



# Epimorphisms in varieties of subidempotent residuated structures

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**Abstract.** A commutative residuated lattice  $\mathbf{A}$  is said to be *subidempotent* if the lower bounds of its neutral element  $e$  are idempotent (in which case they naturally constitute a Brouwerian algebra  $\mathbf{A}^-$ ). It is proved here that epimorphisms are surjective in a variety  $\mathbf{K}$  of such algebras  $\mathbf{A}$  (with or without involution), provided that each finitely subdirectly irreducible algebra  $\mathbf{B} \in \mathbf{K}$  has two properties: (1)  $\mathbf{B}$  is generated by lower bounds of  $e$ , and (2) the poset of prime filters of  $\mathbf{B}^-$  has finite depth. Neither (1) nor (2) may be dropped. The proof adapts to the presence of bounds. The result generalizes some recent findings of G. Bezhanishvili and the first two authors concerning epimorphisms in varieties of Brouwerian algebras, Heyting algebras and Sugihara monoids, but its scope also encompasses a range of interesting varieties of De Morgan monoids.

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## 1. Introduction

In a variety of algebras, if a homomorphism is surjective, then it is an epimorphism, but the converse need not hold. Indeed, rings and distributive lattices each form varieties in which non-surjective epimorphisms arise. As it happens, this reflects the absence of unary terms defining multiplicative inverses in rings, and complements in distributive lattices, despite the uniqueness of those entities when they exist.

Such constructs are said to be *implicitly* (and not *explicitly*) *definable*. In a variety of logic, they embody implicitly definable propositional functions that cannot be explicated in the corresponding logical syntax, and Beth-style ‘definability properties’ *preclude* phenomena of this kind. (The allusion is to E.W. Beth’s theorems for classical propositional and predicate logic in [3]).

In particular, when a logic  $\mathbf{L}$  is algebraized, in the sense of [7], by a variety  $\mathbf{K}$  of algebras, then the *ES property* for  $\mathbf{K}$ —i.e., the demand that *all* epimorphisms in  $\mathbf{K}$  be surjective—amounts to the so-called *infinite Beth definability property* for  $\mathbf{L}$ . The most general version of this ‘bridge theorem’ was formulated and proved by Blok and Hoogland [5, Theorems 3.12, 3.17] (also see [39, Theorem 7.6] and the antecedents cited in both papers).

In this situation, the subvarieties of  $\mathbf{K}$  algebraize the axiomatic extensions of  $\mathbf{L}$ , but the ES property need not persist in subvarieties. It is therefore a well-motivated (but often nontrivial) task to determine which subvarieties of  $\mathbf{K}$  have surjective epimorphisms. The present paper addresses this question in the context of residuated structures, i.e., the algebraic models of substructural logics (see [17, 27]).

We consider algebras called *SRLs*, which are residuated lattice-ordered commutative monoids satisfying the *subidempotent law*  $x \leq e \implies x \cdot x = x$  (where  $\cdot$  is the monoid operation and  $e$  its neutral element) and possibly equipped with an involution (*SIRLs*) and/or lattice-bounds. They need not be *integral*, i.e.,  $e$  need not be the greatest element. They include De Morgan monoids, i.e., the algebraic models of the relevance logic  $\mathbf{R}^t$  of [1]. The *negative cone* of an S[I]RL  $\mathbf{A}$ , which comprises the lower bounds of  $e$ , may be given the structure of a Brouwerian or Heyting algebra  $\mathbf{A}^-$ , to which the Esakia duality of [13] applies. In particular, the *depth* of  $\mathbf{A}$  may be defined as that of  $\mathbf{A}^-$ .

Heyting and Brouwerian algebras model intuitionistic propositional logic and its positive fragment. A result of Kreisel [28] shows (in effect) that every variety consisting of such algebras has a *weak* form of the ES property, whereas Maksimova established that only finitely many enjoy a certain *strong* form; see [16, 32, 33]. Uncountably many varieties of Heyting [Brouwerian] algebras have finite depth [29], and all of these have surjective epimorphisms; the latter claim was proved recently by G. Bezhanishvili and the first two authors [4], using Esakia duality. On the other hand, the ES property fails in uncountably many further varieties of Heyting [Brouwerian] algebras [40].

Relational duals for non-integral S[I]RLs are sometimes available [46], but they are rather complicated. Also, the functor that constructs negative cones (and restricts morphisms accordingly) is not a category equivalence,

except in quite special cases; see [15, 18, 20, 21]. Partly for these reasons, we cannot systematically reduce ES problems for arbitrary varieties of S[I]RLs to an examination of negative cones.

Nevertheless,  $\mathbf{A}^-$  contains enough information about  $\mathbf{A}$  to facilitate the main result of this paper (Theorem 8.1), which is a sufficient condition for the surjectivity of epimorphisms. It states that, in a variety  $\mathbf{K}$  of S[I]RLs, epimorphisms will be surjective if each finitely subdirectly irreducible member of  $\mathbf{K}$  has finite depth and is generated by elements of its negative cone. Neither hypothesis may be dropped.

The assumptions of Theorem 8.1 persist in subvarieties and under varietal joins, so the result is labour-saving. Apart from generalizing the aforementioned findings of [4], it settles the question of epimorphism-surjectivity for many interesting varieties of De Morgan monoids (see Sections 9 and 10), yielding definability results for a corresponding range of relevance logics.

**Notation.** In an indicated partially ordered set  $\langle X; \leq \rangle$ , we define

$$\uparrow x = \{y \in X : x \leq y\} \text{ and } \uparrow U = \bigcup_{u \in U} \uparrow u,$$

for  $U \cup \{x\} \subseteq X$ , and if  $U = \uparrow U$ , we call  $U$  an *up-set* of  $\langle X; \leq \rangle$ . We define  $\downarrow x$  and  $\downarrow U$  dually. For  $x, y \in X$ , the notation  $x \prec y$  (' $y$  covers  $x$ ') signifies that  $x < y$  and there is no  $z \in X$  such that  $x < z < y$ .

If  $Y \subseteq X$  and  $x \in X$ , we sometimes need to refer to  $Y \cap \uparrow x$ , which we then denote as  $\uparrow^Y x$ , even if  $x \notin Y$ .

The universe of an algebra  $\mathbf{A}$  is denoted by  $A$ . Thus, the congruence lattice **Con**  $\mathbf{A}$  of  $\mathbf{A}$  has universe  $Con \mathbf{A}$ . For  $\emptyset \neq X \subseteq A$ , the subalgebra of  $\mathbf{A}$  generated by  $X$  is denoted by  $\mathbf{Sg}^A X$  (and its universe by  $Sg^A X$ ).

The class operator symbols  $\mathbb{H}$ ,  $\mathbb{S}$ ,  $\mathbb{P}$  and  $\mathbb{P}_U$  stand, respectively, for closure under homomorphic images, subalgebras, direct products and ultraproducts, while  $\mathbb{V}$  denotes varietal generation, i.e.,  $\mathbb{V} = \mathbb{H}\mathbb{S}\mathbb{P}$ . We abbreviate  $\mathbb{V}(\{\mathbf{A}\})$  as  $\mathbb{V}(\mathbf{A})$ .

Recall that an algebra  $\mathbf{A}$  is *finitely subdirectly irreducible* (briefly, *FSI*) iff its identity relation  $id_A$  is meet-irreducible in **Con**  $\mathbf{A}$ . For a variety  $\mathbf{K}$ , we use  $\mathbf{K}_{FSI}$  to denote the class of all FSI members of  $\mathbf{K}$ . Of course,  $\mathbf{K} = \mathbb{V}(\mathbf{K}_{FSI})$ , by the Subdirect Decomposition Theorem.

## 2. Epimorphisms

Let  $\mathbf{K}$  be a class of similar algebras. By a *K-morphism*, we mean a homomorphism  $f: \mathbf{A} \rightarrow \mathbf{B}$ , where  $\mathbf{A}, \mathbf{B} \in \mathbf{K}$ . It is called a *K-epimorphism* provided that, whenever  $g, h: \mathbf{B} \rightarrow \mathbf{C}$  are  $\mathbf{K}$ -morphisms with  $g \circ f = h \circ f$ , then  $g = h$ . Clearly, surjective  $\mathbf{K}$ -morphisms are  $\mathbf{K}$ -epimorphisms. We say that  $\mathbf{K}$  has the *epimorphism-surjectivity (ES) property* if all  $\mathbf{K}$ -epimorphisms are surjective.

A subalgebra  $\mathbf{D}$  of an algebra  $\mathbf{E} \in \mathbf{K}$  is said to be *K-epic* (in  $\mathbf{E}$ ) if every  $\mathbf{K}$ -morphism with domain  $\mathbf{E}$  is determined by its restriction to  $\mathbf{D}$ . (This means that the inclusion map  $\mathbf{D} \rightarrow \mathbf{E}$  is a  $\mathbf{K}$ -epimorphism, assuming that  $\mathbf{D} \in \mathbf{K}$ .) Thus, a  $\mathbf{K}$ -morphism is a  $\mathbf{K}$ -epimorphism iff its image is a  $\mathbf{K}$ -epic subalgebra of

its co-domain. And, when  $\mathbf{K}$  is closed under subalgebras, it has the ES property iff none of its members has a  $\mathbf{K}$ -epic proper subalgebra.

Recall that if  $\mathbf{A}$  is a subalgebra of an algebra  $\mathbf{B}$ , and  $\mu \in \text{Con } \mathbf{B}$ , then the relation  $\mu|_A := A^2 \cap \mu$  is a congruence of  $\mathbf{A}$ .

**Lemma 2.1.** *Let  $\mathbf{K}$  be a variety of algebras, and  $\mathbf{A}$  a  $\mathbf{K}$ -epic subalgebra of  $\mathbf{B} \in \mathbf{K}$ . Then, for any  $\mu \in \text{Con } \mathbf{B}$ , the map  $f: \mathbf{A}/(\mu|_A) \rightarrow \mathbf{B}/\mu$  defined by  $a/(\mu|_A) \mapsto a/\mu$  is an injective  $\mathbf{K}$ -epimorphism.*

*Proof.* As  $\mathbf{K}$  is a variety,  $\mathbf{B}/\mu, \mathbf{A}/(\mu|_A) \in \mathbf{K}$ . Let  $i: \mathbf{A} \rightarrow \mathbf{B}$  be the inclusion homomorphism and  $q: \mathbf{B} \rightarrow \mathbf{B}/\mu$  the surjective homomorphism  $b \mapsto b/\mu$ . Clearly,  $\mu|_A$  is the kernel of the homomorphism  $q \circ i: \mathbf{A} \rightarrow \mathbf{B}/\mu$ , so  $f$  is an injective  $\mathbf{K}$ -morphism.

Suppose  $g, h: \mathbf{B}/\mu \rightarrow \mathbf{C} \in \mathbf{K}$  are homomorphisms with  $g \circ f = h \circ f$ . Then  $g \circ q$  and  $h \circ q$  are homomorphisms from  $\mathbf{B}$  to  $\mathbf{C}$ . For each  $a \in A$ ,

$$g(q(a)) = g(a/\mu) = g(f(a/(\mu|_A))) = h(f(a/(\mu|_A))) = h(q(a)),$$

i.e.,  $g \circ q \circ i = h \circ q \circ i$ . Therefore,  $g \circ q = h \circ q$ , as  $i$  is a  $\mathbf{K}$ -epimorphism. Since  $q$  is surjective, it follows that  $g = h$ , as required.  $\square$

A variety  $\mathbf{K}$  is said to have *EDPM* if it is congruence distributive and  $\mathbf{K}_{FSI}$  is a universal class (i.e., subalgebras and ultraproducts of FSI members of  $\mathbf{K}$  are FSI). The acronym stands for ‘equationally definable principal meets’ and is motivated by other characterizations of the notion in [6, 9].

**Theorem 2.2** (Campercholi [8, Theorem 6.8]). *If a congruence permutable variety  $\mathbf{K}$  with EDPM lacks the ES property, then some FSI member of  $\mathbf{K}$  has a  $\mathbf{K}$ -epic proper subalgebra.*

### 3. Residuated structures

General information about residuated structures (and their connection with substructural logics) can be found in [17]. Here, we recall the basic definitions and facts that will be relied on below.

