote the coproduct injections of X by

$$i_x\colon A \to X \qquad (x \in HA)\,.$$

The conical colimit C of D_H is then obtained as the following coequalizer (using Remark 3.24(2)):

$$Y \underset{q}{\overset{p}{\rightrightarrows}} X \xrightarrow{\ c\ } C$$

where the components of p and q corresponding to $f\colon (A, x) \to (B, y)$ are i_x and $i_y \cdot f$, resp. The colimit cocone is $ci_x\colon (A, x) \to C$.

(4) We thus can define a functor

$$L\colon [\mathcal{A}^{\mathrm{op}},\ \mathbf{Pos}] \to \mathcal{K}$$

by assigning to every object H the colimit

$$LH = \ colim D_H\,.$$

We verify that L is a left adjoint of E. Now consider an object $K \in \mathcal{K}$ and the corresponding object $EK = \mathcal{K}(-, K)/\mathcal{A}^{\mathrm{op}}$. To give a natural transformation from H to EK is precisely to give a cocone of D_H with codomain K. We thus obtain the desired natural order-isomorphism

$$\frac{H \longrightarrow EK}{LH \longrightarrow K}$$

proving that L is left adjoint to E. □

4. Varieties as abstract categories

Notation 4.1. Let $\Sigma = (\Sigma)_{n \in \mathbb{N}}$ be a signature. The category

$$\Sigma\text{-}\mathbf{Alg}$$

has as objects ordered Σ-algebras: posets with a structure of a Σ-algebra whose operations are monotone. Morphisms are the monotone homomorphisms.

A *variety* of ordered algebras is a full subcategory of Σ-**Alg** specified by a set of inequations between terms.

Example 4.2. (1) Ordered monoids form a variety of Σ-algebras for $\Sigma = \{\circ, e\}$ specified by the usual monoid equations.

(2) Ordered monoids with the least element e form the subvariety specified by the inequation $e \leq x$.

For a given algebra A a *subalgebra* is represented by a homomorphism $m\colon B \to A$ carried by an order-embedding: $x \leq y$ in B iff $m(x) \leq m(y)$ in A. A *quotient algebra* is represented by a surjective monotone homomorphism $c\colon A \to C$. The following result was sketched by Bloom [5], a detailed proof can be found in [2].

Birkhoff Variety Theorem 4.3. *A full subcategory of Σ-**Alg** is a variety (i.e., can be presented by a set of inequations) iff it is closed under products, subalgebras and quotient algebras.*

Homomorphic images in varieties are precisely the subregular quotients:

Proposition 4.4. *Subregular epimorphisms in a variety of ordered algebras are precisely the surjective homomorphisms.*

Proof. (1) If $h\colon A \to B$ is subregular, say, a coinserter of u_0, $u_1\colon U \to A$, then it is surjective. Indeed, the subalgebra B' of B on $h[A]$ lies in our variety $\mathcal{V}$, and h restricts to a morphism $h'\colon A \to B'$ of $\mathcal{V}$. It clearly satisfies $h'u_0 \leq h'u_1$, thus, there exists $f\colon B \to B'$ with $h' = f.h$. The inclusion $i\colon B' \to B$ fulfils

$$(fi)h' = fh = h',$$

thus $fi = \mathrm{id}$ since h' is surjective. From the universal property of h we deduce, since $(if)h = ih' = h$, that $if = \mathrm{id}$. Thus $i = f^{-1}$, proving that $B' = B$, as stated.

(2) If $h\colon A \to B$ is surjective in $\mathcal{V}$, let E be the subalgebra of $A \times A$ on all (x, y) with $h(x) \leq h(y)$. (This is closed under operations since h is a nonexpanding homomorphism.) From $A \in \mathcal{V}$ we conclude $A \times A \in \mathcal{V}$, hence, $E \in \mathcal{V}$. The restricted projections $u_0, u_1\colon E \to A$ are morphisms of $\mathcal{V}$ with $u_0 \leq u_1$, and they form clearly a reflexive pair (since E contains the diagonal of A). It is easy to see that h is a coinserter of u_0, u_1. □

The concept of effective projective (Definition 2.3) has the following enriched variant:

Definition 4.5. An object whose hom-functor (into **Pos**) preserves coinserters of reflexive pairs is called a *subeffective projective.*

Example 4.6. (1) In every variety $\mathcal{V}$ the free algebra G on one generator is a subeffective projective. Indeed, its hom-functor is naturally isomorphic to the forgetful functor $U\colon \mathcal{V} \to \mathbf{Pos}$.

