



Congruence lattices of connected monounary algebras

Danica Jakubíková-Studenovská and Lucia Janičková

Abstract. The system of all congruences of an algebra (A, F) forms a lattice, denoted $\text{Con}(A, F)$. Further, the system of all congruence lattices of all algebras with the base set A forms a lattice \mathcal{E}_A . We deal with meet-irreducibility in \mathcal{E}_A for a given finite set A . All meet-irreducible elements of \mathcal{E}_A are congruence lattices of monounary algebras. Some types of meet-irreducible congruence lattices were already described. In the case when a monounary algebra (A, f) is connected, we prove necessary and sufficient condition under which $\text{Con}(A, f)$ is \wedge -irreducible.

Mathematics Subject Classification. 08A30, 08A60, 08A62.

Keywords. Monounary algebra, Congruence, Lattice of congruence lattices, Meet-irreducible congruence lattice.

1. Introduction

Congruences and congruence lattices on algebras have been studied by many authors, recently e.g. by [1, 2, 4]. Further, [5] and [10] studied congruences on other types of structures.

Our paper is a continuation of [6] and [7]. As it was mentioned in [7], the subject of the papers is related to the finite representation problem in its concrete version.

For a fixed finite set A we consider all possible congruence lattices of algebras with the base set A . These congruence lattices, ordered by inclusion, form a lattice themselves; it is denoted \mathcal{E}_A . All join-irreducible congruence lattices of \mathcal{E}_A were characterized in [7]. In this paper, we study meet-irreducibility in the lattice \mathcal{E}_A .

Since $F \subseteq G$ implies $\text{Con}(A, G) \subseteq \text{Con}(A, F)$, all \wedge -irreducible elements in \mathcal{E}_A must be of the form $\text{Con}(A, f)$ for a single mapping f , otherwise

$\text{Con}(A, F)$ would be the intersection of all $\text{Con}(A, f)$ where $f \in F$. Therefore, it is sufficient to explore meet-irreducibility of congruence lattices of monounary algebras (i.e. algebras with a single unary operation). While studying properties of monounary algebras, we often benefit from the fact that they can be easily visualized as digraphs which are always planar, hence easy to draw [8].

In general, the problem which unary operations f yield \wedge -irreducibles is difficult. Some partial results exist for special classes of operations f ([6], [7]). Let us remark that maximal \wedge -irreducibles, i.e. the coatoms in \mathcal{E}_A , are described completely in [7].

In this paper, we prove a necessary and sufficient condition under which $\text{Con}(A, f)$ is \wedge -irreducible in \mathcal{E}_A in the case when (A, f) is connected (definitions are in the section Preliminary):

Theorem 1.1. *Let (A, f) be a connected monounary algebra with at least 3 cyclic elements. Then $\text{Con}(A, f)$ is \wedge -irreducible if and only if the set of cyclic elements is covered and there exist distinct noncyclic elements $a, b, c, d \in A$ such that $f(a), f(c)$ are cyclic and $f(b) = a, f(d) = c$.*

Remark 1.2. Let us note that in the case when (A, f) is a monounary algebra with at most two cyclic elements, the necessary and sufficient condition under which $\text{Con}(A, f)$ is \wedge -irreducible was proved in [6].

2. Preliminary

In the following, let A be a fixed, non-empty, finite set and f be a mapping on A . The pair (A, f) is said to be a *monounary algebra*. For the notions concerning monounary algebras, see [8].

Let us denote $\mathbb{N} := \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

For a mapping $f : A \rightarrow A$, $f(a)$ denotes the image of the element $a \in A$ in the mapping f , and if $n \in \mathbb{N}$ then f^n denotes the n -fold composition of f . By convention, f^0 denotes the identity mapping id_A .

The operation $f : A \rightarrow A$ is called *trivial* if it is either identity $x \mapsto x$ or the constant mapping $x \mapsto a$. Otherwise it is called *nontrivial*.

An element $x \in A$ is called *cyclic* if there exists $n \in \mathbb{N}$ such that $f^n(x) = x$, otherwise it is called *noncyclic*. In this case, the set $\{x, f^1(x), f^2(x), \dots, f^{n-1}(x)\}$ is called a *cycle* of (A, f) . Since A is finite, for each $a \in A$ there exists $k \in \mathbb{N}_0$ such that $f^k(a)$ is cyclic. The cycle containing $f^k(a)$ will be denoted $C(a)$.

Definition 2.1. A monounary algebra (A, f) is called *connected*, if for every $x, y \in A$ there exist $m, n \in \mathbb{N}_0$ such that $f^m(x) = f^n(y)$.

Apparently, a connected monounary algebra (A, f) contains exactly one cycle. In [6], Definitions 2.2, 2.3 were introduced.

Definition 2.2. A monounary algebra (A, f) is called a *permutation-algebra* if the operation f is a permutation on A .