**Definition 3.1.** An algebra  $\mathbf{A} = \langle A; \wedge, \vee, \cdot, \rightarrow, e \rangle$  will be called a (commutative) *subidempotent residuated lattice*, or briefly an *SRL*, if  $\langle A; \wedge, \vee \rangle$  is a lattice,  $\langle A; \cdot, e \rangle$  is a commutative monoid and  $\rightarrow$  is a binary operation (called the *residual*) such that  $\mathbf{A}$  satisfies

$$x \cdot y \leq z \iff x \leq y \rightarrow z \quad (\text{the law of residuation}) \tag{3.1}$$

$$x \leq e \implies x = x^2 \quad (:= x \cdot x) \quad (\text{subidempotence}), \tag{3.2}$$

where  $\leq$  is the lattice order. An equational paraphrase of (3.2) is

$$(x \wedge e)^2 = x \wedge e.$$

Every SRL satisfies the postulates below. Here and subsequently,  $x \leftrightarrow y$  abbreviates  $(x \rightarrow y) \wedge (y \rightarrow x)$ .

$$x \cdot (y \vee z) = (x \cdot y) \vee (x \cdot z) \tag{3.3}$$

$$x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z) \tag{3.4}$$

$$(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z) \tag{3.5}$$

$$x \leq y \iff e \leq x \rightarrow y \tag{3.6}$$

$$x = y \iff e \leq x \leftrightarrow y \tag{3.7}$$

$$e \leq x \rightarrow x \text{ and } e \rightarrow x = x \tag{3.8}$$

$$(x \leq e \ \& \ y \leq e) \implies x \wedge y = x \cdot y. \tag{3.9}$$

Of these properties, only (3.9) relies on subidempotence. It follows from (3.3) that  $\cdot$  is isotone in both arguments, and from (3.4) and (3.5) that  $\rightarrow$  is isotone in its second argument and antitone in its first. In particular, the implication  $x \leq e \implies x^2 \leq x$  does not rely on (3.2), so we could express (3.2) as

$$x \leq e \implies x \leq x^2.$$

The key step in the proof of (3.9) is  $x \wedge y \wedge e = (x \wedge y \wedge e)^2 \leq x \cdot y$ .

An SRL  $\mathbf{A}$  is said to be

- *square-increasing* if it satisfies  $x \leq x^2$ ;
- *idempotent* if it satisfies  $x = x^2$ ;
- *distributive* if its lattice reduct  $\langle A; \wedge, \vee \rangle$  is distributive; and
- *integral* if  $e$  is its greatest element.

By (3.9), an SRL is integral iff its operations  $\cdot$  and  $\wedge$  coincide.

**Definition 3.2.** A *Brouwerian algebra* is an integral SRL  $\mathbf{A}$ ; it is normally identified with its reduct  $\langle A; \wedge, \vee, \rightarrow, e \rangle$  (in view of the previous remark).

It follows that a Brouwerian algebra is idempotent, distributive (by (3.3)) and determined by its lattice reduct, and that it satisfies  $x \rightarrow e = e = x \rightarrow x$ .

**Definition 3.3.** An *involutive SRL*, briefly an *SIRL*, is the expansion  $\mathbf{A}$  of an SRL by an *involution*, i.e., a unary operation  $\neg$  such that  $\mathbf{A}$  satisfies  $\neg\neg x = x$  and  $x \rightarrow \neg y = y \rightarrow \neg x$ . In these algebras, we define  $f = \neg e$ .

An SIRL satisfies  $x \rightarrow y = \neg(x \cdot \neg y)$  and  $\neg x = x \rightarrow f$ . Every SRL can be embedded into an SIRL, in such a way that distributivity and the square-increasing law  $x \leq x^2$  are preserved (see [34, 19] and Section 9 below).

The class of all S[I]RLs is a finitely axiomatizable variety. It is arithmetical, i.e., congruence distributive and congruence permutable. (See [17, p. 94], where a termwise-equivalent formulation is used.) The variety of Brouwerian algebras has the ES property [33] (also see [14, 16, 32]).

Distributive square-increasing SIRLs—a.k.a. *De Morgan monoids*—algebraize the principal relevance logic  $\mathbf{R}^t$  of [1] (also see [10, 36, 37, 38]), while distributive square-increasing SRLs—a.k.a. *Dunn monoids*—algebraize the ‘positive’ (negationless) fragment of  $\mathbf{R}^t$ . Their non-distributive counterparts similarly match the subsystem  $\mathbf{LR}^t$  of  $\mathbf{R}^t$  and its positive fragment. ( $\mathbf{LR}^t$  adds the double-negation axiom to the system  $\mathbf{FL}_{ec}$  of [17]; also see [35, 45].) The square-increasing law embodies the logical *contraction* axiom

$$(p \rightarrow (p \rightarrow q)) \rightarrow (p \rightarrow q).$$

The idempotent De Morgan monoids are called *Sugihara monoids*. They model the logic  $\mathbf{RM}^t$  (a.k.a.  $\mathbf{R}\text{-mingle}$ ), and their structure is better understood than that of other De Morgan monoids; see [1, 11, 20, 21, 41]. In all of these cases, there is a transparent lattice anti-isomorphism between the axiomatic extensions of the logic and the subvarieties of the model class.

A *bounded S[I]RL* is the expansion of an S[I]RL by a distinguished element  $\perp$ , which is the least element of the order, whence  $\perp \rightarrow \perp$  is the greatest element. A *Heyting algebra* is a bounded Brouwerian algebra. For simplicity, we defer further discussion of bounded S[I]RLs until Section 10.4.

### 4. Filters

Let  $\mathbf{A}$  be an S[I]RL. By a *filter* of  $\mathbf{A}$ , we mean a filter of the lattice  $\langle A; \wedge, \vee \rangle$ , i.e., a non-empty subset  $F$  of  $A$  that is upward closed and closed under the binary operation  $\wedge$ . It is called a *deductive filter* of  $\mathbf{A}$  if, moreover,  $e \in F$ . In that case,  $F$  is a submonoid of  $\langle A; \cdot, e \rangle$  and, whenever  $b \in A$  and  $a, a \rightarrow b \in F$ , then  $b \in F$ . The respective explanations are that  $\mathbf{A}$  satisfies

$$x \cdot y \geq (x \wedge e) \cdot (y \wedge e) = x \wedge y \wedge e \text{ (by (3.9)) and} \\ x \cdot (x \rightarrow y) \leq y \text{ (by (3.1)).}$$

The lattice of deductive filters of  $\mathbf{A}$  is isomorphic to  $\mathbf{Con} \mathbf{A}$ . The isomorphism and its inverse are given by

$$F \mapsto \Omega^A F := \{ \langle a, b \rangle \in A^2 : a \leftrightarrow b \in F \}; \\ \theta \mapsto \{ a \in A : a \wedge e \equiv_\theta e \}.$$

We abbreviate  $\mathbf{A}/\Omega^A F$  as  $\mathbf{A}/F$ , and  $a/\Omega^A F$  as  $a/F$ , noting that

$$a \rightarrow b \in F \text{ iff } a/F \leq b/F \text{ in } \mathbf{A}/F.$$

Whenever  $\mathbf{B}$  is a subalgebra of  $\mathbf{A}$ , and  $F$  is a deductive filter of  $\mathbf{A}$ , then  $B \cap F$  is a deductive filter of  $\mathbf{B}$  and

$$\Omega^B (B \cap F) = (\Omega^A F)|_B. \tag{4.1}$$

Because of the above isomorphism,  $\mathbf{A}$  is FSI iff its smallest deductive filter  $\{a \in A : e \leq a\}$  is meet-irreducible in its lattice of deductive filters, and that amounts to the join-irreducibility of  $e$  in  $\langle A; \wedge, \vee \rangle$ . Since this last condition is expressible as a universal first order sentence, every variety of S[I]RLs has EDPM. Therefore, Theorem 2.2 applies to all such varieties.

For any subset  $X$  of  $\mathbf{A}$ , the smallest deductive filter of  $\mathbf{A}$  containing  $X$  is denoted by  $Fg^A X$ . Thus,  $Fg^A X$  consists of all  $a \in A$  such that

$$a \geq x_1 \wedge \cdots \wedge x_n \text{ for some } x_1, \dots, x_n \in X \cup \{e\}, \text{ where } 0 < n < \omega.$$

In particular, if  $e \geq b \in A$ , then  $Fg^A \{b\} = \{a \in A : b \leq a\}$ .

It follows that the deductive filters of a subalgebra  $\mathbf{B}$  of  $\mathbf{A}$  are just the sets  $B \cap F$  such that  $F$  is a deductive filter of  $\mathbf{A}$ . Note that the deductive filters of a Brouwerian algebra are exactly its (lattice-) filters.

A filter  $F$  of a lattice  $\langle L; \wedge, \vee \rangle$  is said to be *prime* if its complement  $L \setminus F$  is closed under the binary operation  $\vee$ . For any filters  $F, G, H$  of  $\langle L; \wedge, \vee \rangle$ ,

$$\text{if } F \cap G \subseteq H \text{ and } H \text{ is prime, then } F \subseteq H \text{ or } G \subseteq H. \tag{4.2}$$

The *Prime Filter Extension Theorem* asserts that, when  $\langle K; \wedge, \vee \rangle$  is a sublattice of a distributive lattice  $\langle L; \wedge, \vee \rangle$ , then the prime filters of  $\langle K; \wedge, \vee \rangle$  are exactly the non-empty sets  $K \cap F$  such that  $F$  is a prime filter of  $\langle L; \wedge, \vee \rangle$  [2, Theorem III.6.5].

We use  $\text{Pr}(\mathbf{A})$  to denote the set of all prime deductive filters of an S[I]RL  $\mathbf{A}$ , including  $\mathbf{A}$  itself. We always consider  $\text{Pr}(\mathbf{A})$  to be partially ordered by set inclusion. For a deductive filter  $F$  of  $\mathbf{A}$ , we write

$$\uparrow^{\mathbf{A}} F = \{H \in \text{Pr}(\mathbf{A}) : F \subseteq H\},$$

i.e.,  $\uparrow^{\mathbf{A}} F$  abbreviates  $\uparrow^{\text{Pr}(\mathbf{A})} F$ , where  $\text{Pr}(\mathbf{A})$  is considered as a subset of the lattice of deductive filters of  $\mathbf{A}$ .

**Remark 4.1.** Suppose  $h: \mathbf{A} \rightarrow \mathbf{B}$  is a surjective homomorphism of S[I]RLs. The kernel of  $h$  is  $\Omega^{\mathbf{A}} K$  for some deductive filter  $K$  of  $\mathbf{A}$ . If  $G$  is a deductive filter of  $\mathbf{A}$ , with  $K \subseteq G$ , then  $h[G] := \{h(g) : g \in G\}$  is a deductive filter of  $\mathbf{B}$ , and by the Correspondence Theorem of Universal Algebra,

$$H \mapsto h^{\leftarrow}[H] := \{a \in \mathbf{A} : h(a) \in H\}$$

is a lattice isomorphism from the deductive filter lattice of  $\mathbf{B}$  onto the lattice of deductive filters of  $\mathbf{A}$  that contain  $K$ ; the inverse isomorphism is given by  $G \mapsto h[G]$ . In particular,

$$h^{\leftarrow}[h[G]] = G \text{ for all deductive filters } G \text{ of } \mathbf{A} \text{ such that } K \subseteq G. \tag{4.3}$$

Clearly, a deductive filter  $H$  of  $\mathbf{B}$  is prime iff the filter  $h^{\leftarrow}[H]$  of  $\mathbf{A}$  is prime, so  $H \mapsto h^{\leftarrow}[H]$  also defines an isomorphism of partially ordered sets from  $\text{Pr}(\mathbf{B})$  onto  $\uparrow^{\mathbf{A}} K$  (both ordered by inclusion).

