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The spectrum problem for Abelian ℓ-groups and MV-algebras

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Abstract. This paper deals with the problem of characterizing those topological spaces which are homeomorphic to the prime spectra of MV-algebras or Abelian ℓ-groups. As a first main result, we show that a topological space X is the prime spectrum of an MV-algebra if and only if X is spectral, and the lattice $K(X)$ of compact open subsets of X is a closed epimorphic image of the lattice of "cylinder rational polyhedra" (a natural generalization of rational polyhedra) of $[0,1]^Y$ for some set Y. As a second main result we extend our results to Abelian ℓ-groups. That is, a topological space X is the prime spectrum of an Abelian ℓ-group if and only if X is generalized spectral, and the lattice $K(X)$ is a closed epimorphic image of the lattice of "cylinder rational cones" (a generalization of rational cones) in $\mathbb{R}^Y$ for some set Y. Finally, we axiomatize, in monadic second order logic, the Belluce lattices of free MV-algebras (equivalently, the lattice of cylinder rational polyhedra) of dimension 1, 2 and infinite, and we study the problem of describing Belluce lattices in certain fragments of second order logic.

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1. Introduction

Abelian ℓ-groups (Abelian lattice ordered groups) are an important kind of algebraic structures; they find applications in functional analysis, economy, etc. see e.g. [1,2,4,14,15,16,33,37,39,50].

Since many years it has been understood that, in the study of algebraic structures, it is useful to attach topological invariants to the structures. As a first example, Stone duality (see [54]) gives a correspondence between Boolean

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algebras and a well-known kind of topological spaces (Stone spaces, i.e. compact Hausdorff spaces with a basis of closed and open sets). The map realizing this duality is the prime spectrum map. One can ask something similar for Abelian ℓ-groups: which kind of topological spaces are prime spectra of Abelian ℓ-groups? We call this problem the *spectrum problem for Abelian ℓ-groups*. Progress on this problem has been made, for instance, in [22], where necessary conditions and sufficient conditions are obtained for spectra of Abelian ℓ-groups.

A kind of structures deeply related with Abelian ℓ-groups is given by *MV-algebras*. They can be considered as generalized, non-idempotent Boolean algebras. They give the algebraic counterpart to Łukasiewicz infinitely- many valued logic, in the same way as Boolean algebras give the algebraic counterpart of classical logic. They have been introduced by Chang in [18] in order to prove a completeness theorem for Łukasiewicz logic. They find applications in as diverse areas as Abelian ℓ-groups themselves (see [5,10,11,30,35,42,43]), probability and measure theory (see [49]), C^*-algebras (see [44]) etc. The *spectrum problem for MV-algebras* has been posed, for instance, in [45]. In [45], page 235, there is a list of eleven problems in MV-algebra theory. The second problem asks to characterize the topological spaces which are the prime spectrum of some MV-algebra.

Mundici in [44] discovered a categorial equivalence between Abelian ℓ-groups *with strong unit* and MV-algebras. Thanks to this equivalence, we know that spectra of Abelian ℓ-groups with strong unit and spectra of MV-algebras *coincide*.

Much literature has been devoted to this problem, see e.g. [7,6,19,31,23, 22,25] and the recent [57] which solves the problem for countable MV-algebras, in the sense that proves the following: a topological space X is homeomorphic to the spectrum of some *countable* Abelian ℓ-group with strong unit (or equivalently MV-algebra) if and only if X is spectral, has a countable basis of open sets, and for any points x and y in the closure of a singleton $\{z\}$, either x is in the closure of $\{y\}$ or y is in the closure of $\{x\}$.

In [6], for every MV-algebra A, Belluce defines a lattice, here denoted by $\beta(A)$, and called the Belluce lattice of A, which is the quotient of A modulo the equivalence relation of lying in the same prime ideals, and the lattice operations between equivalence classes are given by $[a] \vee [b] = [a \vee b]$ and $[a] \wedge [b] = [a \wedge b]$.

Up to lattice isomorphism, we can also define $\beta(A)$ as follows:

- the set of all principal ideals of A, and the lattice operations are defined as follows: $\mathrm{id}(f) \wedge \mathrm{id}(g) = \mathrm{id}(f \wedge g)$, and $\mathrm{id}(f) \vee \mathrm{id}(g) = \mathrm{id}(f \vee g)$;
- the set of all compact open subsets of the prime spectrum of A, where the lattice operations are union and intersection.

In the first sense, the Belluce lattice is an example of a general construction called reticulation, see for rings [52] and, for lattices, [47].