Definition 2.3. Let (\bar{A}, \bar{f}) be a monounary algebra. Then (\bar{A}, \bar{f}) is said to be a *permutation-algebra with short tails* if there is a subalgebra (B, f) of (\bar{A}, \bar{f}) such that (B, f) is a permutation-algebra and $\bar{f}(x) \in B$ for each $x \in \bar{A}$. In this case (B, f) is called a *permutation-algebra corresponding to (\bar{A}, \bar{f})* .

Notation 2.4. Let $t_f(a) := \min\{n \in \mathbb{N}_0 : f^n(a) \in C(a)\}$. Clearly, if a is cyclic then $t_f(a) = 0$. For $k \in \mathbb{N}$ we denote $C_k := \{x \in A : t_f(x) = k\}$.

Definition 2.5. For every element $x \in A$ such that $t_f(x) = t$, there is a single cyclic element $y \in A$ such that $f^t(x) = f^t(y)$. We will call this element a *colleague* of x and we denote it x' . It is clear that $(f(x))' = f(x')$.

Notation 2.6. For $a \in A$, we denote $\langle a \rangle_f := \{f^k(a) : k \in \mathbb{N}_0\}$ and $(a)_f := \{f^k(a) : k \in \mathbb{N}\}$.

Let us denote the least and the greatest congruence on the set A as $\Delta := \{(x, x) : x \in A\}$ and $\nabla := A \times A$, respectively.

For $x, y \in A$ let $\theta_f(x, y)$ be the smallest congruence on (A, f) such that $(x, y) \in \theta_f(x, y)$. In [3], the following notations were introduced. For $a \in A$ cyclic, $d \in \mathbb{N}$, we denote $\theta^{(d)}$ the smallest congruence on (A, f) such that $(a, f^d(a)) \in \theta^{(d)}$.

Notice that if $m, n \in \mathbb{N}$ such that m divides n then $\theta^{(n)} \subseteq \theta^{(m)}$.

Let x belong to a cycle with n elements and let $y = f^i(x), i \in \mathbb{N}$. Then $\theta_f(x, y) = \theta^{(d)}$, where $d = \gcd(i, n)$.

Let $\text{Eq}(A)$ denote the set of all *equivalence relations* on a given set A (i.e., reflexive, symmetric and transitive relations). For the sake of simplification, we will use the following notation to denote the equivalence classes.

Notation 2.7. For $\varkappa \in \text{Eq}(A)$ consider the corresponding partition A/\varkappa into equivalence classes. If $A_1 = \{a_{11}, a_{12}, \dots\}$, $A_2 = \{a_{21}, a_{22}, \dots\}$, ..., $A_k = \{a_{k1}, a_{k2}, \dots\}$ are the equivalence classes of \varkappa with at least two elements, then we use the notation

$$\begin{aligned} \varkappa &= [a_{11}, a_{12}, \dots] [a_{21}, a_{22}, \dots] \dots [a_{k1}, a_{k2}, \dots] \text{ or} \\ \varkappa &= [A_1] [A_2] \dots [A_k]. \end{aligned}$$

Definition 2.8. If L is lattice, then a non-unit element $a \in L$ is called *meet-irreducible* (shortly \wedge -irreducible) if $a = b_1 \wedge b_2$ implies $a \in \{b_1, b_2\}$. Similarly, nonzero element $a \in L$ is called *join-irreducible* (\vee -irreducible) if $a = b_1 \vee b_2$ implies $a \in \{b_1, b_2\}$ (see e.g. [9]).

Lemmas 2.9, 2.10 present some properties of operations $f, g \in A^A$ with $\text{Con}(A, f) \subseteq \text{Con}(A, g)$, (see [7]).

Lemma 2.9. Let $f, g \in A^A$ be nontrivial and $\text{Con}(A, f) \subseteq \text{Con}(A, g)$. Then we have

- (i) $\forall x, y \in A : (x, y) \in \varkappa \in \text{Con}(A, f) \implies (g(x), g(y)) \in \varkappa$,
in particular we have $(g(x), g(y)) \in \theta_f(x, y)$ and $\theta_g(x, y) \subseteq \theta_f(x, y)$.
- (ii) Let B be a subalgebra of (A, f) . Then either B is also a subalgebra of (A, g) or g is constant on B , where the constant does not belong to B .

Lemma 2.10. *Let $f, g \in A^A$ be nontrivial. Then*

$$\text{Con}(A, f) \subseteq \text{Con}(A, g) \Leftrightarrow \forall x, y \in A : (g(x), g(y)) \in \theta_f(x, y).$$

Corollary 2.11. *Let $g_i, i \in I$ be nontrivial operations on A . Then*

$$\text{Con}(A, f) = \bigcap_{i \in I} \text{Con}(A, g_i) \Leftrightarrow \forall x, y \in A : \theta_f(x, y) = \bigvee_{i \in I} \theta_{g_i}(x, y).$$

Lemma 2.12. *Assume that a is a noncyclic element of A and a' is its colleague. Let $d = \min\{m \in \mathbb{N} : f^m(a) = a'\}$. Then $|C(a)|$ divides d .*