### 5. Negative cones

**Definition 5.1.** The *negative cone* of an S[I]RL  $\mathbf{A} = \langle \mathbf{A}; \wedge, \vee, \cdot, \rightarrow, e, [, \neg] \rangle$  is the Brouwerian algebra

$$\mathbf{A}^- = \langle \mathbf{A}^-; \wedge|_{(\mathbf{A}^-)^2}, \vee|_{(\mathbf{A}^-)^2}, \rightarrow^-, e \rangle,$$

where  $\mathbf{A}^- = \{a \in \mathbf{A} : a \leq e\}$  and  $a \rightarrow^- b = (a \rightarrow b) \wedge e$  for all  $a, b \in \mathbf{A}^-$ .

**Lemma 5.2.** Let  $\mathbf{A}$  and  $\mathbf{B}$  be S[I]RLs, and  $F$  a deductive filter of  $\mathbf{A}$ .

- (i) If  $h: \mathbf{A} \rightarrow \mathbf{B}$  is a homomorphism, then  $h|_{\mathbf{A}^-}$  is a homomorphism from  $\mathbf{A}^-$  to  $\mathbf{B}^-$ . If, moreover,  $h$  is surjective, then so is  $h|_{\mathbf{A}^-}$ .
- (ii)  $\mathbf{A}^- \cap F$  is a filter of  $\mathbf{A}^-$ , and  $(\mathbf{A}/F)^- \cong \mathbf{A}^- / (\mathbf{A}^- \cap F)$ , the isomorphism and its inverse being

$$a/F \mapsto (a \wedge e) / (\mathbf{A}^- \cap F) \text{ and } a / (\mathbf{A}^- \cap F) \mapsto a/F.$$

*Proof.* (i) Since homomorphisms between S[I]RLs are isotone maps that preserve  $e$ , we have  $h[A^-] \subseteq B^-$ . Also, as  $h$  is a homomorphism, it is clear from the definitions of the operations on the negative cone that  $h|_{A^-}$  is a homomorphism from  $A^-$  to  $B^-$ . Now suppose  $h$  is onto. For each  $b \in B^-$ , there exists  $a \in A$  with  $h(a) = b$ , and since  $b \leq e$ , we have  $b = h(a) \wedge e = h(a \wedge e)$ . As  $a \wedge e \in A^-$ , this shows that  $B^- = h[A^-]$ .

(ii) Clearly,  $A^- \cap F$  is a filter of  $A^-$ . Let  $q: A \rightarrow A/F$  be the canonical surjection. By (i),  $q|_{A^-}: A^- \rightarrow (A/F)^-$  is a surjective homomorphism. For all  $a, b \in A^-$ , we have  $a \leftrightarrow b \in F$  iff  $(a \leftrightarrow b) \wedge e \in A^- \cap F$  (since  $F$  is upward closed and contains  $e$ ), i.e., the kernel of  $q|_{A^-}$  is  $\Omega^{A^-}(A^- \cap F)$ . Thus, by the Homomorphism Theorem,  $a/(A^- \cap F) \mapsto a/F$  defines an isomorphism from  $A^-/(A^- \cap F)$  onto  $(A/F)^-$ . For any  $a \in A$ , if  $a/F \in (A/F)^-$ , then  $a \wedge e \in A^-$  and  $(a \wedge e)/F = (a/F) \wedge (e/F) = a/F$ , so  $a/F \mapsto (a \wedge e)/(A^- \cap F)$  defines the inverse of the above isomorphism.  $\square$

Given an S[I]RL  $A$ , if  $F$  is a filter of  $A^-$ , then

$$Fg^A F = \{a \in A : a \geq b \text{ for some } b \in F\}, \text{ so } A^- \cap Fg^A F = F. \tag{5.1}$$

**Definition 5.3.** An S[I]RL  $A$  is *negatively generated* if  $A = Sg^A(A^-)$ .

As surjective homomorphisms always map generating sets onto generating sets, the following lemma applies.

**Lemma 5.4.** *If  $h: A \rightarrow B$  is a surjective homomorphism of S[I]RLs and  $A$  is negatively generated, then so is  $B$ .*

### 6. Duality for Brouwerian algebras

A well-known duality between Heyting algebras and Esakia spaces was established in [12, 13]. It entails an analogous duality between the variety BRA of Brouwerian algebras (considered as a concrete category, equipped with all algebraic homomorphisms) and the category PESP of ‘pointed Esakia spaces’ defined below, i.e., there is a category equivalence between BRA and the opposite category of PESP. This is explained, for instance, in [4, Section 3], but we recall the key definitions here.

A structure  $X = \langle X; \tau, \leq, m \rangle$  is a *pointed Esakia space* if  $\langle X; \leq \rangle$  is a partially ordered set with a greatest element  $m$ , and  $\langle X; \tau \rangle$  is a compact Hausdorff space in which

- (i) every open set is a union of clopen (i.e., closed and open) sets;
- (ii)  $\uparrow x$  is closed, for all  $x \in X$ ;
- (iii)  $\downarrow V$  is clopen, for all clopen  $V \subseteq X$ .

In this case, the *Priestley separation axiom* of [42] holds: for all  $x, y \in X$ ,

- (iv) if  $x \not\leq y$ , then  $x \in U$  and  $y \notin U$ , for some clopen up-set  $U \subseteq X$ .

The morphisms of PESP are the so-called *Esakia morphisms* between these objects. They are the isotone continuous functions  $g: X \rightarrow Y$  such that,

$$\text{whenever } x \in X \text{ and } g(x) \leq y \in Y, \text{ then } y = g(z) \text{ for some } z \in \uparrow x. \tag{6.1}$$



It follows that  $g(m) = m$ , and if  $g$  is bijective then  $g^{-1}: Y \rightarrow X$  is also an Esakia morphism, so  $g$  is a (categorical) isomorphism.

Given  $A \in \text{BRA}$  and  $a \in A$ , let  $\varphi^A(a)$  denote  $\{F \in \text{Pr}(A) : a \in F\}$  and  $\varphi^A(a)^c$  its complement  $\{F \in \text{Pr}(A) : a \notin F\}$ . The dual (in PESP) of  $A$  is  $A_* = \langle \text{Pr}(A); \tau, \subseteq, A \rangle$ , where  $\tau$  is the topology on  $\text{Pr}(A)$  with sub-basis

$$\{\varphi^A(a) : a \in A\} \cup \{\varphi^A(a)^c : a \in A\}.$$

The dual of a morphism  $h: A \rightarrow B$  in BRA is the Esakia morphism  $h_*: B_* \rightarrow A_*$ , defined by  $F \mapsto h^{-1}[F]$ . Thus, the contravariant functor  $(-)_*: \text{BRA} \rightarrow \text{PESP}$  is given by  $A \mapsto A_*$ ;  $h \mapsto h_*$ .

The contravariant functor  $(-)^*: \text{PESP} \rightarrow \text{BRA}$  works as follows. Let  $g: X \rightarrow Y$  be an Esakia morphism, where  $X, Y \in \text{PESP}$ . Then

$$X^* = \langle \text{Cpu}(X); \cap, \cup, \rightarrow, X \rangle \in \text{BRA},$$

where  $\text{Cpu}(X)$  is the set of all *non-empty* clopen up-sets of  $X$ , and

$$U \rightarrow V := X \setminus \downarrow(U \setminus V)$$

for all  $U, V \in \text{Cpu}(X)$ . The homomorphism  $g^*: Y^* \rightarrow X^*$  is given by  $U \mapsto g^{-1}[U]$ . We refer to  $X^*$  [resp.  $g^*$ ] as the dual of  $X$  [resp.  $g$ ] in BRA.

The functors  $(-)_*$  and  $(-)^*$  establish the aforementioned duality. For  $A \in \text{BRA}$  and  $X \in \text{PESP}$ , the respective canonical isomorphisms from  $A$  to  $A_*^*$  and from  $X$  to  $X^*_*$  are given by  $a \mapsto \varphi^A(a)$  and

$$x \mapsto \{U \in \text{Cpu}(X) : x \in U\}.$$

In PESP, there is a notion of substructure: an *E-subspace* of  $X \in \text{PESP}$  is a non-empty closed up-set  $U$  of  $X$ . It is the universe of a pointed Esakia space  $U$ , with the restricted order and the subspace topology, so the inclusion  $U \rightarrow X$  is an Esakia morphism.

**Lemma 6.1** [13].

- (i) A homomorphism  $h$  between Brouwerian algebras is surjective iff  $h_*$  is injective. Also,  $h$  is injective iff  $h_*$  is surjective.
- (ii) The image of a morphism in PESP is an *E-subspace* of the co-domain.

The ES property for BRA is relied on in the right-to-left implication of the first assertion of Lemma 6.1(i). The forward implication in the second assertion of (i) employs the Prime Filter Extension Theorem. In the absence of a convenient reference, a proof of the next lemma is supplied below; the result is presumably well-known.

**Lemma 6.2.** Let  $F$  be a filter of a Brouwerian algebra  $A$ , and  $q: A \rightarrow A/F$  the canonical surjection. Then  $q_*$  is an isomorphism from  $(A/F)_*$  onto the *E-subspace*  $q_*[(A/F)_*]$  of  $A_*$  whose universe is  $\uparrow^A F$ . Also, the map

$$\varphi_F^A: a/F \mapsto \{H \in \text{Pr}(A) : F \cup \{a\} \subseteq H\}$$

is an isomorphism from  $A/F$  onto  $(q_*[(A/F)_*])^*$  and the following diagram commutes, where  $i_1: q_*[(A/F)_*] \rightarrow A_*$  is the inclusion map.

$$\begin{array}{ccc}
 \mathbf{A} & \xrightarrow{q} & \mathbf{A}/F \\
 \downarrow \varphi^{\mathbf{A}} & & \downarrow \varphi_F^{\mathbf{A}} \\
 \mathbf{A}_*^* & \xrightarrow{i_1^*} & (q_*[(\mathbf{A}/F)_*])^* = (\uparrow^{\mathbf{A}}F)^*
 \end{array}$$

Furthermore, if  $G$  is a filter of  $\mathbf{A}$ , with  $F \subseteq G$ , then the following diagram commutes, where  $q' : a/F \mapsto a/G$ , and  $i_2$  is the inclusion map.

$$\begin{array}{ccc}
 \mathbf{A}/F & \xrightarrow{q'} & \mathbf{A}/G \\
 \downarrow \varphi_F^{\mathbf{A}} & & \downarrow \varphi_G^{\mathbf{A}} \\
 (\uparrow^{\mathbf{A}}F)^* & \xrightarrow{i_2^*} & (q_*[(\mathbf{A}/G)_*])^* = (\uparrow^{\mathbf{A}}G)^*
 \end{array}$$

*Proof.* As  $q : \mathbf{A} \rightarrow \mathbf{A}/F$  is a surjective BRA-morphism, Lemma 6.1(i) shows that  $q_* : (\mathbf{A}/F)_* \rightarrow \mathbf{A}_*$  is an injective Esakia morphism; its image is the E-subspace  $\uparrow^{\mathbf{A}}F$  of  $\mathbf{A}_*$ , by Lemma 6.1(ii) and Remark 4.1. Let  $k = q_*^{-1}|_{\uparrow^{\mathbf{A}}F}$ , so  $k : \uparrow^{\mathbf{A}}F \cong (\mathbf{A}/F)_*$  is defined by

$$k(H) = q[H] = H/F := \{a/F : a \in H\} \quad (H \in \uparrow^{\mathbf{A}}F),$$

and  $k^* : (\mathbf{A}/F)_*^* \cong (\uparrow^{\mathbf{A}}F)^*$ . Since  $\varphi^{\mathbf{A}/F} : \mathbf{A}/F \cong (\mathbf{A}/F)_*^*$ , we have

$$k^* \circ \varphi^{\mathbf{A}/F} : \mathbf{A}/F \cong (\uparrow^{\mathbf{A}}F)^*.$$

For each  $a \in A$ ,

$$\begin{aligned}
 (k^* \circ \varphi^{\mathbf{A}/F})(a/F) &= k^*(\{H \in \text{Pr}(\mathbf{A}/F) : a/F \in H\}) \\
 &= \{G \in \uparrow^{\mathbf{A}}F : a/F \in G/F\} \\
 &= \{G \in \uparrow^{\mathbf{A}}F : a \in G\} \text{ (by (4.3))} = \varphi_F^{\mathbf{A}}(a/F),
 \end{aligned}$$

so  $k^* \circ \varphi^{\mathbf{A}/F} = \varphi_F^{\mathbf{A}}$ , whence  $\varphi_F^{\mathbf{A}} : \mathbf{A}/F \cong (\uparrow^{\mathbf{A}}F)^* = (q_*[(\mathbf{A}/F)_*])^*$ .