As proved in [2], U is a monadic functor preserving reflexive coinserters.

(2) Moreover, G is finitely presentable in the enriched sense since U is finitary. And it is a subregular generator. Indeed, the universal property of $P \otimes G$ implies for every poset P that

$$P \otimes G \quad \text{is a free algebra on} P \text{in} \mathcal{V}\,.$$

For every algebra K in $\mathcal{V}$ the canonical homomorphism

$$c_K\colon \mathcal{V}(G, K) \otimes G \to K$$

is the unique extension of id_K to a homomorphism from the free algebra on UK to K, and this is a subregular epimorphism by Proposition 4.4.

(3) Finally, G is a strong generator (Definition 3.17). Indeed,

$$U\colon \mathcal{V} \to \mathbf{Pos}$$

is conservative, thus so is $E\colon \mathcal{V} \to \mathbf{Pos}^{\mathcal{V}(G,G)^{\mathrm{op}}}$ because $U = V \cdot E$ for the forgetful $V\colon \mathbf{Pos}^{\mathcal{V}(G,G)^{\mathrm{op}}} \to \mathbf{Pos}$.

Remark 4.7. Every subeffective projective is a subregular projective, provided that subkernel pairs exist. Indeed, every subregular epimorphism is the coinserter of its subkernel pair (which is reflexive).

Theorem 4.8. *A category with reflexive coinserters is equivalent to a variety of ordered algebras iff it has a subregular generator which is an abstractly finite subeffective projective.*

Proof. In view of the previous example we need to prove only the sufficiency: if $\mathcal{K}$ has subkernel pairs and reflexive coinserters and a generator G with the above properties, then it is equivalent to a variety. Recall from Proposition 3.25 that $\mathcal{K}$ has weighted limits and colimits.

(1) $\mathcal{K}$ has a factorization system with $\mathcal{E}$ all subregular epimorphisms (coinserters of some pairs) and $\mathcal{M}$ all embeddings m (Definition 3.16). Indeed, let $f\colon X \to Y$ be a morphism and choose a subkernel pair p_0, $p_1\colon P \to X$. It is clearly reflexive. Let $c\colon X \to Z$ be a coinserter of p_0, p_1 and $m\colon Z \to Y$

the unique morphism with $f = m \cdot c$. The proof that m is an embedding is completely analogous to point (1) of the proof of Theorem 3.23.

(2) We now define a full embedding $E\colon \mathcal{K} \to \Sigma$-**Alg** for the following signature Σ:

$$\Sigma_n = \mathcal{K}_0(G, n.G) \qquad (n \in \mathbb{N})\,.$$

That is, an n-ary operation symbol is precisely a morphism $\sigma\colon G \to n.G$ of $\mathcal{K}_0$, the ordinary category underlying $\mathcal{K}$.

The algebra EK assigned to an object K has the underlying poset $\mathcal{K}(G, K)$. Given an n-ary operation σ and an n-tuple $(f_0, \dots, f_{n-1})$ in $\mathcal{K}(G, K)$ we form the morphism $[f_i]\colon n.G \to K$, and define the result of σ_{EK} in our n-tuple as the following composite

$$\sigma_{EK}(f_i) \equiv G \xrightarrow{\sigma} n.G \xrightarrow{[f_i]} K\,.$$

To every morphism $h\colon K \to L$ we assign the homomorphism

$$Eh = \mathcal{K}(G, h)$$

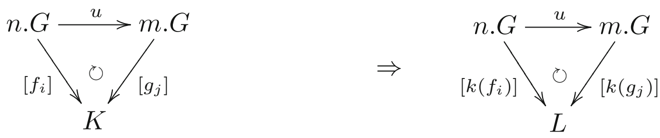
of post-composition with h. Then Eh is clearly monotone. Preservation of $\sigma\colon G \to n.G$ is clear:

$$Eh\big(\sigma_{EK}(f_i)\big) = h \cdot [f_i] \cdot \sigma = [h \cdot f_i] \cdot \sigma = \sigma_{EL}(h \cdot f_i)\,.$$

(2a) E is a fully faithful functor. To prove that it is full, let $k\colon EK \to EL$ be a homomorphism. By Theorem 3.23 it is sufficient to verify the naturality of the following transformation

$$\frac{n.G \xrightarrow{[f_i]} K}{n.G \xrightarrow{k(f_i)} L}$$

That is, we need to prove the following implication for all morphisms $u\colon n.G \to m.G$ $(n, m \in \mathbb{N})$:



From this it follows that there exists $h\colon K \to L$ with $h \cdot [f_i] = [k(f_i)]$ for all $[f_i]\colon n.G \to K$. The case $n = 1$ then yields

$$h \cdot f = k(f) \quad \text{for all} \quad f\colon G \to K$$

in other words, $Eh = k$, as desired.