Proof. Let $|C(a)| = c$ and $k = t_f(a)$. Since a is noncyclic, $a' = f^{i-1}(f^k(a))$, where $i = \min\{n \in \mathbb{N}, n + k - 1 \equiv 0 \pmod{c}\}$, i.e., $a' = f^{k+i-1}(a)$. Clearly, $d = k + i - 1 \equiv 0 \pmod{c}$, hence c divides d . □

Corollary 2.13. *Let $x, y \in A$ be noncyclic, $C(x) = C(y)$. Then $(x', y') \in \theta_f(x, y)$ i.e., $\theta_f(f(x'), f(y')) \subseteq \theta_f(f(x), f(y))$.*

Proof. Let $|C(x)| = c$. Lemma 2.12 yields that there exist $k, l \in \mathbb{N}$ such that $f^{ck}(x) = x'$ and $f^{cl}(y) = y'$. Hence $f^{ck+cl}(x) = x'$ and $f^{ck+cl}(y) = y'$, which implies $\theta_f(x, y) = [x, y][f(x), f(y)] \dots [x', y'] \dots$ □

Lemma 2.14. *Let a be a noncyclic element of A . Then $(a', a) \in \theta_f(a, x)$ for each $x \in (a)_f$.*

Proof. Let $|C(a)| = c$ and $d \in \mathbb{N}$ be such that $f^d(a) = x$. Clearly, there exists $j \in \mathbb{N}$ such that $f^{jd}(a), f^{jd}(x) \in C(a)$, which implies that

$$\theta_f(f^{jd}(a), f^{jd}(x)) = \theta^{\text{gcd}(d,c)} \subseteq \theta_f(a, x).$$

We denote $m = \text{gcd}(d, c)$, hence there exist $k, l \in \mathbb{N}$ such that

$$d = km, c = lm.$$

According to Lemma 2.12, there exists $i \in \mathbb{N}$ such that $f^{ic}(a) = a'$. Apparently, $(f^{ic}(a), f^{ic}(x)) = (a', f^{iml}(x)) \in \theta_f(a, x)$. Moreover, $f^{nd}(x) = f^{nkm}(x), f^{ic}(x) = f^{nkm}(x) \in C(a)$ imply that

$$(f^{nkm}(x), f^{ilm}(x)) \in \theta^{(m)} \subseteq \theta_f(a, x).$$

It is easy to see that the following elements are congruent in $\theta_f(a, x)$:

$$\begin{aligned} a, f^d(a) = x, f^{2d}(a), \dots \\ x, f^d(x), f^{2d}(x), \dots, f^{nd}(x), \dots \end{aligned}$$

which yields that $(a, f^{nd}(x)) = (a, f^{nkm}(x)) \in \theta_f(a, x)$.

Hence, we get $(a', f^{iml}(x)), (f^{ilm}(x), f^{nkm}(x)), (f^{nkm}(x), a) \in \theta_f(a, x)$ and from transitivity, it follows that $(a, a') \in \theta_f(a, x)$. □

Corollary 2.15. *Let (A, f) be a connected monounary algebra. Let $x, y \in A$ such that x is cyclic, y is noncyclic. Then $(b, b') \in \theta_f(x, y)$ for each $b \in \langle y \rangle_f$.*

Proof. Holds trivially if b is cyclic.

Assume that b is noncyclic. For $b = y$, the assertion holds according to Lemma 2.14.

If $b \in (y)_f$ then there is $n \in \mathbb{N}$ such that $f^n(y) = b$. Then $(f^n(x), b) = (f^n(x), f^n(y)) \in \theta_f(x, y)$ which implies that $\theta_f(b, f^n(x)) \subseteq \theta_f(x, y)$. Moreover, $f^n(x)$ is cyclic, hence $f^n(x) \in (b)_f$ and from Lemma 2.14 we get $(b, b') \in \theta_f(b, f^n(x)) \subseteq \theta_f(x, y)$. \square

If (A, f) is either a permutation-algebra with short tails or its cycle contains at most 2 elements, then the necessary and sufficient conditions under which $\text{Con}(A, f)$ is meet-irreducible were already proved in [6] and [7].

Definition 2.16. Let (A, f) be a connected algebra with the cycle C such that $|C| = n, n \geq 3$. Next, let $C = \{0, 1, \dots, n - 1\}$ where $f(0) = 1, f(1) = 2, \dots, f(n - 1) = 0$ (on C , we compute modulo n). We say that a cyclic element c is *covered* if there exists a noncyclic $x \in A \setminus C_1$ such that $c = x'$.

Further, we denote the canonical decomposition of the number of elements of C as $n = p_1^{\alpha_1} \dots p_k^{\alpha_k}$; the numbers $p_l^{\alpha_l}$ for $l \in \{1, \dots, k\}$ are said to be elementary divisors of n . The cycle C is called *covered* if each equivalence class modulo σ , where σ is an elementary divisor of n , contains at least one covered $c \in C$.