Commutativity of the second diagram subsumes that of the first (after identification of  $\mathbf{A}/\{e\}$  with  $\mathbf{A}$ , and  $\varphi_{\{e\}}^{\mathbf{A}}$  with  $\varphi^{\mathbf{A}}$ ).

Accordingly, let  $G$  be a filter of  $\mathbf{A}$ , with  $F \subseteq G$ , so  $q' : a/F \mapsto a/G$  is a homomorphism from  $\mathbf{A}/F$  onto  $\mathbf{A}/G$ . For each  $a \in A$ , the respective left and right hand sides of the equation  $\varphi_G^{\mathbf{A}}(q'(a/F)) = i_2^*(\varphi_F^{\mathbf{A}}(a/F))$  are, by definition,

$$\{H \in \text{Pr}(\mathbf{A}) : G \cup \{a\} \subseteq H\} \text{ and } (\uparrow^{\mathbf{A}}G) \cap \{H \in \text{Pr}(\mathbf{A}) : F \cup \{a\} \subseteq H\},$$

which are clearly equal, so  $i_2^* \circ \varphi_F^{\mathbf{A}} = \varphi_G^{\mathbf{A}} \circ q'$ . □

### 7. Depth

For  $\mathbf{X} = \langle X; \tau, \leq, m \rangle \in \text{PESP}$  and  $x \in X$ , we define  $\text{depth}(x)$  (the *depth* of  $x$  in  $\mathbf{X}$ ) to be the greatest  $n \in \omega$  (if it exists) such that there is a chain

$$x = x_0 < x_1 < \dots < x_n = m$$

in  $\mathbf{X}$ . (Thus,  $m$  has depth 0 in  $\mathbf{X}$ .) If no greatest such  $n$  exists, we set  $\text{depth}(x) = \infty$ . We define  $\text{depth}(\mathbf{X}) = \sup \{ \text{depth}(x) : x \in X \}$ .

For  $\mathbf{A} \in \text{BRA}$  and any subvariety  $\mathbf{K}$  of  $\text{BRA}$ , we define

$$\text{depth}(\mathbf{A}) = \text{depth}(\mathbf{A}_*) \text{ and } \text{depth}(\mathbf{K}) = \sup \{ \text{depth}(\mathbf{B}) : \mathbf{B} \in \mathbf{K} \}.$$

If a subvariety of  $\text{BRA}$  is finitely generated (i.e., of the form  $\mathbb{V}(\mathbf{A})$  for some finite  $\mathbf{A} \in \text{BRA}$ ) then it has finite depth, and if it has finite depth then it is locally finite (i.e., its finitely generated members are finite). Both converses are false. For each  $n \in \omega$ , the class  $\text{BRA}_n = \{ \mathbf{A} \in \text{BRA} : \text{depth}(\mathbf{A}) \leq n \}$  is a finitely axiomatizable variety. These claims are explained in [4, Section 4], where their antecedents are also discussed.

**Definition 7.1.** For any S[I]RL  $\mathbf{A}$  and any variety  $\mathbf{K}$  of S[I]RLs, we define the *depth* of  $\mathbf{A}$  to be the depth of its negative cone  $\mathbf{A}^-$ , and the *depth* of  $\mathbf{K}$  to be  $\sup \{ \text{depth}(\mathbf{B}) : \mathbf{B} \in \mathbf{K} \}$ .

For each  $n \in \omega$ , an S[I]RL has depth at most  $n$  iff it satisfies the equations that result from the axioms for  $\text{BRA}_n$  when we replace  $\rightarrow$  by  $\rightarrow^-$  and  $x$  by  $x \wedge e$ , for every apparent variable  $x$ . Thus, the S[I]RLs of depth at most  $n$  also form a finitely axiomatizable variety.

Consequently, every finitely generated variety of S[I]RLs has finite depth, and the class of S[I]RLs of depth greater than (any fixed)  $n$  is closed under ultraproducts. Since the class of S[I]RLs that are FSI is also closed under ultraproducts, it follows that, when each FSI member of a variety  $\mathbf{K}$  of S[I]RLs has finite depth, then so does  $\mathbf{K}$ .

## 8. The ES property

We can now formulate the main result of this paper.

**Theorem 8.1.** *Let  $\mathbf{K}$  be a variety of S[I]RLs, such that each FSI member of  $\mathbf{K}$  has finite depth and is negatively generated. Then every  $\mathbf{K}$ -epimorphism is surjective.*

The proof of Theorem 8.1 is by contradiction, and it proceeds via a sequence of claims. Let  $\mathbf{K}$  be as postulated, and suppose that  $\mathbf{K}$  lacks the ES property. By Theorem 2.2, some  $\mathbf{A} \in \mathbf{K}_{FSI}$  has a proper  $\mathbf{K}$ -epic subalgebra. Now  $\mathbf{A}/\theta \in \mathbf{K}$  for all  $\theta \in \text{Con } \mathbf{A}$ , as  $\mathbf{K}$  is a variety. We shall define a congruence  $\theta$  of  $\mathbf{A}$  such that the following is true.

**Claim 1.** *There exist  $a \in A$  and a  $\mathbf{K}$ -epic proper subalgebra  $\mathbf{C}$  of  $\mathbf{A}/\theta$  such that  $\mathbf{C}$  is negatively generated and  $\mathbf{A}/\theta$  is generated by  $C^- \cup \{a/\theta\}$ , and  $a/\theta \prec e/\theta$  in  $\mathbf{A}/\theta$ .*

Once  $\theta$  has been identified and Claim 1 proved, we shall contradict the fact that  $\mathbf{C}$  is  $\mathbf{K}$ -epic in  $\mathbf{A}/\theta$ , by constructing a non-identity homomorphism  $\ell: \mathbf{A}/\theta \rightarrow \mathbf{A}/\theta$ , such that  $\ell|_C = \text{id}_C$ , as follows.

Let  $b = a/\theta$ . Then  $Fg^{A/\theta}\{b\} = \{d \in A/\theta : b \leq d\}$ , because  $b < e/\theta$ . Let  $\alpha = \Omega^{A/\theta} Fg^{A/\theta}\{b\}$ . For any  $u, v \in A/\theta$ , we have

$$u \equiv_\alpha v \quad \text{iff} \quad b \leq u \leftrightarrow v. \tag{8.1}$$

In particular,  $e/\theta \equiv_\alpha b$ , by (3.6) and (3.8).

Let  $\{a_i : i \in I\}$  be an indexing of  $A/\theta$ , and  $\vec{c} = c_0, c_1, \dots$  a well-ordering of the elements of  $C^-$ . Since  $C \neq A/\theta$  and  $\mathbf{A}/\theta$  is generated by  $C^- \cup \{b\}$ , it follows that  $b \notin C$  and, for each  $i \in I$ , we have  $a_i = t_i^{A/\theta}(b, \vec{c})$  for a suitable S[I]RL-term  $t_i(x, \vec{y})$ , where  $\vec{y} = y_0, y_1, \dots$ . When  $a_i$  is some  $c_j \in C^-$ , we can (and do) choose  $t_i$  to be the variable  $y_j$ .

We define  $\ell: A/\theta \rightarrow A/\theta$  by

$$\ell(a_i) = t_i^{A/\theta}(e/\theta, \vec{c}) \text{ for all } i \in I.$$

By the above choice,  $\ell(c_j) = c_j$  for  $j = 0, 1, \dots$ , while  $\ell[A/\theta] \subseteq C$ , because  $C$  is a subalgebra of  $\mathbf{A}/\theta$ . We claim that  $\ell$  is a homomorphism.

To see this, let  $\sigma$  be a fundamental S[I]RL-operation symbol, and consider  $a_{i_1}, \dots, a_{i_n} \in A/\theta$ , where  $n$  is the rank of  $\sigma$ . Then  $\sigma^{A/\theta}(a_{i_1}, \dots, a_{i_n}) = a_j$  for some  $j \in I$ . For this  $j$ , we perform the following calculation, where every term is evaluated in  $\mathbf{A}/\theta$ :

$$\begin{aligned} \sigma(\ell(a_{i_1}), \dots, \ell(a_{i_n})) &= \sigma(t_{i_1}(e/\theta, \vec{c}), \dots, t_{i_n}(e/\theta, \vec{c})) \\ &\equiv_\alpha \sigma(t_{i_1}(b, \vec{c}), \dots, t_{i_n}(b, \vec{c})) \\ &= \sigma(a_{i_1}, \dots, a_{i_n}) = a_j = t_j(b, \vec{c}) \\ &\equiv_\alpha t_j(e/\theta, \vec{c}) = \ell(a_j) = \ell(\sigma(a_{i_1}, \dots, a_{i_n})). \end{aligned}$$

By (8.1), therefore,

$$b \leq (\sigma(\ell(a_{i_1}), \dots, \ell(a_{i_n})) \leftrightarrow \ell(\sigma(a_{i_1}, \dots, a_{i_n}))) \wedge (e/\theta).$$

Note that

$$(\sigma(\ell(a_{i_1}), \dots, \ell(a_{i_n})) \leftrightarrow \ell(\sigma(a_{i_1}, \dots, a_{i_n}))) \wedge (e/\theta) \in C$$

(because  $\ell[A/\theta] \subseteq C$ ), but  $b \notin C$ , so

$$b < (\sigma(\ell(a_{i_1}), \dots, \ell(a_{i_n})) \leftrightarrow \ell(\sigma(a_{i_1}, \dots, a_{i_n}))) \wedge (e/\theta) \leq e/\theta.$$

Since  $b < e/\theta$  in  $\mathbf{A}/\theta$ , this forces

$$e/\theta = (\sigma(\ell(a_{i_1}), \dots, \ell(a_{i_n})) \leftrightarrow \ell(\sigma(a_{i_1}, \dots, a_{i_n}))) \wedge (e/\theta),$$

i.e.,  $e/\theta \leq \sigma(\ell(a_{i_1}), \dots, \ell(a_{i_n})) \leftrightarrow \ell(\sigma(a_{i_1}, \dots, a_{i_n}))$ . Then, by (3.7),

$$\sigma(\ell(a_{i_1}), \dots, \ell(a_{i_n})) = \ell(\sigma(a_{i_1}, \dots, a_{i_n})),$$

confirming that  $\ell$  is a homomorphism.

For each  $c \in C$ , we have  $c = t^{A/\theta}(\vec{c})$  for some S[I]RL-term  $t$  (as  $C$  is generated by  $C^-$ ), so

$$\ell(c) = \ell(t(\vec{c})) = t(\ell(c_0), \ell(c_1), \dots) = t(c_0, c_1, \dots) = c.$$

This shows that  $\ell|_C = \text{id}_C$ , but  $\ell(b) \neq b$ , since  $\ell(b) \in C$ . As intended, this contradicts the fact that  $C$  is K-epic in  $\mathbf{A}/\theta$ .

It remains to construct  $\theta$  and to prove Claim 1. The construction of  $\theta$  exploits the assumption that members of  $\mathbf{K}_{FSI}$  have finite depth.

Recall that  $\mathbf{A}$  has a proper  $\mathbf{K}$ -epic subalgebra,  $\mathbf{B}$  say. As  $\mathbf{A}$  is FSI, so is  $\mathbf{B}$  (because  $\mathbf{K}$  has EDPM, as we noted after (4.1)). By assumption, therefore,  $\mathbf{A}$  and  $\mathbf{B}$  are both negatively generated, so  $\mathbf{B}^- \neq \mathbf{A}^-$ , because  $\mathbf{B} \neq \mathbf{A}$ .

For each  $F \in \text{Pr}(\mathbf{A}^-)$ , we clearly have  $B \cap F = B^- \cap F = i_*(F)$ , where  $i$  is the inclusion map  $i: \mathbf{B}^- \rightarrow \mathbf{A}^-$ , considered as a BRA-morphism.