The above implication is clear if $n = 1$: here u is an m-ary operation symbol in Σ and $f_0 = [g_j] \cdot u = u_{EK}(g_j)$. Since k is a homomorphism, we deduce

$$k(f_0) = u_{EL}\big(k(g_j)\big) = \big[k(g_j)\big] \cdot u\,.$$

For $n > 1$ that implication follows by considering the n components of u separately.

To prove that E is faithful, that is given $k_0, k_1 : K \to L$ with $Ek_0 \le Ek_1$, we conclude $k_0 \le k_1$, use that fact that by Lemma 3.22 the canonical morphism $c_K : \coprod_{\mathcal{K}(G,K)} G \to K$ is a subregular epimorphism.

(2b) E preserves limits, filtered colimits, and reflexive coinserters. In fact, if $U : \Sigma$-**Alg** $\to$ **Pos** denotes the forgetful functor, then

$$U \cdot E = \mathcal{K}(G, -) : \mathcal{K} \to \mathbf{Pos}\,.$$

U creates limits, filtered colimits and reflexive coinserters by Example 4.6. Since $\mathcal{K}(G, -)$ preserves all those three types of constructions, so does E.

(3) $\mathcal{K}$ is equivalent to a variety. For that denote by $\bar{\mathcal{K}}$ the closure of the image of E under isomorphism in Σ-**Alg**. From (2a) we know that $\mathcal{K}$ is equivalent to $\bar{\mathcal{K}}$. We now use the Birkhoff Variety Theorem 4.3 to prove that $\bar{\mathcal{K}}$ is a variety.

(3a) $\bar{\mathcal{K}}$ is closed under products because $\mathcal{K}$ has products by Proposition 3.25 and E preserves them.

(3b) $\bar{\mathcal{K}}$ is closed under subalgebras. It is sufficient to prove this for finitely generated subalgebras. Indeed, $\bar{\mathcal{K}}$ is closed under directed colimits (since $\mathcal{K}$ has them and E preserves them), and every subalgebra is a directed colimit of finitely generated subalgebras.

Thus our task is, for every object K of $\mathcal{K}$ and every finite subposet $m : M \hookrightarrow \mathcal{K}(G, K)$, to prove that the least subalgebra of EK containing m lies in $\bar{\mathcal{K}}$. Factorize $\hat{m} : M \otimes G \to K$ as a subregular epimorphism c followed by an embedding u :

$$\begin{array}{ccc} M \otimes G & \xrightarrow{\ \hat{m}\ } & K \\ & {\scriptstyle c}\searrow \quad \nearrow{\scriptstyle u} & \\ & L & \end{array}$$

We will prove that Eu represents the least subalgebra containing m. That this is a subalgebra of EK is clear: we know that Eu is a monotone homomorphism, and we have: $Eu(x_0) \le Eu(x_1)$ implies $x_0 \le x_1$ because u is an embedding with $u \cdot x_0 \le u \cdot x_1$.

We verify that every subalgebra B of K containing M:

$$M \subseteq UB \subseteq \mathcal{K}(G, K)$$

also contains the image of Eu. That is:

$$u \cdot z \in B \quad \text{for all} \quad z : G \to L\,.$$

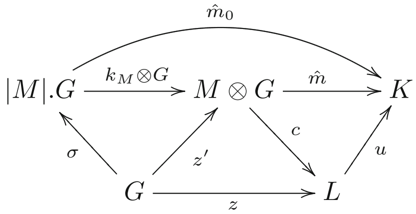
We know that since G is a subeffective projective, $\mathcal{K}(G, c)$ is surjective, thus, z factorizes as $z = c \cdot z'$ for some $z' : G \to M \otimes G$. We next use that $\mathcal{K}(G, -)$ preserves the following reflexive coinserter

$$M^{(2)}.G \underset{\pi_1.G}{\overset{\pi_0.G}{\rightrightarrows}} |M|.G \xrightarrow{\ k_M \otimes G\ } M \otimes G$$

of Example 3.14. Thus, $z'\colon G \to M \otimes G$ factorizes through $k_M \otimes G$ via an operation

$$\sigma\colon G \to |M| \otimes G$$

of arity card $|M|$:



The morphism $m_0 = m \cdot k_M\colon |M| \to \mathcal{K}(G, K)$ fulfils $\hat{m}_0 = \hat{m} \cdot k_M \otimes G$ by Remark 3.12(2), therefore,

$$u \cdot z = \hat{m}_0 \cdot \sigma = \sigma_{EK}(m_0)\,.$$

Since B is closed under σ_{EK} and contains m_0, this proves $u \cdot z \in B$.

(3c) $\bar{\mathcal{K}}$ is closed under quotient algebras. Let $e\colon EK \to A$ be a surjective homomorphism. Form its subkernel pair $u_0, u_1\colon Z \to UEK$ in **Pos**. Then Z is closed in $UEK \times UEK \simeq UE(K \times K)$ under operations. Indeed, let σ be an n-ary operation. If an n-tuple $f_0, \dots, f_{n-1}\colon G \to K \times K$ lies in Z, more precisely, it factorizes through $\langle \pi_0, \pi_1 \rangle\colon Z \hookrightarrow UE(K \times K)$, we have $f_i = \langle \pi_0, \pi_1 \rangle . g_i$ from which we deduce that $\sigma_{E(K\times K}(f_i)$ lies in Z:

$$\sigma_{E(K\times K}(f_i) = [f_i].\sigma = \langle u_0, u_1 \rangle \cdot [g_i] \cdot \sigma\,.$$

By (3a) and (3b) there exists a subobject $m = [m_0, m_1]\colon M \to K \times K$ with Em_0, Em_1 forming the prekernel pair of e. Since m_0, m_1 is a reflexive pair, E preserves its coinserter $k\colon K \to L$. Thus $A \simeq EL$ lies in $\bar{\mathcal{K}}$.

□

5. Varieties as free completions

The aim of this section is to prove a parallel result to Theorem 2.11: varieties of ordered algebras are precisely the free completions of $\mathcal{T}^{\mathrm{op}}$ under sifted (weighted) colimits, where $\mathcal{T}$ ranges over discrete Lawvere theories. This latter concept was introduced by Power [16] for algebras of countable arities. The corresponding finitary variant uses $\mathcal{N}$ of Notation 2.5, now considered as a trivially enriched category: all hom-sets are discrete.

Recall that all categories, functors etc. are enriched over **Pos**.

Definition 5.1 ([16]). A (finitary) *discrete Lawvere theory* is a small enriched category $\mathcal{T}$ with specified finite products together with a functor $I\colon \mathcal{N}^{\mathrm{op}} \to \mathcal{T}$ which is identity on objects and strictly preserves finite products.

The category **Mod**$\mathcal{T}$ of models is now defined analogously to the ordinary case: it consists of (enriched) functors $A\colon \mathcal{T} \to$ **Pos** preserving finite products and natural transformations.

Example 5.2. The discrete Lawvere theory $\mathcal{T}_{\mathcal{V}}$ associated to a variety of ordered algebras $\mathcal{V}$ has as objects natural numbers, the hom-poset $\mathcal{T}_{\mathcal{V}}(n,1)$ is the underlying poset of the free algebra of $\mathcal{V}$ on n, and $\mathcal{T}_{\mathcal{V}}(n,k) = \mathcal{T}_{\mathcal{V}}(n,1)^k$. Thus n is the power of 1: the projection $\pi_i \colon n \to 1$ corresponds to i as an element of the free algebra on n.