The following conditions are the conditions mentioned in Theorem 1.1.

Notation 2.17. Consider the following two conditions:

- (*1) the set C is covered,
- (*2) there exist distinct noncyclic elements $a, b, c, d \in A$ with $f(a), f(c) \in C$ and $f(b) = a, f(d) = c$.

3. The necessary condition

In this section, we prove that if either of the conditions (*1), (*2) is not satisfied, then $\text{Con}(A, f)$ is \wedge -reducible.

Lemma 3.1. *Let (A, f) be a cycle with $\alpha \cdot \beta$ elements, where α and β are mutually prime. If $g_1 = f^\alpha, g_2 = f^\beta$, then $\text{Con}(A, f) = \text{Con}(A, g_1, g_2)$.*

Proof. Clearly, $\text{Con}(A, f) \subseteq \text{Con}(A, g_1, g_2)$.

By number theory there exist positive integers ι, κ, τ such that

$$\tau\alpha\beta + 1 = \iota\alpha + \kappa\beta.$$

Let $\theta \in \text{Con}(A, g_1, g_2), (x, y) \in \theta$. Then

$$\begin{aligned} (f(x), f(y)) &= (f^{\tau\alpha\beta+1}(x), f^{\tau\alpha\beta+1}(y)) = (f^{\iota\alpha+\kappa\beta}(x), f^{\iota\alpha+\kappa\beta}(y)) \\ &= (g_1^\iota(g_2^\kappa(x)), g_1^\iota(g_2^\kappa(y))) \in \text{Con}(A, g_1, g_2), \end{aligned}$$

hence $\text{Con}(A, g_1, g_2) \subseteq \text{Con}(A, f)$. \square

From now on, we use the notations from Definition 2.16.

Until Proposition 3.5 let us assume that $(*)_1$ is not satisfied and that (A, f) is not a cycle. This yields that there are $l \in \{1, \dots, k\}$ and $i \in C$ such that $c \equiv i \pmod{p_l^{\alpha_l}}$ fails to hold for any covered element $c \in C$. Without loss of generality, let $i = 0$. Thus if c is covered then $p_l^{\alpha_l}$ does not divide c . Notice that then $f^{p^\alpha}(y) \in C$ for each $y \in A$.

Now we set $p = p_l$, $\alpha = \alpha_l$, $\beta = \frac{n}{p^\alpha}$, $j = p^{\alpha-1} \cdot \beta$.

We define the following operations on A :

$$\begin{aligned} g_3(x) &:= x' + p^\alpha, \\ g_4(x) &:= x' + \beta \end{aligned}$$

for each $x \in A$, and

$$g(x) := \begin{cases} x' + j + 1 & \text{if } x \in C \cup C_1, p^\alpha \text{ divides } x', \\ f(x) & \text{otherwise.} \end{cases}$$

Notice that the operations g_3, g_4 and g are nontrivial, distinct and not equal to f . Therefore, if $\text{Con}(A, f) = \text{Con}(A, g_3) \cap \text{Con}(A, g_4) \cap \text{Con}(A, g)$, then $\text{Con}(A, f)$ is \wedge -reducible.

Lemma 3.2. *Let $x, y \in A$. Then $\theta_{g_3}(x, y) \vee \theta_{g_4}(x, y) \subseteq \theta_f(x, y)$.*

Proof. Let us show that $\theta_{g_3}(x, y) \subseteq \theta_f(x, y)$. The other inclusion is analogous. There is $t \in \mathbb{N}$ such that $\{f^t(x), f^t(y)\} \subseteq C$. This implies

$$\begin{aligned} (g_3(x), g_3(y)) &= (x' + p^\alpha, y' + p^\alpha) \in \theta_f(x' + p^\alpha, y' + p^\alpha) \\ &= \theta_f(x', y') = \theta_f(x' + t, y' + t) \\ &= \theta_f(f^t(x), f^t(y)) \subseteq \theta_f(x, y). \end{aligned} \quad \square$$

Lemma 3.3. *Let $x, y \in A$. Then $\theta_g(x, y) \subseteq \theta_f(x, y)$.*

Proof. It is sufficient to show that $(g(x), g(y)) \in \theta_f(x, y)$. Let $g(x) \neq f(x)$. Thus $x \in C \cup C_1$, p^α divides x' , $g(x) = x' + j + 1$. If $g(y) = y' + j + 1$, then we can proceed analogously as in the previous proof.