As  $i$  is not surjective, its dual  $i_*: (\mathbf{A}^-)_* \rightarrow (\mathbf{B}^-)_*$  is not injective, by Lemma 6.1(i), i.e., the following set is not empty:

$$W := \{ \langle F_1, F_2 \rangle \in (\text{Pr}(\mathbf{A}^-))^2 : F_1 \neq F_2 \text{ and } F_1 \cap B = F_2 \cap B \}.$$

By assumption,  $\mathbf{A}^-$  has finite depth, so

$$\{ \min \{ \text{depth}(F_1), \text{depth}(F_2) \} : \langle F_1, F_2 \rangle \in W \}$$

is a non-empty subset of  $\omega$ , and therefore has a least element,  $n$  say. Pick  $F_1 \in \text{Pr}(\mathbf{A}^-)$  such that  $\text{depth}(F_1) = n$  and  $\langle F_1, G \rangle \in W$  for some  $G$ . Now,

$$\text{whenever } \langle F'_1, F'_2 \rangle \in W \text{ then } \text{depth}(F_1) \leq \text{depth}(F'_1), \text{depth}(F'_2). \tag{8.2}$$

Having fixed  $F_1$  in this way, we similarly choose  $F_2 \in \text{Pr}(\mathbf{A}^-) \setminus \{F_1\}$  such that  $F_1 \cap B = F_2 \cap B$  and

$$\text{whenever } \langle F_1, F'_2 \rangle \in W, \text{ then } \text{depth}(F_2) \leq \text{depth}(F'_2). \tag{8.3}$$

As  $\langle F_1, F_2 \rangle \in W$ , we have  $\text{depth}(F_1) \leq \text{depth}(F_2)$  (by (8.2)), so  $F_1$  is not a proper subset of  $F_2$ .

**Lemma 8.2.** *If  $F_1 \subsetneq G \in \text{Pr}(\mathbf{A}^-)$ , then  $F_2 \subsetneq G$ .*

*Proof.* Let  $F_1 \subsetneq G \in \text{Pr}(\mathbf{A}^-)$ , so  $\text{depth}(G) < \text{depth}(F_1)$ . As  $i_*$  is an Esakia morphism and  $i_*(F_2) = i_*(F_1) \subseteq i_*(G)$ , there exists  $H \in \text{Pr}(\mathbf{A}^-)$  such that  $F_2 \subseteq H$  and  $i_*(G) = i_*(H)$ , by (6.1), i.e.,  $G \cap B = H \cap B$ . Therefore, if  $G \neq H$ , then  $\text{depth}(F_1) \leq \text{depth}(G)$ , by (8.2). This is a contradiction, so  $G = H$ , whence  $F_2 \subseteq G$ . If  $F_2 = G$ , then

$$\text{depth}(F_2) = \text{depth}(G) < \text{depth}(F_1),$$

contradicting the fact that  $\text{depth}(F_1) \leq \text{depth}(F_2)$ . Therefore,  $F_2 \subsetneq G$ . □

**Lemma 8.3.** *If  $F_2 \subsetneq G \in \text{Pr}(\mathbf{A}^-)$ , then  $F_1 \subseteq G$ .*

*Proof.* Let  $F_2 \subsetneq G \in \text{Pr}(\mathbf{A}^-)$ . Again,  $i_*(F_1) = i_*(F_2) \subseteq i_*(G)$ , so there exists  $H \in \text{Pr}(\mathbf{A}^-)$  such that  $F_1 \subseteq H$  and  $i_*(G) = i_*(H)$ . Suppose  $G \neq H$ . If  $F_1 = H$ , then  $F_1 \cap B = G \cap B$ , so, by (8.3),  $\text{depth}(F_2) \leq \text{depth}(G)$ , contradicting the fact that  $F_2 \subsetneq G$ . Therefore,  $F_1 \subsetneq H$ , so  $\text{depth}(H) < \text{depth}(F_1)$ . Then, by (8.2),  $\text{depth}(F_1) \leq \text{depth}(H)$ , since  $G \cap B = H \cap B$ . This is a contradiction, so  $G = H$ , whence  $F_1 \subseteq G$ . □

Recalling that  $F_1, F_2$  are distinct and that  $F_1$  is not properly contained in  $F_2$ , we make the following claim:

**Claim 2.** *There are just two possibilities:*

- (A)  $F_2 \subsetneq F_1$ , in which case  $F_2 \prec F_1$  (in fact,  $F_1$  is the least strict upper bound of  $F_2$  in  $\text{Pr}(\mathbf{A}^-)$ );
- (B)  $F_1$  and  $F_2$  are incomparable, in which case they have the same depth, the same strict upper bounds and, therefore, the same covers in  $\text{Pr}(\mathbf{A}^-)$ .

*Proof.* If  $F_2 \subsetneq F_1$ , then  $F_1$  is the least strict upper bound of  $F_2$ , by Lemma 8.3, i.e.,  $F_1$  is the unique cover of  $F_2$ . We may therefore assume that  $F_2$  is not a proper subset of  $F_1$ , i.e., that  $F_1$  and  $F_2$  are incomparable. Then, by Lemmas 8.2 and 8.3,  $F_1$  and  $F_2$  have the same strict upper bounds, and hence the same covers in  $\text{Pr}(\mathbf{A}^-)$ , from which it clearly follows that they also have the same depth.  $\square$

By (5.1),  $Fg^A(F_1 \cap F_2) = \{d \in A : d \geq c \text{ for some } c \in F_1 \cap F_2\}$ . We define

$$\theta = \Omega^A Fg^A(F_1 \cap F_2).$$

**Claim 3.** *The following diagram commutes, where the maps will be defined in the proof.*

$$\begin{array}{ccccc}
 (\mathbf{B}/(\theta|_B))^- & \xleftarrow{j} & & & (\mathbf{A}/\theta)^- \\
 \downarrow \wr_{i_2} & & & & \downarrow \wr_{i_1} \\
 \mathbf{B}^-/(F_1 \cap B) & & \mathbf{X}^* & \xleftarrow{\varphi_{F_1 \cap F_2}^{\mathbf{A}^-}} & \mathbf{A}^-/(F_1 \cap F_2) \\
 \downarrow \wr_{\varphi_{F_1 \cap B}^{\mathbf{B}^-}} & & \downarrow (i_Y)^* & & \downarrow q \\
 \mathbf{Z}^* & \xrightarrow{(i_*|_Y)^*} & \mathbf{Y}^* & \xleftarrow{\varphi_{F_1}^{\mathbf{A}^-}} & \mathbf{A}^-/F_1
 \end{array}$$

*Proof.* The map  $j: \mathbf{B}/(\theta|_B) \rightarrow \mathbf{A}/\theta$ , defined by  $b/(\theta|_B) \mapsto b/\theta$ , is an injective K-epimorphism, by Lemma 2.1. By Lemma 5.2(i), the restriction of  $j$  to  $(\mathbf{B}/(\theta|_B))^-$  is a BRA-morphism from  $(\mathbf{B}/(\theta|_B))^-$  into  $(\mathbf{A}/\theta)^-$ . We shall not distinguish notationally between  $j$  and this restriction. Whenever  $b \in B$  and  $b/(\theta|_B) \leq e/(\theta|_B)$ , then  $b/(\theta|_B) = (b \wedge e)/(\theta|_B)$  and  $b/\theta = (b \wedge e)/\theta$ , so

$$(\mathbf{B}/(\theta|_B))^- = \{b/(\theta|_B) : b \in B^-\} \tag{8.4}$$

and  $j[(\mathbf{B}/(\theta|_B))^-] = \{b/\theta : b \in B^-\}$ .

Let  $K = F_1 \cap F_2$ , so  $\theta = \Omega^A Fg^A K$ . By (5.1),  $A^- \cap Fg^A K = K$ , so  $B^- \cap Fg^A K = B^- \cap K = B \cap K = B \cap F_1$  (since  $B \cap F_1 = B \cap F_2$ ). By (4.1),  $\theta|_B = \Omega^B(B \cap Fg^A K)$ , so Lemma 5.2(ii) supplies isomorphisms

$$i_1: (\mathbf{A}/\theta)^- \cong \mathbf{A}^-/(F_1 \cap F_2) \text{ and } i_2: (\mathbf{B}/(\theta|_B))^- \cong \mathbf{B}^-/(F_1 \cap B),$$

defined by  $a/\theta \mapsto (a \wedge e)/(F_1 \cap F_2)$  and  $b/(\theta|_B) \mapsto (b \wedge e)/(F_1 \cap B)$ .

By Lemma 6.2,  $\uparrow^{\mathbf{B}^-}(F_1 \cap B)$  is the universe of an E-subspace,  $\mathbf{Z}$  say, of  $\mathbf{B}^-$ , and  $\varphi_{F_1 \cap B}^{\mathbf{B}^-}: \mathbf{B}^-/(F_1 \cap B) \cong \mathbf{Z}^*$ . Also,  $q: \mathbf{A}^-/(F_1 \cap F_2) \mapsto \mathbf{A}^-/F_1$  defines a homomorphism from  $\mathbf{A}^-/(F_1 \cap F_2)$  onto  $\mathbf{A}^-/F_1$ . Let  $\mathbf{X}$  [resp.  $\mathbf{Y}$ ] be the E-subspace of  $\mathbf{A}^-$  with universe  $\uparrow^{\mathbf{A}^-}(F_1 \cap F_2)$  [resp.  $\uparrow^{\mathbf{A}^-}F_1$ ]. Let  $i_Y: \mathbf{Y} \rightarrow \mathbf{X}$  be the inclusion morphism in PESP. By Lemma 6.2, the following diagram commutes.

$$\begin{array}{ccc}
 \mathbf{A}^- / (F_1 \cap F_2) & \xrightarrow{\varphi_{F_1 \cap F_2}^{\mathbf{A}^-}} & \mathbf{X}^* \\
 \downarrow q & & \downarrow (i_Y)^* \\
 \mathbf{A}^- / F_1 & \xrightarrow{\varphi_{F_1}^{\mathbf{A}^-}} & \mathbf{Y}^*
 \end{array}$$

Recall that the BRA-morphism  $i: \mathbf{B}^- \rightarrow \mathbf{A}^-$  is the inclusion map. As  $i$  is injective, its dual  $i_*: \mathbf{A}^* \rightarrow \mathbf{B}^*$  is surjective, by Lemma 6.1(i).

The above definitions clearly imply that  $i_*[Y] \subseteq Z$ . To establish the reverse inclusion, let  $G \in Z$ . By the Prime Filter Extension Theorem,  $G = H \cap B$  for some  $H \in \text{Pr}(\mathbf{A}^-)$ . Now,  $i_*[F_1] = B \cap F_1 \subseteq G = i_*[H]$ , so by (6.1),  $i_*[H] = i_*[H']$  for some  $H' \in \uparrow^{\mathbf{A}^-} F_1 = Y$ , whence  $G = i_*[H'] \in i_*[Y]$ . Therefore,  $Z = i_*[Y]$ .

We claim that  $i_*|_Y$  is injective. Suppose, by way of contradiction, that  $H_1, H_2 \in Y$ , with  $H_1 \neq H_2$  and  $i_*[H_1] = i_*[H_2]$ . For each  $k \in \{1, 2\}$ , (8.2) shows that  $\text{depth}(F_1) \leq \text{depth}(H_k)$ , but  $F_1 \subseteq H_k$ , so  $H_k = F_1$ , whence  $H_1 = H_2$ . This contradiction confirms that  $i_*|_Y$  is injective, whence  $i_*|_Y: \mathbf{Y} \cong \mathbf{Z}$  in PESP. In BRA, therefore,  $(i_*|_Y)^*: \mathbf{Z}^* \cong \mathbf{Y}^*$ .