Every algebra A of $\mathcal{V}$ defines a model $\widehat{A}$ of $\mathcal{T}_{\mathcal{V}}$: to each n it assigns the underlying poset of A^n. To every morphism $f \in \mathcal{T}_{\mathcal{V}}(n,1)$ it assigns the monotone map $\widehat{f} \colon A^n \to A$ which, given an n-tuple $h \colon n \to A$, extends it to the unique homomorphism $h^\sharp \colon \mathcal{T}_{\mathcal{V}}(n,1) \to A$ and yields

$$\widehat{f}(h) = h^\sharp(f)\,.$$

We now turn to sifted colimits in the enriched setting (cf. Definition 2.8) following the dissertation of Bourke [8] (where the base category of categories was considered) and the paper [11] in which the appropriate adaptation to **Pos** was made explicit:

Definition 5.3 ([8]). A weight $W \colon \mathcal{D}^{\mathrm{op}} \to$ **Pos** is called *sifted* if colimits of diagrams in **Pos** weighted by W commute with finite products: given diagrams D_1, $D_2 \colon \mathcal{D} \to$ **Pos**, the canonical morphism $\mathit{colim}_W(D_1 \times D_2) \to (\mathit{colim}_W D_1) \times (\mathit{colim}_W D_2)$ is an isomorphism.

Weighted colimits in a category are called *sifted colimits* if the weight is sifted.

Example 5.4. (1) Filtered colimits (those where $\mathcal{D}$ has a filtered underlying category) are sifted.

(2) Reflexive coinserters are sifted colimits [2].

(3) Also split coequalizers are sifted colimit. This follows from [12] 1.3, because split coequalizers are absolute colimits (i.e. every functor preserves split coequalizers). This can be directly verified as follows. Let

$$A \underset{d_1}{\overset{d_0}{\rightrightarrows}} B \xrightarrow{\;e\;} C$$

be a coequalizer split by $t : B \to A$ and $s : C \to B$. That is $es = \mathrm{id}_C$, $d_0 t = \mathrm{id}_B$ and $d_1 t = se$. We have

$$d_1 t d_1 = s e d_1 = s e d_0 = d_1 t d_0.$$

Conversely, having $t \colon B \to A$ such that $d_0 t = \mathrm{id}_B$ and $d_1 t d_0 = d_1 t d_1$, we get a unique $s \colon C \to B$ such that $se = d_1 t$.

Hence C is a colimit of the filtered diagram consisting of d_0, d_1 and t.

Remark 5.5. (1) Analogously to the ordinary categories (see Example 2.9), the first two cases above are in a way exhaustive. For example, an endofunctor of **Pos** preserves sifted colimits iff it preserves filtered colimits and reflexive coinserters ([8], 8.45).

(2) For every variety $\mathcal{K}$ the forgetful functor $U \colon \mathcal{K} \to$ **Pos** preserves (indeed: creates) filtered colimits. This is easy to verify.

Remark 5.6. Following [2, Theorem 4.5], varieties of ordered algebras are, up to concrete isomorphism, precisely categories of algebras $\mathbf{Pos}^T$ for enriched monads T on $\mathbf{Pos}$ preserving sifted colimits. Moreover, these are up to equivalence precisely the categories of models of discrete Lawvere theories: see Theorem 7.7 cf [12]. (That result is more general, dealing with a closed category $\mathcal{V}$ and a class ϕ of limits. Applying it to $\mathcal{V} = \mathbf{Pos}$ and $\phi =$ finite products yields the above special case.)

The equivalence of op.cit. assigns to every variety $\mathcal{V}$ the theory $\mathcal{T}_{\mathcal{V}}$: it turns out that all models of $\mathcal{T}_{\mathcal{V}}$ are naturally equivalent to $\widehat{A}$ for algebras $A \in \mathcal{V}$.

Remark 5.7. Let $\mathcal{K}$ be an enriched category. By its *free completion under sifted colimits* is meant an enriched category $\mathbf{Sind}\mathcal{K}$ with sifted colimits containing $\mathcal{K}$ as a full subcategory having the expected universal property: for every functor $F: \mathcal{K} \to \mathcal{L}$ where $\mathcal{L}$ has sifted colimits there exists an extension to $\mathbf{Sind}\mathcal{K}$ preserving sifted colimits, unique up to natural isomorphism.

It follows that the functor category $[\mathcal{K}, \mathcal{L}]$ is equivalent (via the domain-restriction functor) to the full subcategory of $[\mathbf{Sind}\mathcal{K}, \mathcal{L}]$ formed by functors preserving sifted colimits.

Definition 5.8. An object A of $\mathcal{K}$ is called *perfectly presentable* if its hom-functor $\mathcal{K}(A, -) : \mathcal{K} \to \mathbf{Pos}$ preserves sifted colimits.

Remark 5.9. By Example 5.4 every perfectly presentable object is finitely presentable and a subeffective projective.