Suppose that $g(y) = f(y)$. Then p^α does not divide y' . Clearly,

$$(x' + 1, f(y)) = (f(x'), f(y)) = (f(x), f(y)) \in \theta_f(x, y).$$

Since $(x', y') \in \theta_f(x, y)$, it holds that $\theta_f(x', y') \subseteq \theta_f(x, y)$. Moreover,

$$\theta_f(x', y') = \theta^{(d)}, d = \text{gcd}(x' - y', p^\alpha \beta)$$

Further, p^α does not divide $x' - y'$ and d divides $p^\alpha \beta$, thus d divides $p^{\alpha-1} \beta = j$. This implies

$$(x' + j + 1, x' + 1) = (f^{x'+1}(j), f^{x'+1}(0)) \in \theta_f(j, 0) \subseteq \theta^{(d)} \subseteq \theta_f(x, y).$$

By transitivity, $(g(x), g(y)) = (x' + j + 1, f(y)) \in \theta_f(x, y)$. □

Lemma 3.4. *Let $x, y \in A$. Then $\theta_f(x, y) \subseteq \theta_g(x, y) \vee \theta_{g_3}(x, y) \vee \theta_{g_4}(x, y)$.*

Proof. We want to prove that $(f(x), f(y)) \in \theta_g(x, y) \vee \theta_{g_3}(x, y) \vee \theta_{g_4}(x, y)$. Let $\phi = \theta_g(x, y) \vee \theta_{g_3}(x, y) \vee \theta_{g_4}(x, y)$. Without loss of generality, $f(x) \neq g(x)$, i.e., $x \in C \cup C_1$, p^α divides x' , $g(x) = x' + j + 1$, $f(x) = x' + 1$.

If $g(y) = y' + j + 1$ then $f(y) = y' + 1$ and

$$(f(x), f(y)) = (x' + 1, y' + 1) \in \theta_f(x' + 1, y' + 1) = \theta_f(x' + j + 1, y' + j + 1).$$

For cyclic elements we will use Lemma 3.1, therefore

$$\begin{aligned} &\theta_f(x' + j + 1, y' + j + 1) \\ &= \theta_{g_3}(x' + j + 1, y' + j + 1) \vee \theta_{g_4}(x' + j + 1, y' + j + 1) \\ &= \theta_{g_3}(g(x), g(y)) \vee \theta_{g_4}(g(x), g(y)) \subseteq \phi, \end{aligned}$$

hence $(f(x), f(y)) \in \phi$.

Now let $g(y) = f(y)$. Then p^α does not divide y' . We have

$$(x' + j + 1, f(y)) = (g(x), g(y)) \in \phi.$$

Next, $(x' + p^\alpha, y' + p^\alpha) = (g_3(x), g_3(y)) \in \phi$. The elements $x' + p^\alpha, y' + p^\alpha$ are cyclic, hence

$$\begin{aligned} &\theta_{g_3}(x' + p^\alpha, y' + p^\alpha) \vee \theta_{g_4}(x' + p^\alpha, y' + p^\alpha) = \theta_f(x' + p^\alpha, y' + p^\alpha) \\ &= \theta_f(x', y') = \theta_{g_3}(x', y') \vee \theta_{g_4}(x', y'). \end{aligned}$$

This implies that $(x', y') \in \phi$ (and analogously $(x' + 1, y' + 1) \in \phi$), thus

$$(x' + j + 1, y' + 1) = (g(x'), g(y')) \in \phi.$$

So, the congruence ϕ contains the pairs

$$(x' + 1, y' + 1), (y' + 1, x' + j + 1), (x' + j + 1, f(y)),$$

so by transitivity it also contains the pair $(x' + 1, f(y)) = (f(x), f(y))$. □

Proposition 3.5. *If (A, f) does not satisfy the condition $(*)_1$, then $\text{Con}(A, f)$ is \wedge -reducible.*

Proof. For a single cycle, $\text{Con}(A, f)$ is \wedge -reducible (see [6], Theorem 5.19). If (A, f) fails to be a cycle, then the assertion is valid according to Lemmas 3.2–3.4. □

In 3.6–3.11, let us suppose that the condition $(*)_2$ is not valid and that C_2 is nonempty.

Let k be the least positive integer such that there exist distinct noncyclic elements $a, b, c, d, e, t \in A$ such that $b \in C_k$, $f(b) = a$, $f(c) = f(e) = b$, $f(d) = c$, $f(t) = e$. If such k does not exist, then put $k = 1$ and $a = f(b)$ for $c \in C_2, b = f(c)$.

Notice that if $k = 1$ then $a \in C$.

First we will investigate the case $k > 1$.

We define the operations g_5 and g_6 on A by putting

$$g_5(x) := \begin{cases} b' & \text{if } f(x) = b, \\ f(x) & \text{otherwise,} \end{cases}$$

$$g_6(x) := \begin{cases} a' & \text{if } f(x) = a, \\ f(x) & \text{otherwise.} \end{cases}$$

Lemma 3.6. *Let $x, y \in A$. Then $\theta_{g_5}(x, y) \vee \theta_{g_6}(x, y) \subseteq \theta_f(x, y)$.*

Proof. The proof is easy; let us show it e.g. for g_5 , $g_5(y) \neq g_5(x) \neq f(x)$. Then $f(x) = b$, $g_5(x) = b'$, and $g_5(y) = f(y)$. It is sufficient to show that,

$$(b', f(y)) = (g_5(x), g_5(y)) \in \theta_f(f(x), f(y)) = \theta_f(b, f(y)).$$

Clearly, $f(y) \neq b$, hence one of the following holds:

- 1) If $f(y) \in (b)_f$ then Lemma 2.14 yields $(b, b') \in \theta_f(b, f(y))$.
- 2) If $b \in (f(y))_f$ then Lemma 2.14 and Corollary 2.13 imply that $(f(y)', f(y)) \in \theta_f(b, f(y))$ and $(f(y)', b') \in \theta_f(b, f(y))$, respectively.