A composition of isomorphisms in BRA is an isomorphism, so

$$g := (i_*|_Y)^* \circ \varphi_{F_1 \cap B}^{\mathbf{B}^-}: \mathbf{B}^- / (F_1 \cap B) \cong \mathbf{Y}^*. \tag{8.5}$$

To show that the diagram in Claim 3 commutes, it remains to prove that  $g \circ i_2 = \varphi_{F_1}^{\mathbf{A}^-} \circ q \circ i_1 \circ j$ . And indeed, if  $b \in B$  and  $b/(\theta|_B) \in (B/(\theta|_B))^-$ , then

$$\begin{aligned}
 (g \circ i_2)(b/(\theta|_B)) &= g((b \wedge e)/(F_1 \cap B)) \\
 &= (i_*|_Y)^* (\{H \in \text{Pr}(\mathbf{B}^-) : (F_1 \cap B) \cup \{b \wedge e\} \subseteq H\}) \\
 &= \{H \in \text{Pr}(\mathbf{A}^-) : F_1 \subseteq H \text{ and } (F_1 \cap B) \cup \{b \wedge e\} \subseteq H \cap B\} \\
 &= \{H \in \text{Pr}(\mathbf{A}^-) : F_1 \cup \{b \wedge e\} \subseteq H\} \\
 &= \varphi_{F_1}^{\mathbf{A}^-}((b \wedge e)/F_1) = (\varphi_{F_1}^{\mathbf{A}^-} \circ q \circ i_1 \circ j)(b/(\theta|_B)).
 \end{aligned}$$

□

**Claim 4.** Suppose  $k \in \{1, 2\}$  and  $a \in A^-$  and  $b \in B^-$ , where  $a \equiv_{\theta} b$ . Then  $a \in F_k$  iff  $b \in F_k$ . Consequently,  $a \notin (F_1 \setminus F_2) \cup (F_2 \setminus F_1)$ .

*Proof.* As  $a \equiv_{\theta} b$ , we have  $a \leftrightarrow b \in Fg^{\mathbf{A}}(F_1 \cap F_2)$ , so

$$a \leftrightarrow^- b := (a \leftrightarrow b) \wedge e \in A^- \cap Fg^{\mathbf{A}}(F_1 \cap F_2) = F_1 \cap F_2,$$

by (5.1). As  $a \rightarrow^- b, b \rightarrow^- a \geq a \leftrightarrow^- b$ , it follows that  $a \rightarrow^- b, b \rightarrow^- a \in F_k$ . Thus,  $a \in F_k$  iff  $b \in F_k$ . In particular, if  $a \in (F_1 \setminus F_2) \cup (F_2 \setminus F_1)$  then  $b \in B \cap ((F_1 \setminus F_2) \cup (F_2 \setminus F_1))$ , contradicting the fact that  $B \cap F_1 = B \cap F_2$ .

□

By Claim 3,  $h := \varphi_{F_1 \cap F_2}^{\mathbf{A}^-} \circ i_1 : (\mathbf{A}/\theta)^- \cong \mathbf{X}^*$  and, for each  $a \in A$  such that  $a/\theta \in (A/\theta)^-$ , we have

$$h(a/\theta) = \{H \in \text{Pr}(\mathbf{A}^-) : (F_1 \cap F_2) \cup \{a \wedge e\} \subseteq H\}.$$

By Claim 2,  $F_1 \setminus F_2 \neq \emptyset$ . In fact,

$$h(d/\theta) = \uparrow^{A^-} F_1 \quad \text{for all } d \in F_1 \setminus F_2.$$

To confirm this, let  $d \in F_1 \setminus F_2$ . Clearly,  $\uparrow^{A^-} F_1 \subseteq h(d/\theta)$ . Conversely, let  $H \in h(d/\theta)$ . If  $F_1 \not\subseteq H$ , then since  $F_1 \cap F_2 \subseteq H \in \text{Pr}(\mathbf{A}^-)$ , we have  $F_2 \subseteq H$ , by (4.2). In that case,  $F_2 \subsetneq H$ , because  $d \in H \setminus F_2$ , but this contradicts Lemma 8.3. Thus,  $F_1 \subseteq H$ , and so  $h(d/\theta) = \uparrow^{A^-} F_1$ , as claimed.

In Case (A) of Claim 2, we have  $\uparrow^{A^-} F_2 = \uparrow^{A^-} (F_1 \cap F_2) = X = h(e/\theta)$ .

In Case (B), we have  $F_2 \setminus F_1 \neq \emptyset$ , and we claim that

$$h(d/\theta) = \uparrow^{A^-} F_2 \quad \text{for every } d \in F_2 \setminus F_1.$$

To see this, let  $d \in F_2 \setminus F_1$ . It is clear that  $\uparrow^{A^-} F_2 \subseteq h(d/\theta)$ , so consider  $H \in h(d/\theta)$ . If  $F_2 \not\subseteq H$  then, since  $F_1 \cap F_2 \subseteq H$ , we have  $F_1 \subseteq H$ , by (4.2). In that case,  $F_1 \subsetneq H$ , as  $d \in H \setminus F_1$ , but this contradicts Lemma 8.2. Thus,  $F_2 \subseteq H$ , and so  $h(d/\theta) \subseteq \uparrow^{A^-} F_2$ .

We define

$$M = (A/\theta)^- \setminus j[(B/(\theta|_B))^-].$$

**Claim 5.** Fix any  $a_1 \in F_1 \setminus F_2$ . Choose  $a_2$  to be  $e$  in Case (A) of Claim 2, and an arbitrary element of  $F_2 \setminus F_1$  in Case (B). Then

$$M = \{a_1/\theta\} \text{ in Case (A), and } M = \{a_1/\theta, a_2/\theta\} \text{ in Case (B).}$$

Moreover,  $a/\theta < e/\theta$  (in  $A/\theta$ ) for all  $a \in A$  such that  $a/\theta \in M$ .

*Proof.* Observe that  $a_1, a_2 \leq e$  and, as we showed above,

$$h(a_1/\theta) = \uparrow^{A^-} F_1 \text{ and } h(a_2/\theta) = \uparrow^{A^-} F_2.$$

By Claim 4 and (8.4), we have  $a_1/\theta \in M$  and, in Case (B),  $a_2/\theta \in M$ . In Case (A),  $a_2/\theta \notin M$ , since  $e \in B$ .

Because  $h$  is an isomorphism,  $h[M] = X^* \setminus h[j[(B/(\theta|_B))^-]]$  and, for the first assertion of Claim 5, it suffices to prove that  $h[M] \subseteq \{\uparrow^{A^-} F_1, \uparrow^{A^-} F_2\}$ .

Suppose, with a view to contradiction, that there exists  $\mathcal{U} \in h[M]$  with  $\mathcal{U} \notin \{\uparrow^{A^-} F_1, \uparrow^{A^-} F_2\}$ . Then  $\mathcal{U} \subseteq X$ , but  $\mathcal{U} \neq X$ , because

$$X = h(j(e/(\theta|_B))) \in h[j[(B/(\theta|_B))^-]].$$

We show first that  $\mathcal{U} \subsetneq \uparrow^{A^-} F_1$ .

Suppose  $F_1, F_2 \in \mathcal{U}$ . For each  $H \in X$ , we have  $F_1 \subseteq H$  or  $F_2 \subseteq H$ , by (4.2), so  $H \in \mathcal{U}$  (since  $\mathcal{U}$  is upward closed). This shows that  $X \subseteq \mathcal{U}$ , a contradiction. Therefore,  $F_1$  and  $F_2$  don't both belong to  $\mathcal{U}$ .

Suppose  $F_2 \in \mathcal{U}$ . Then  $F_1 \notin \mathcal{U}$  and  $\uparrow^{A^-} F_2 \subseteq \mathcal{U}$ , as  $\mathcal{U}$  is upward closed. If  $H \in \mathcal{U}$ , then  $H \neq F_1$  and  $F_1 \cap F_2 \subseteq H$  (as  $\mathcal{U} \subseteq X$ ). In that case,  $F_2 \subseteq H$  (otherwise,  $F_1 \subseteq H$ , by (4.2), whence  $F_1 \subsetneq H$ , but then  $F_2 \subsetneq H$ , by Lemma 8.2). This shows that  $\mathcal{U} \subseteq \uparrow^{A^-} F_2$ , so  $\mathcal{U} = \uparrow^{A^-} F_2$ , contrary to our initial assumptions about  $\mathcal{U}$ . Therefore,  $F_2 \notin \mathcal{U}$ .

We claim that  $\mathcal{U} \subseteq \uparrow^{A^-} F_1$ . For otherwise,  $F_1 \not\subseteq H$  for some  $H \in \mathcal{U}$ , whence  $H \neq F_2$  and, by (4.2),  $F_2 \subseteq H$ , i.e.,  $F_2 \subsetneq H$ , whereupon Lemma 8.3



delivers the contradiction  $F_1 \subseteq H$ . Thus,  $\mathcal{U} \subsetneq \uparrow^{A^-} F_1$  (since  $\mathcal{U} \neq \uparrow^{A^-} F_1$ , by assumption).

Now we shall argue that  $\mathcal{U} \in h[j[(B/\theta|_B)^-]]$  (contradicting the fact that  $\mathcal{U} \in h[M]$ , and thereby confirming the relation  $h[M] \subseteq \{\uparrow^{A^-} F_1, \uparrow^{A^-} F_2\}$ ).

As  $\mathcal{U} \in X^*$  and  $\mathcal{U} \subsetneq \uparrow^{A^-} F_1 = Y$ , we have

$$\mathcal{U} = \mathcal{U} \cap Y = (i_Y)^*(\mathcal{U}) \in Y^*,$$

so by Claim 3, there exists  $b \in B$  with  $b/(\theta|_B) \in (B/(\theta|_B))^-$  such that

$$\mathcal{U} = g(i_2(b/(\theta|_B))) = (i_Y)^*(h(j(b/(\theta|_B)))) = Y \cap h(b/\theta) = Y \cap \mathcal{V},$$

where  $g$  is as in (8.5), and  $\mathcal{V} := h(b/\theta)$ . By (8.4), we may assume that  $b \leq e$ .

Now  $\mathcal{V} \in X^*$ , so  $\mathcal{V}$  is an upward-closed subset of  $X$ . Note that  $F_1 \notin \mathcal{V}$  (otherwise  $Y = \uparrow^{A^-} F_1 \subseteq \mathcal{V}$ , yielding the contradiction  $\mathcal{U} \subsetneq Y = Y \cap \mathcal{V} = \mathcal{U}$ ). It follows that  $Y \not\subseteq \mathcal{V}$  (as  $F_1 \in Y$ ).

For any  $H \in \mathcal{V}$ , if  $F_1 \subseteq H$ , then  $F_1 \subsetneq H$  (as  $F_1 \notin \mathcal{V}$ ), whence  $F_2 \subsetneq H$  (by Lemma 8.2), whereas if  $F_1 \not\subseteq H$ , then  $F_2 \subseteq H$  (by (4.2)). This shows that  $\mathcal{V} \subseteq \uparrow^{A^-} F_2$ .

We now argue that  $\mathcal{V} \subseteq Y$ .

Suppose, on the contrary, that there exists  $H \in \mathcal{V} \setminus Y$ . Then  $F_1 \not\subseteq H$  (by definition of  $Y$ ), so  $F_2 \subseteq H$ , by (4.2). Now Lemma 8.3 prevents  $F_2$  from being a proper subset of  $H$ , so  $F_2 = H$ . In particular,  $F_2 \in \mathcal{V}$ , so  $\uparrow^{A^-} F_2 \subseteq \mathcal{V}$ , whence  $\mathcal{V} = \uparrow^{A^-} F_2$ .

In Case (A) of Claim 2, it would follow that  $Y = \uparrow^{A^-} F_1 \subseteq \uparrow^{A^-} F_2 = \mathcal{V}$ , a contradiction.

In Case (B), we have  $e \geq a_2 \in F_2 \setminus F_1$  and  $h(a_2/\theta) = \uparrow^{A^-} F_2 = \mathcal{V} = h(b/\theta)$ . Then, since  $h$  is injective,  $a_2/\theta = b/\theta$ , contradicting Claim 4.

This confirms that  $\mathcal{V} \subseteq Y$ , and so  $\mathcal{V} = Y \cap \mathcal{V} = \mathcal{U}$ . Therefore,

$$\mathcal{U} = h(b/\theta) = h(j(b/(\theta|_B))) \in h[j[(B/\theta|_B)^-]],$$

completing the proof that  $M$  is  $\{a_1/\theta\}$  in Case (A), and is  $\{a_1/\theta, a_2/\theta\}$  in Case (B).