Lemma 5.10. *Perfectly presentable objects are closed under finite coproducts and retracts.*

Proof. The proof of the first claim is analogous to [12], 5.6.9 The second statement is easy. □

Proposition 5.11. *Let $\mathcal{K}$ be a variety of ordered algebras. The following properties of an arbitrary algebra A of $\mathcal{K}$ are equivalent:*

(1) *A is perfectly presentable,*
(2) *A is finitely presentable and a subregular projective, and*
(3) *A is a retract of a free algebra on a finite discrete poset.*

Proof. (1) ⇒ (2) follows from Example 5.4.

(2) ⇒ (3): Since A is finitely presentable, there is a finite subposet P of A such that for the free-algebra functor $F: \mathbf{Pos} \to \mathcal{K}$ a surjective homomorphism $e: FP \to A$ exists. The canonical coinserter of P (Example 3.14) is preserved by F, thus, $Fk_p: F|P| \to FP$ is also surjective. As in Example 3.6 (2), we can prove that surjective homomorphisms are coinserters of their subkernel pairs. Thus $e \cdot Fk_p: F|P| \to A$ is a subregular epimorphism. Since A is a subregular projective, this implies that $e \cdot Fk_p$ is a split epimorphism. Thus A is a retract of $F|P|$.

(3) ⇒ (1): Since by Remark 5.5 the forgetful functor $U : \mathcal{K} \to \mathbf{Pos}$ preserves sifted colimits, its left adjoint $F : \mathbf{Pos} \to \mathcal{K}$ preserves perfectly presentable objects. Thus (1) holds due to Remark 4.7. □

The following theorem is due to Kurz and Velebil ([11], 6.9 and 6.12). We present a full proof because it is simpler than that in op.cit.

Theorem 5.12. *Let $\mathcal{K}$ be a variety of ordered algebras and $\mathcal{P}$ its full subcategory on free algebras on finite discrete posets. Then $\mathcal{K} =$ **Sind**$\mathcal{P}$.*

Proof. Following [10, Proposition 4.2] and 5.11, all we have to show is that $\mathcal{K}$ is the closure of $\mathcal{P}$ under sifted colimits.

(1) A finite poset P is a reflexive coinserter as in Example 3.14. This yields the following reflexive coinserter in $\mathcal{K}$

$$FP^{(2)} \underset{F\pi_0}{\overset{F\pi_1}{\rightrightarrows}} F|P| \xrightarrow{Fk_p} FP$$

with FP in $\mathcal{P}$.

(2) A free algebra on an arbitrary poset X is a filtered colimit of free algebras over finite posets. This follows from the fact that the free-algebra functor from **Pos** to $\mathcal{K}$ preserves colimits:

(3) Finally, every algebra A in $\mathcal{K}$ is a split coequalizer of free algebras via its canonical presentation

$$FUFUA \underset{FU\varepsilon_A}{\overset{\varepsilon_{FUA}}{\rightrightarrows}} FUA \xrightarrow{\varepsilon_A} A$$

Following Example 5.4, this is a filtered colimit. □

Remark 5.13. A concrete category over **Pos** is a category $\mathcal{K}$ together with a faithful (enriched) 'forgetful' functor $U: \mathcal{K} \to$ **Pos**.

Given concrete categories $(\mathcal{K}, U)$ and $(\mathcal{K}', U')$, they are (concretely) *equivalent* if there exists an equivalence functor $E: \mathcal{K} \to \mathcal{K}'$ with $U = U'E$. Analogously, they are *isomorphic* if E is an isomorphism.

Theorem 5.14. *The following statements are equivalent for an enriched category $\mathcal{K}$ up to concrete equivalence:*

(1) *$\mathcal{K}$ is a variety of ordered algebras,*
(2) *$\mathcal{K} =$ **Sind**$\mathcal{T}^{\mathrm{op}}$ for a discrete Lawvere theory $\mathcal{T}$, and*
(3) *$\mathcal{K} =$ **Mod**$\mathcal{T}$ for a discrete Lawvere theory $\mathcal{T}$.*

Proof. (1)⇔(3) follows from 5.6.

(1)⇒(2) follows from 5.12.