In both cases, the transitivity yields $(b', f(y)) \in \theta_f(b, f(y))$. □

Lemma 3.7. *Let $x, y \in A$. Then $\theta_f(x, y) \subseteq \theta_{g_5}(x, y) \vee \theta_{g_6}(x, y)$.*

Proof. Denote the relation on the right side by ϕ . Suppose that $f(y) \neq f(x) \neq g_6(x)$. This implies $f(x) = a = g_5(x)$, $g_6(x) = a'$, $g_6(y) = f(y)$. If $g_5(y) = f(y)$, the assertion is obvious. Thus $g_5(y) = b'$ and $f(y) = b$. Then $(a', b) = (g_6(x), g_6(y)) \in \phi$ and also $(f(a'), a) = (g_5(a'), g_5(b)) \in \phi$. From the definition of g_5 , it follows that $(b', a') \in \phi$. Next, we have $(a, b') = (g_5(x), g_5(y)) \in \phi$ and $(a', b) = (g_6(x), g_6(y)) \in \phi$, therefore $(a, b) = (f(x), f(y)) \in \phi$. □

As a corollary of Lemmas 3.6 and 3.7 we obtain:

Corollary 3.8. *If $k \neq 1$ then $\text{Con}(A, f)$ is \wedge -reducible.*

Now we deal with the case $k = 1$.

Let g_7 and g_8 be the following operations on A :

$$g_7(x) := \begin{cases} b' & \text{if } f(x) = b, \\ f(x) & \text{otherwise,} \end{cases}$$

$$g_8(x) := \begin{cases} b & \text{if } f(x) \in C, \\ b' & \text{otherwise.} \end{cases}$$

Lemma 3.9. *Let $x, y \in A$. Then $\theta_{g_7}(x, y) \vee \theta_{g_8}(x, y) \subseteq \theta_f(x, y)$.*

Proof. Suppose that $g_8(x) \neq g_8(y) \neq f(y)$, hence without loss of generality: $g_8(x) = b, g_8(y) = b'$. Therefore $f(x) \in C$ and $f(y) \notin C$. From Corollary 2.15 we get $(g_8(x), g_8(y)) = (b, b') \in \theta(f(x), f(y))$.

The proof for g_7 is analogous as in Lemma 3.6. □

Lemma 3.10. *Let $x, y \in A$. Then $\theta_f(x, y) \subseteq \theta_{g_7}(x, y) \vee \theta_{g_8}(x, y)$.*

Proof. Denote the relation on the right side by ϕ . Let $f(y) \neq f(x) \neq g_7(x)$. Then $f(x) = b, g_7(x) = b' = g_8(x), g_7(y) = f(y)$. We have $(b', f(y)) = (g_7(x), g_7(y)) \in \phi, (b', g_8(y)) = (g_8(x), g_8(y)) \in \phi$.

Let us show that $(b, f(y)) = (f(x), f(y)) \in \phi$, i.e., that $(b, b') \in \phi$.

If $f(y) \in C$ then $g_8(y) = b$ and the assertion is valid. Otherwise $f(y) \neq f(x)$ yields $f(y) \neq b$ hence $f^2(y) \notin C$. Therefore $(b, b') = (g_8(b'), g_8(f(y))) = (g_8(g_7(x)), g_8(g_7(x))) \in \phi$. □

From Lemmas 3.9 and 3.10 it follows:

Corollary 3.11. *If $k = 1$ then $\text{Con}(A, f)$ is \wedge -reducible.*

Proposition 3.12. *If (A, f) does not satisfy the condition (*2), then $\text{Con}(A, f)$ is \wedge -reducible.*

Proof. If (A, f) is a permutation-algebra with short tails, then $\text{Con}(A, f)$ is \wedge -reducible (see [6], Theorem 5.19). Otherwise the assertion holds due to Corollary 3.8 and 3.11. □

4. The sufficient condition

In this section (until Proposition 4.7), we suppose that the condition (*1) (with notations from Definition 2.16) and the condition (*2) are satisfied. Our aim is to prove that $\text{Con}(A, f)$ is \wedge -irreducible.

Notice that since the condition (*2) is satisfied, there exist distinct non-cyclic elements $a, b, c, d \in A$ such that $f(a), f(c) \in C$ and $f(b) = a, f(d) = c$.