It remains to show that  $a/\theta \prec e/\theta$  in  $\mathbf{A}/\theta$ , whenever  $a/\theta \in M$ .

To establish that  $a_1/\theta \prec e/\theta$  in  $\mathbf{A}/\theta$  (i.e., in  $(\mathbf{A}/\theta)^-$ ), it suffices to show that  $\uparrow^{A^-} F_1 \prec X$  in  $\mathbf{X}^*$ , because  $h$  is an isomorphism.

Suppose  $\uparrow^{A^-} F_1 \subsetneq \mathcal{W} \subsetneq X$ , where  $\mathcal{W} \in X^*$ . Then  $F_1 \not\subseteq H$  for some  $H \in \mathcal{W}$ , whence  $F_2 \subseteq H$ , by (4.2). We cannot have  $F_2 = H$ , otherwise  $\uparrow^{A^-} F_2 \subseteq \mathcal{W}$ , in which case every element  $G$  of  $X$  belongs to  $\mathcal{W}$  (as  $G$  contains  $F_1$  or  $F_2$ , again by (4.2)). Therefore,  $F_2 \subsetneq H$ , and so  $F_1 \subseteq H$ , by Lemma 8.3. This contradiction confirms that  $a_1/\theta \prec e/\theta$  in  $\mathbf{A}/\theta$ .

We may now assume that Case (B) applies. The desired conclusion (that  $a_2/\theta \prec e/\theta$ ) amounts similarly to the claim that  $\uparrow^{A^-} F_2 \prec X$  in  $\mathbf{X}^*$ . Suppose  $\uparrow^{A^-} F_2 \subsetneq \mathcal{W} \subsetneq X$ , where  $\mathcal{W} \in X^*$ . Then  $F_2 \not\subseteq H$  for some  $H \in \mathcal{W}$ . Now  $H \in X$ , so  $F_1 \subseteq H$ , by (4.2). Then  $F_1 = H$ , by Lemma 8.2, so  $F_1 \in \mathcal{W}$ , whence

$\uparrow^{A^-} F_1 \subseteq \mathcal{W}$ . By (4.2) again,  $X \subseteq (\uparrow^{A^-} F_1) \cup (\uparrow^{A^-} F_2) \subseteq \mathcal{W}$ , contradicting the fact that  $\mathcal{W} \subsetneq X$ . Therefore,  $a_2/\theta \prec e/\theta$  in  $\mathbf{A}/\theta$ .  $\square$

We are now in a position to prove Claim 1 (and hence Theorem 8.1).

*Proof of Claim 1.* Since  $\mathbf{A}$  and  $\mathbf{B}$  are negatively generated, so are  $\mathbf{A}/\theta$  and  $\mathbf{B}/(\theta|_B)$ , by Lemma 5.4. The subalgebra

$$\mathbf{J} := j[\mathbf{B}/(\theta|_B)]$$

of  $\mathbf{A}/\theta$  is isomorphic to  $\mathbf{B}/(\theta|_B)$ , so  $\mathbf{J}$  is also negatively generated. By Lemma 5.2(i),  $J^- = j[(\mathbf{B}/(\theta|_B))^-]$ , whence  $M = (A/\theta)^- \setminus J^-$ . As  $\mathbf{B}$  is  $\mathbf{K}$ -epic in  $\mathbf{A}$ , Lemma 2.1 shows that  $\mathbf{J}$  is  $\mathbf{K}$ -epic in  $\mathbf{A}/\theta$ . Moreover,

$$\mathbf{S} := \mathbf{S}g^{A/\theta}(J^- \cup \{a_1/\theta\})$$

is negatively generated (since  $J^- \cup \{a_1/\theta\} \subseteq S^-$ ), and  $\mathbf{J}$  is a subalgebra of  $\mathbf{S}$  (as  $J = \mathbf{S}g^{A/\theta}(J^-)$ ), so  $\mathbf{S}$  is  $\mathbf{K}$ -epic in  $\mathbf{A}/\theta$  (because  $\mathbf{J}$  is).

Observe that  $J \neq A/\theta$ , because  $a_1/\theta \notin J$  (by Claim 4 and (8.4), since  $a_1 \in F_1 \setminus F_2$ ), and that  $A/\theta = \mathbf{S}g^{A/\theta}((A/\theta)^-) = \mathbf{S}g^{A/\theta}(J^- \cup M)$ .

We choose  $\mathbf{C} = \mathbf{J}$  and  $a = a_1$  in Case (A). We make the same choices in Case (B) if  $a_2/\theta \in S$ . Under these conditions,  $J^- \cup M = J^- \cup \{a_1/\theta\}$  (by Claim 5) and  $A/\theta = \mathbf{S}g^{A/\theta}(J^- \cup M) \subseteq S = \mathbf{S}g^{A/\theta}(C^- \cup \{a/\theta\})$ , so  $\mathbf{A}/\theta$  is generated by  $C^- \cup \{a/\theta\}$ , as required.

In Case (B), if  $a_2/\theta \notin S$  (whence  $S \neq A/\theta$ ), we choose  $\mathbf{C} = \mathbf{S}$  and  $a = a_2$ , whereupon  $J^- \cup M = J^- \cup \{a_1/\theta, a_2/\theta\}$  (by Claim 5) and

$$A/\theta = \mathbf{S}g^{A/\theta}(J^- \cup M) \subseteq \mathbf{S}g^{A/\theta}(S^- \cup \{a_2/\theta\}) = \mathbf{S}g^{A/\theta}(C^- \cup \{a/\theta\}),$$

so again,  $\mathbf{A}/\theta$  is generated by  $C^- \cup \{a/\theta\}$ .  $\square$

## 9. Reflections and De Morgan monoids

Given an SRL  $\mathbf{A}$ , let  $A' = \{a' : a \in A\}$  be a disjoint copy of  $A$ , and let  $\perp, \top$  be distinct non-elements of  $A \cup A'$ . The *reflection*  $\mathbf{R}(\mathbf{A})$  of  $\mathbf{A}$  is the SIRL with universe  $\mathbf{R}(\mathbf{A}) = A \cup A' \cup \{\perp, \top\}$  such that  $\mathbf{A}$  is a subalgebra of the SRL-reduct of  $\mathbf{R}(\mathbf{A})$  and, for all  $a, b \in A$  and  $x, y \in \mathbf{R}(\mathbf{A})$ ,

$$x \cdot \perp = \perp < a < b' < \top = a' \cdot b', \text{ and if } x \neq \perp, \text{ then } x \cdot \top = \top;$$

$$a \cdot b' = (a \rightarrow b)';$$

$$\neg a = a' \text{ and } \neg(a') = a \text{ and } \neg\perp = \top \text{ and } \neg\top = \perp;$$

$$x \rightarrow y = \neg(x \cdot \neg y).$$

Since  $f = e'$ , we have  $\top = f^2$  and  $\perp = \neg(f^2)$ , so  $\perp, \top$  belong to every subalgebra of  $\mathbf{R}(\mathbf{A})$ .

The reflection construction originates with Meyer [34]; also see [19, Section 9]. It preserves (and reflects) distributivity and the square-increasing law, so  $\mathbf{A}$  is a Dunn monoid iff  $\mathbf{R}(\mathbf{A})$  is a De Morgan monoid. Also,  $\mathbf{A}$  is FSI iff  $\mathbf{R}(\mathbf{A})$  is.

The reflection of a variety  $\mathbf{K}$  of SRLs is the variety

$$\mathbb{R}(\mathbf{K}) := \mathbb{V}\{\mathbf{R}(\mathbf{A}) : \mathbf{A} \in \mathbf{K}\}$$

of SURLs. The structure of a member of  $\mathbb{R}(\mathbf{K})$  is illuminated by the following lemma, which is proved in [38, Section 6]. (The extra assumptions there were not relied on in the proof.)

**Lemma 9.1** (cf. [38, Lemma 6.5]). *Let  $\mathbf{A}$  be an SRL.*

- (i) *If  $\mathbf{B}$  is a subalgebra of  $\mathbf{A}$ , then  $B \cup \{b' : b \in B\} \cup \{\perp, \top\}$  is the universe of a subalgebra of  $\mathbf{R}(\mathbf{A})$  that is isomorphic to  $\mathbf{R}(\mathbf{B})$ , and every subalgebra of  $\mathbf{R}(\mathbf{A})$  arises in this way from a subalgebra of  $\mathbf{A}$ .*
- (ii) *If  $\theta$  is a congruence of  $\mathbf{A}$ , then*

$$R(\theta) := \theta \cup \{\langle a', b' \rangle : \langle a, b \rangle \in \theta\} \cup \{\langle \perp, \perp \rangle, \langle \top, \top \rangle\}$$

*is a congruence of  $\mathbf{R}(\mathbf{A})$ , and  $\mathbf{R}(\mathbf{A})/R(\theta) \cong \mathbf{R}(\mathbf{A}/\theta)$ . Also, every proper congruence of  $\mathbf{R}(\mathbf{A})$  has the form  $R(\theta)$  for some  $\theta \in \text{Con } \mathbf{A}$ .*

- (iii) *If  $\{\mathbf{A}_i : i \in I\}$  is a family of SRLs and  $\mathcal{U}$  is an ultrafilter over  $I$ , then  $\prod_{i \in I} \mathbf{R}(\mathbf{A}_i)/\mathcal{U} \cong \mathbf{R}(\prod_{i \in I} \mathbf{A}_i/\mathcal{U})$ .*

Jónsson's Theorem [24, 25] states that, for any subclass  $\mathbf{L}$  of a congruence distributive variety,  $\mathbb{V}(\mathbf{L})_{FSI} \subseteq \mathbb{HSP}_{\mathcal{U}}(\mathbf{L})$ . Together with Lemma 9.1, this yields the next corollary (as every variety is generated by its FSI members).

**Corollary 9.2.** *Let  $\mathbf{K}$  be a variety of SRLs, with  $\mathbf{E} \in \mathbb{R}(\mathbf{K})$ . Then  $\mathbf{E}$  is FSI iff  $\mathbf{E} \cong \mathbf{R}(\mathbf{D})$  for some  $\mathbf{D} \in \mathbf{K}_{FSI}$ .*

**Theorem 9.3.** *Let  $\mathbf{K}$  be a variety of SRLs, let  $\mathbf{B}$  be a subalgebra of  $\mathbf{A} \in \mathbf{K}$ , and identify  $\mathbf{R}(\mathbf{B})$  with the subalgebra of  $\mathbf{R}(\mathbf{A})$  given in Lemma 9.1(i). Then*

- (i)  *$\mathbf{B}$  is  $\mathbf{K}$ -epic in  $\mathbf{A}$  iff  $\mathbf{R}(\mathbf{B})$  is  $\mathbb{R}(\mathbf{K})$ -epic in  $\mathbf{R}(\mathbf{A})$ ;*
- (ii)  *$\mathbf{K}$  has the ES property iff  $\mathbb{R}(\mathbf{K})$  has the ES property;*
- (iii)  *$\mathbf{K}$  is locally finite iff  $\mathbb{R}(\mathbf{K})$  is locally finite.*

*Proof.* (i) ( $\Rightarrow$ ): Let  $g, h: \mathbf{R}(\mathbf{A}) \rightarrow \mathbf{E} \in \mathbb{R}(\mathbf{K})$  be homomorphisms that agree on  $\mathbf{R}(\mathbf{B})$ . In showing that  $g = h$ , we may assume that  $\mathbf{E}$  is subdirectly irreducible (by the Subdirect Decomposition Theorem), whence  $\mathbf{E} = \mathbf{R}(\mathbf{D})$  for some  $\mathbf{D} \in \mathbf{K}_{FSI}$ , by Corollary 9.2. Since  $g, h$  preserve  $e, \cdot, \neg$ , they preserve  $\perp, \top$ . If  $a, b \in \mathbf{A}$ , then  $g(a), h(a) \neq \perp$  (otherwise, the kernel of  $g$  or  $h$  would identify  $\top = a \cdot \top$  with  $\perp \cdot \top = \perp$ ), and  $g(a), h(a) \neq \top$  (because the kernels don't identify  $\top = \top \rightarrow \top$  with  $\top \rightarrow a = \perp$ ), while  $g(a), h(a) \neq b'$  (because the kernels don't identify  $a^2 \in \mathbf{A}$  with  $\top = (b')^2$ ). Thus,  $g[A], h[A] \subseteq \mathbf{D}$ , and so  $g|_A, h|_A$  are homomorphisms from  $\mathbf{A}$  to  $\mathbf{D}$ , which agree on  $\mathbf{B}$ . As  $\mathbf{D} \in \mathbf{K}$  and  $\mathbf{B}$  is  $\mathbf{K}$ -epic in  $\mathbf{A}$ , we conclude that  $g|_A = h|_A$ . Then  $g|_{A'} = h|_{A'}$ , since  $g, h$  preserve  $\neg$ . Consequently,  $g = h$ . ( $\Leftarrow$ ): Let  $g, h: \mathbf{A} \rightarrow \mathbf{D} \in \mathbf{K}$  be homomorphisms that agree on  $\mathbf{B}$ . Then  $\mathbf{R}(\mathbf{D}) \in \mathbb{R}(\mathbf{K})$ . Let  $\bar{g}, \bar{h}: \mathbf{R}(\mathbf{A}) \rightarrow \mathbf{R}(\mathbf{D})$  be the respective extensions of  $g, h$ , preserving  $\perp, \top$ , such that  $\bar{g}(a') = g(a')$  and  $\bar{h}(a') = h(a')$  for all  $a \in \mathbf{A}$ . Then  $\bar{g}, \bar{h}$  are homomorphisms that agree on  $\mathbf{R}(\mathbf{B})$ , so by assumption,  $\bar{g} = \bar{h}$ , whence  $g = h$ .