(2)⇒(1): Following [10, Proposition 8.1], **Sind**$\mathcal{T}^{\mathrm{op}} \subseteq$ **Mod**$\mathcal{T}$. Due to [12, Theorem 7.7], $\mathcal{T}^{\mathrm{op}}$ is the category of free $\mathcal{T}$-algebras on finite discrete posets. Due to 5.12, **Sind**$\mathcal{T}^{\mathrm{op}} =$ **Mod**$\mathcal{T}$. And we already know that **Sind**$\mathcal{T}^{\mathrm{op}}$ is a concretely equivalent to variety of ordered algebras. □

6. Varieties as concrete categories

In Section 4 we have characterized varieties $\mathcal{V}$ of ordered algebras as abstract categories enriched over **Pos**. In the present section we derive a characterization of the forgetful functors $U: \mathcal{V} \to$ **Pos**, i.e., varieties as concrete categories.

As mentioned in Related Work, Bloom and Wright presented a characterization in [6]. In this section we try and compare this with our results. For categories which are exact (in the enriched sense over **Pos**) another such characterization is due to Kurz and Velebil ([11], Theorem 5.7), however, we are not working with exactness in our paper.

There is not much difference between characterizing varieties abstractly or concretely:

Remark 6.1. Let $\mathcal{K}$ be a category with tensors.

(1) Every generator G defines a faithful functor

$$U = \mathcal{K}(G, -) \colon \mathcal{K} \to \mathbf{Pos}$$

with a left adjoint

$$\phi = - \otimes G \colon \mathbf{Pos} \to \mathcal{K}\,.$$

(2) Every faithful functor $U \colon \mathcal{K} \to \mathbf{Pos}$ with a left adjoint F has the above form for the generator $G = \phi 1$.

Thus Theorem 4.8 has the following

Corollary 6.2. *A concrete category $(\mathcal{K}, U)$ over **Pos** is equivalent to a variety iff*

(1) *$\mathcal{K}$ has tensors, subkernel pairs and reflexive coinserters, and*
(2) *U is a finitary right adjoint which reflects isomorphisms and preserves reflexive coinserters.*

Proof. (a) Let U be the forgetful functor of a variety $\mathcal{K}$. From Example 4.6 we know that U is a right adjoint preserving reflexive coinserters. Following Proposition 4.4, U reflects subregular epimorphisms. Hence it reflects isomorphisms.

(b) Conversely, let (1) and (2) hold. Denote by $\phi \colon \mathbf{Pos} \to \mathcal{K}$ the left adjoint of U and by $T = U\phi$ the corresponding monad. $\varepsilon \colon \phi U \to \mathrm{Id}$

The object $G = \phi 1$ is a strong generator: the functor

$$E \colon \mathcal{K} \to \mathbf{Pos}^{\mathcal{K}(G,G)^{\mathrm{op}}}$$

of Notation 3.15(2) reflects isomorphisms because U does, and we have $U \cong V.E$ for the forgetful functor $V \colon \mathbf{Pos}^{\mathcal{K}(G,G)^{\mathrm{op}}} \to \mathbf{Pos}$. Indeed, the posets $\mathcal{K}(G, K)$ are isomorphic to UK (naturally in $K \in \mathcal{K}$).

Since U is finitary, G is finitely presentable, thus abstractly finite (Example 3.20(3)), and since U preserves reflexive coinserters, G is an effective projective.

In the proof of Theorem 4.8 we have presented a variety $\bar{\mathcal{K}}$ and an equivalence $E \colon \mathcal{K} \to \bar{\mathcal{K}}$ assigning to every object K an algebra on the poset $\mathcal{K}(G, K) = UK$ and to every morphism $f \colon K \to L$ a homomorphism carried by $\mathcal{K}(G, f) = Uf$. Thus, E is an equivalence of concrete categories. □

Remark 6.3. The above corollary is related to the characterization of varieties of ordered algebras due to Bloom and Wright [6]. Their main theorem works with possibly infinitary signatures, but we present now the formulation just for the finitary ones to make the comparison clearer.

Theorem 6.4 (Bloom and Wright [6]). *A concrete category* $(\mathcal{K}, U)$ *over* ***Pos*** *is isomorphic to a variety iff*

(1) $\mathcal{K}$ *has coinserters and*
(2) U *is a finitary right adjoint which*
  a. *preserves and reflects subregular epimorphisms,*
  b. *reflects subkernel pairs, and*
  c. *creates isomorphisms.*

Condition (2c) is clearly related to the fact that the theorem deals with isomorphic categories rather than equivalent ones.

There does not seem to be a direct proof of one of the above results from the other one.

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