Lemma 4.1. *Let g be a nontrivial operation on A with $\text{Con}(A, f) \subseteq \text{Con}(A, g)$. Then $g(a) = g(a') \in C, g(c) = g(c') \in C$ and either*

- (a) $g(b) = a, g(d) = c, g(b') = a', g(d') = c',$ or
- (b) $g(b) = g(b') \in C, g(d) = g(d') \in C.$

Proof. Let $g(a) \neq g(a')$. Since $\theta_g(a, a') \subseteq \theta_f(a, a') = [a, a']$, this implies $g(a) = a', g(a') = a$. Then by Lemma 2.9, $g(x) = a$ for each $x \in C$. Thus

$$(g(c), a) = (g(c), g(c')) \in \theta_g(c, c') \subseteq \theta_f(c, c') = [c, c'],$$

hence $g(c) = a$, and

$$(a, a') = (g(c), g(a)) \in \theta_f(a, c) = [a, c] \dots,$$

which is a contradiction. It follows that $g(x) = g(x')$ for each $x \in C_1$.

Let $g(a) = g(a') = y \notin C$. Then

$$(g(y), y) = (g(y), g(y')) \in \theta_f(y, y') = [y, y'] \dots,$$

implies that $g(y) \in \{y, y'\}$. Let $g(y) = y'$ and without loss of generality, let $y \neq a$ (otherwise, let $y \neq c$). It follows that

$$(y, y') = (g(a), g(y)) \in \theta_f(a, y) = [a, y] \dots,$$

which is a contradiction. Therefore, $g(x) = g(x') \in C$ for each $x \in C_1$.

Next, $(g(b), g(d)) \in \theta_f(b, d) = [b, d][a, c] \dots$

Clearly, if $g(b) \in C$ then $g(d) \in C$. Moreover,

$$\begin{aligned} (g(b), g(b')) &\in \theta_f(b, b') = [b, b'] [a, a'], \\ (g(d), g(d')) &\in \theta_f(d, d') = [d, d'] [c, c'], \end{aligned}$$

imply that $g(b) = g(b')$ and $g(d) = g(d')$, hence the condition (b) is valid.

If $g(b) \notin C$ then $g(b) \in \{a, b\}$. Since

$$(g(b), g(c)) \in \theta_f(b, c) = [b, c] [a, \dots] \dots,$$

and $g(a) \in C$, we get $g(b) = a$. Similarly, $g(d) = c$ and (a) holds. □

Lemma 4.2. *Suppose that $\text{Con}(A, f)$ is \wedge -reducible. Then there exists a non-trivial operation g on A such that $\text{Con}(A, f) \subsetneq \text{Con}(A, g)$ and the condition (a) of Lemma 4.1 is valid.*

Proof. $\text{Con}(A, f)$ is \wedge -reducible implies that there is a set G of nontrivial operations on A such that $|G| \geq 2$ and

$$\text{Con}(A, f) = \bigcap_{g \in G} \text{Con}(A, g), \quad \text{Con}(A, f) \neq \text{Con}(A, g) \text{ for } g \in G.$$

Assume that for every $g \in G$ the condition (a) fails to hold. Then according to Lemma 4.1, the condition (b) is valid and $g(b) = g(b')$. This yields

$$\bigvee_{g \in G} \theta_g(b, b') = [b, b'] \neq [b, b'] [a, a'] = \theta_f(b, b'),$$

which is a contradiction. □

In Lemmas 4.3–4.6 we will suppose that g is a nontrivial operation on A such that $\text{Con}(A, f) \subseteq \text{Con}(A, g)$ and the condition (a) of Lemma 4.1 is satisfied.

Lemma 4.3. $g(x) = f(x)$ for each $x \in A \setminus (C \cup C_1)$.

Proof. Let $x \in A \setminus (C \cup C_1)$. Then either $a \notin \langle x \rangle_f$ or $c \notin \langle x \rangle_f$. Without loss of generality, let $a \notin \langle x \rangle_f$. Then

$$(g(x), a) = (g(x), g(b)) \in \theta_f(x, b) = [x, b] [f(x), a] \dots,$$

implies that $g(x) \in \{f(x), a\}$. Further,

$$g(x') \in C \quad \text{and} \quad (g(x), g(x')) \in \theta_f(x, x') \subseteq [\langle x \rangle_f]$$

imply $g(x) \in \langle x \rangle_f$, hence $g(x) \in \{f(x), a\} \cap \langle x \rangle_f$. Moreover, $a \notin \langle x \rangle_f$, which yields $g(x) = f(x)$. □

Lemma 4.4. $g(t) = f(t)$ for each covered element $t \in C$.