- (ii) Obviously,  $\mathbf{B} = \mathbf{A}$  iff  $R(\mathbf{B}) = R(\mathbf{A})$ . Therefore, the implication from right to left follows from (i). For the converse, use Theorem 2.2, Corollary 9.2, Lemma 9.1(i) and item (i) of the present theorem.
- (iii)  $(\Rightarrow)$ : As  $\mathbf{K}$  is locally finite, there is a function  $p: \omega \rightarrow \omega$  such that, for each  $n \in \omega$ , every  $n$ -generated member of  $\mathbf{K}_{FSI}$  has at most  $p(n)$  elements. It suffices to show that, for each  $n \in \omega$ , every  $n$ -generated  $\mathbf{E} \in \mathbb{R}(\mathbf{K})_{FSI}$  has at most  $2 + 2p(n)$  elements. By Corollary 9.2, any such  $\mathbf{E}$  may be assumed to be  $R(\mathbf{D})$  for some  $\mathbf{D} \in \mathbf{K}_{FSI}$ . Let  $G$  be an irredundant generating set for  $\mathbf{E}$ , with  $|G| \leq n$ . Then  $\perp, \top \notin G$ . Let  $H = (G \cap D) \cup \{-g : g \in G \cap D'\}$ , so  $|H| \leq n$  and  $\mathbf{C} := Sg^D H$  has at most  $p(n)$  elements. By Lemma 9.1(i),  $R(\mathbf{C})$  may be identified with a subalgebra of  $\mathbf{E}$ , but then  $G \subseteq R(\mathbf{C})$ , so  $R(\mathbf{C}) = \mathbf{E}$ , whence  $|E| \leq 2 + 2p(n)$ .
- $(\Leftarrow)$ : Use the fact that an SIRL of the form  $R(\mathbf{A})$  is generated by  $\mathbf{A}$ .  $\square$

As a function from the lattice of varieties of SRLs into that of SIRLs, the operator  $\mathbb{R}$  is obviously isotone. Using Corollary 9.2, we can show that  $\mathbb{R}$  is also  $\subseteq$ -reflecting (and therefore injective); the proof is the same as that of [38, Lemma 6.7] (where again, the extra assumptions play no role).

Consequently, as all varieties of Brouwerian algebras of finite depth have the ES property (by [4, Theorem 5.4] or by Theorem 8.1) and since there are  $2^{\aleph_0}$  such distinct varieties (even of depth 3) [29], the following can be inferred from Theorem 9.3(ii),(iii).

**Theorem 9.4.** *There are  $2^{\aleph_0}$  distinct locally finite varieties of De Morgan monoids with the ES property.*

## 10. Further examples and applications

### 10.1. The weak ES property

The *weak ES property* for a variety  $\mathbf{K}$  rules out non-surjective  $\mathbf{K}$ -epimorphisms  $h: \mathbf{A} \rightarrow \mathbf{B}$  in all cases where  $\mathbf{B}$  is generated by the union of  $h[A]$  and a *finite* set. By [39, Theorem 5.4], it is equivalent to the demand that no finitely generated member of  $\mathbf{K}$  has a  $\mathbf{K}$ -epic proper subalgebra. In varieties of logic, it amounts to the so-called *finite Beth property* for the corresponding deductive system [5, Theorem 3.14, Corollary 3.15] (also see [39, Theorem 7.9]).

Therefore, by an argument of Kreisel [28], every variety of Brouwerian (or Heyting) algebras has the weak ES property. It follows from a result of Campercholi [8, Corollary 6.5] that, in any finitely generated variety with a majority term (e.g., one generated by a finite lattice-based algebra), the weak ES property entails the ES property. This provides a different explanation of the slightly earlier finding that all finitely generated varieties of Brouwerian (or Heyting) algebras have the ES property [4, Corollary 5.5, Theorem 7.2].

Every variety with the weak ES property and the amalgamation property has (a strong form of) the ES property; see [23, 26, 43] and [22, Section 2.5.3]. Consequently, in all varieties of Brouwerian (or Heyting) algebras, amalgamability entails epimorphism-surjectivity, and the amalgamable varieties of these kinds have been classified completely by Maksimova; see [16, 30, 31].

The situation is different for varieties of (possibly non-integral) S[I]RLs, as they may lack the weak ES property, even when they are finitely generated (see Section 10.2 below).

**10.2. Hypotheses of the main theorem**

For subvarieties  $K$  and  $L$  of a congruence distributive variety, we have

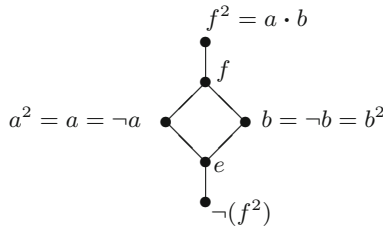
$$\mathbb{V}(K \cup L)_{FSI} = K_{FSI} \cup L_{FSI},$$

by Jónsson’s Theorem. The hypotheses in Theorem 8.1 therefore persist in (binary) varietal joins, and of course in subvarieties. This is helpful in applications, as the ES property itself is not hereditary.

Neither of the two hypotheses in Theorem 8.1 can be dropped.

To see that the finite depth assumption cannot be dropped for varieties of SRLs, it suffices to exhibit a variety of Brouwerian algebras (of infinite depth) without the ES property. This was done in [4, Section 6]. It was subsequently shown in [40] that there are  $2^{\aleph_0}$  distinct locally finite varieties of Brouwerian algebras that lack the ES property and that have *width* 2 (i.e., 2 is the maximum cardinality of an anti-chain in the dual of an FSI member of the variety). It follows, as in Section 9, that there are  $2^{\aleph_0}$  locally finite varieties of De Morgan monoids without the ES property, but this could alternatively be deduced from older findings, discussed below.

The demand for negative generation is not redundant either, because some finitely generated varieties of De Morgan monoids (and of Dunn monoids) lack the ES property. Indeed, by an argument of Urquhart [47] (also see [5, Corollary 4.15]), a variety of De Morgan monoids lacks even the weak ES property if it contains a certain six-element algebra  $\mathcal{C}$ , called the *crystal lattice* (which is not negatively generated). That algebra is depicted below.



(Deletion of  $b$  leaves an epic subalgebra behind, owing to the uniqueness of existent relative complements in distributive lattices.) The argument adapts to Dunn monoids, using the SRL-reduct of  $\mathcal{C}$ . In particular,  $\mathcal{C}$  is absent from each of the  $2^{\aleph_0}$  varieties  $K$  of De Morgan monoids in Theorem 9.4, while the corresponding locally finite varieties  $\mathbb{V}(K \cup \{\mathcal{C}\})$  lack the weak ES property, and by Jónsson’s Theorem, they are distinct.

While the finite depth assumption in Theorem 8.1 cannot be dropped, it is not a *necessary* condition for the ES property. Indeed, epimorphisms are surjective in BRA (hence in its reflection), and in the locally finite variety of *relative Stone algebras* (i.e., subdirect products of totally ordered Brouwerian algebras). Also, the smallest variety containing the De Morgan monoids

(alternatively, the idempotent SRLs) that are totally ordered and negatively generated has the ES property and is locally finite [48]. All of the varieties mentioned in this paragraph have infinite depth.

Likewise, the demand that FSI members be negatively generated is not entailed by the ES property, even in varieties of S[I]RLs of finite depth. Witnessing examples can also be found in [48].

### 10.3. More varieties of De Morgan monoids

*Sugihara monoids*—i.e., idempotent De Morgan monoids—are always negatively generated, and the same applies to the SRLs that can be embedded into them (these are the *positive Sugihara monoids* of [41]). The ES property was established recently for *all* varieties of [positive] Sugihara monoids [4, Theorems 8.5, 8.6]. For the finitely generated varieties of this kind, the surjectivity of epimorphisms could alternatively be deduced (immediately) from Theorem 8.1.

Apart from reflections and idempotent cases, Theorem 8.1 yields further examples as follows.

By [37, Theorem 6.1], the lattice of varieties of De Morgan monoids has just four atoms, each of which is a finitely generated variety satisfying the hypotheses of Theorem 8.1. One of them is generated by the reflection of a trivial SRL, i.e., by the non-idempotent De Morgan monoid  $\mathbf{C}_4$  on the chain

$$\neg(f^2) < e < f < f^2.$$

The covers of  $\mathbb{V}(\mathbf{C}_4)$  are distinctive, as  $\mathbf{C}_4$  is the only 0-generated nontrivial algebra onto which FSI De Morgan monoids may be mapped by non-injective homomorphisms [44, Theorem 1].

There is a largest variety  $\mathbf{U}$  of De Morgan monoids consisting of homomorphic *pre-images* of  $\mathbf{C}_4$  (along with trivial algebras), and in the subvariety lattice of  $\mathbf{U}$ , the variety  $\mathbb{V}(\mathbf{C}_4)$  has just ten covers [38, Sections 4, 8], only two of which are generated by reflections of Dunn monoids.

Each of these ten varieties is generated by a finite De Morgan monoid that is itself generated by one of the lower bounds of its neutral element (and is thus negatively generated). Therefore, the conditions of Theorem 8.1 obtain in all ten covers, and hence in their varietal join, so all subvarieties of this join have the ES property.

### 10.4. Bounds

The original Esakia duality of [13] supplies an equivalence between the category HA of Heyting algebras (and their homomorphisms—which must preserve  $\perp$ ) and the opposite of the category ESP of *Esakia spaces*.

The objects of ESP are like those of PESP, except that they need not have maximum elements; the definition of morphisms is unaffected. For  $\mathbf{A} \in \mathbf{HA}$  and  $\mathbf{X} \in \mathbf{ESP}$ , we *re-define*  $\text{Pr}(\mathbf{A})$  as the set of prime *proper* filters of  $\mathbf{A}$ , and  $\text{Cpu}(\mathbf{X})$  as the set of *all* clopen up-sets of  $\mathbf{X}$ , including  $\emptyset$ . After these changes, the definitions of  $\mathbf{A}_*$ ,  $\mathbf{X}^*$ , the duals of morphisms, and the canonical isomorphisms remain the same (but note that  $\mathbf{A}_*$  is empty when  $|\mathbf{A}| = 1$ ). The definition of depth is adjusted so that a Heyting algebra and its Brouwerian

reduct have the same depth. (In particular, the depth of the Esakia space reduct of a pointed Esakia space  $\mathbf{X}$  exceeds that of  $\mathbf{X}$  by 1.)

Theorem 8.1 remains true for varieties of bounded S[I]RLs; its proof requires no further alteration. In this form, it generalizes the recent finding that every variety of Heyting algebras of finite depth has surjective epimorphisms [4, Theorem 5.3].

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