Proof. Let $t \in C$ be covered. If $x \in A \setminus (C \cup C_1)$ is an element covering t , i.e. $t = x'$, then we can assume that $x \in C_i$ for $2 \leq i \leq n + 1$ with $t = x'$. From Lemma 4.3, it follows that $g(x) = f(x)$. If $i \leq n$ then

$$(g(t), f(x)) = (g(t), g(x)) \in \theta_f(t, x) = [t, x] [f(t), f(x)] \dots [f^{i-1}(t), f^{i-1}(x)].$$

Hence, $g(t) \in \{f(t), f(x)\}$. Moreover, the assumption that the condition (a) of Lemma 4.1 is satisfied implies that $g(a') \in C$ and from Lemma 2.9 (ii) it follows that $f(t) \in C$. Therefore we get $g(t) = f(t)$.

If $i = n + 1$, then

$$\begin{aligned} (g(t), f(x)) &= (g(t), g(x)) \in \theta_f(t, x) \\ &= [t, x, f^n(x)][f(t), f(x)] \dots [f^{n-1}(t), f^{n-1}(x)]. \end{aligned}$$

Analogously as in the previous case, we get $g(t) = f(t)$. □

Lemma 4.5. $g(i) = f(i)$ for each $i \in C$.

Proof. Let $i \in C$. The condition (*1) implies that for each $l \in \{1, \dots, k\}$ there exists a covered $c_l \in C$ such that $c_l \equiv i \pmod{p_l^{\alpha_l}}$. By Lemma 4.4

$$(g(i), c_l + 1) = (g(i), f(c_l)) = (g(i), g(c_l)) \in \theta_f(i, c_l) \subseteq \theta^{(p_l^{\alpha_l})},$$

thus $g(i) \equiv c_l + 1 \pmod{p_l^{\alpha_l}}$ for each l . According to Chinese Remainder Theorem, the system of congruences $x \equiv c_l + 1 \pmod{p_l^{\alpha_l}}, l \in \{1, \dots, k\}$ has a solution, and the solution is unique modulo $n = p_1^{\alpha_1} \dots p_k^{\alpha_k}$. Moreover, $i + 1$ is a solution of this system, what yields that $g(i) = i + 1 = f(i)$. □

Lemma 4.6. $g(t) = f(t)$ for each $t \in C_1$.

Proof. Using Lemma 4.5 we have

$$(g(t), f(t')) = (g(t), g(t')) \in \theta_f(t, t') = [t, t'],$$

hence $g(t) = f(t') = f(t)$. □

Proposition 4.7. *If the conditions (*1) and (*2) are valid, then $\text{Con}(A, f)$ is \wedge -irreducible.*

Proof. Suppose that the conditions (*1) and (*2) are valid and that $\text{Con}(A, f)$ is \wedge -reducible. Due to Lemma 4.2 there exists a nontrivial operation g on A such that $\text{Con}(A, f) \subsetneq \text{Con}(A, g)$ and the condition (a) of Lemma 4.1 is valid. Then Lemmas 4.3–4.6 imply that $g(x) = f(x)$ for each $x \in A$, which is a contradiction. □

From Proposition 3.12 and Proposition 4.7 we obtain:

Theorem 4.8. *Let (A, f) be a connected monounary algebra with at least 3 cyclic elements. Then $\text{Con}(A, f)$ is \wedge -irreducible if and only if the conditions (*1) and (*2) are satisfied.*

Hence Theorem 1.1 is valid.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

- [1] Aichinger, E.: Congruence lattices forcing nilpotency. *J. Algebra Appl.* **17**(2), 1850033 (2018)
- [2] Badawy, A.: Characterization of congruence lattices of principal p -algebras. *Math. Slovaca.* **67**(3), 803–810 (2017)
- [3] Berman, J.: On the congruence lattice of unary algebras. *Proc. Am. Math. Soc.* **36**, 34–38 (1972)
- [4] Czédli, G.: Lattices with many congruences are planar. *Algebra Univ.* **80**(1), 16 (2019)
- [5] Hyndman, J., Nation, J.B., Nishida, J.: Congruence lattices of semilattices with operators. *Stud. Log.* **104**(2), 305–316 (2016)
- [6] Jakubíková-Studenovská, D., Janičková, L.: Meet-irreducible congruence lattices. *Algebra Univ.* **79**, 4 (2018)
- [7] Jakubíková-Studenovská, D., Pöschel, R., Radeleczki, S.: The lattice of congruence lattices of algebra on a finite set. *Algebra Univ.* **79**(2), 4 (2018)
- [8] Jakubíková-Studenovská, D., Pócs, J.: Monounary Algebras. P.J. Šafárik Univ. Košice, Košice (2009)
- [9] Roman, S.: *Lattices and Ordered Sets*. Springer, New York (2008)
- [10] Veldsman, S.: Congruences on topological spaces with an application to radical theory. *Algebra Univ.* **80**(2), 25 (2019)

Danica Jakubíková-Studenovská and Lucia Janičková

Institute of Mathematics

P.J. Šafárik University in Košice

Košice

Slovakia

e-mail [D. Jakubíková-Studenovská]: danica.studenovska@upjs.sk

e-mail [L. Janičková]: lucia.janickova@upjs.sk

Received: 13 March 2020.

Accepted: 26 September 2020.