The usefulness of the Belluce lattice for our aims is witnessed by the following lemma:

Lemma 1.1 (see [6]). *For every MV-algebra A, the prime spectra of $\beta(A)$ and A are homeomorphic.*

One can also note that a topological space X which is the spectrum of some Abelian ℓ-group is the spectrum of an Abelian ℓ-group *with strong unit* if and only if X is *compact*. So, if we understand the complexity of spectra of Abelian ℓ-groups, we have information also about spectra of Abelian ℓ-groups with strong unit or MV-algebras.

The order-theoretic variant of the spectrum problem for MV-algebras (regarding the structure of prime spectra as partial orders under inclusion) has been solved by Cignoli and Torrens in [23] as follows:

Theorem 1.2 (see [23]). *A partially ordered set $(X, \leq)$ is order isomorphic to the prime spectrum of an MV-algebra ordered by inclusion if and only if it is a spectral root system, that is:*

- *every nonempty chain of X has both a supremum and an infimum;*
- *if $x, y \geq z$ then $x \leq y$ or $y \leq x$;*
- *if $x < y$ then there are z, t such that $x \leq z < t \leq y$ and there is no element between z and t.*

Note also that for Boolean algebras prime ideals and maximal ideals coincide, whereas this is not the case for MV-algebras: actually *maximal* spectra of MV-algebras are known: they are exactly compact Hausdorff topological spaces. In fact, it is not difficult to show that if X is a compact Hausdorff space, then X is homeomorphic to the maximal spectrum of the MV-algebra of all continuous functions from X to $[0, 1]$.

Another historically important spectrum problem is the one for *commutative rings with unit*. This problem has been solved in [38] from the topological point of view, and also from the order-theoretical point of view. In fact we have:

Theorem 1.3 (see [38]). *A topological space X is homeomorphic to the prime spectrum of a commutative ring with unit if and only if X is spectral (see Section 2 for the definitions).*

Moreover, the proof of [38] implies (according to [53], where details on this issue are omitted) that a partially ordered set S is isomorphic to the prime spectrum of some commutative ring with unit if and only if S is an inverse limit of finite partially ordered sets in the category of partially ordered sets.

The spectrum problem for Abelian ℓ-groups (or MV-algebras) can also be formulated in terms of lattice theory, in the following sense:

Theorem 1.4 (see [57]). *A topological space X is homeomorphic to the spectrum of an Abelian ℓ-group if and only if:*

- *X is generalized spectral, and*
- *there is an Abelian ℓ-group G such that $\mathrm{Id_c}(G) \cong K(X)$, where $\mathrm{Id_c}(G)$ is the lattice of principal ℓ-ideals of G, $K(X)$ is the set of open compact sets of X ordered by inclusion, and $\cong$ denotes lattice isomorphism.*

Moreover, a topological space X is generalized spectral if and only if it is homeomorphic to the spectrum of a distributive lattice D with minimum. If this holds, then $D \cong K(X)$.

Following [57], we say that a lattice L is *ℓ-representable* if there is an Abelian ℓ-group G such that $\mathrm{Id_c}(G) \cong L$, where $\mathrm{Id_c}(G)$ is the lattice of principal ℓ-ideals of G. So the previous theorem can be reformulated as follows:

Theorem 1.5. *A topological space X is homeomorphic to the spectrum of an Abelian ℓ-group if and only if $K(X)$ is an ℓ-representable lattice under union and intersection and X is homeomorphic to the spectrum of $K(X)$.*

People are looking since several years for a satisfactory topological characterization of spectra, see e.g. [31,24]. Given the importance of the notion of ℓ-representable lattice, in view of Theorem 1.5, we think it interesting to measure the logical complexity of the notion of ℓ-representable lattice.

The logical approach to the spectrum problem has begun with [57], where it is shown that ℓ-representable lattices are not definable in first order logic (actually not even in $L_{\infty,\omega}$, the extension of first order logic with infinitary conjunctions and disjunctions).

On the other hand, [57] also shows that *countable* ℓ-representable lattices are definable in first order logic:

Theorem 1.6 (see [57]). *A countable lattice is ℓ-representable if and only if it has a minimum 0 and is completely normal, that is, for every a, b there are x, y with $a \vee b = x \vee b = y \vee a$ and $x \wedge y = 0$. So, ℓ-representability of countable lattices is definable by a first order sentence.*

In this paper, we observe that a topological space is a prime spectrum of an MV-algebra if and only if it is spectral and its lattice of compact open sets is a closed epimorphic image of the Belluce lattice of a free MV-algebra. In this way, the task of describing prime spectra or Belluce lattices of MV-algebras is reduced to the (hopefully simpler) task of describing prime spectra or Belluce lattices of *free* MV-algebras. For instance, we show that Belluce lattices of free MV-algebras correspond to lattices of cylinder polyhedra (i.e. zeros of McNaughton functions, see Section 2) of some hypercube (Theorem 5.11).

We give also a monadic second order axiomatization of Belluce lattices of free MV-algebras in dimension 1, 2 and infinite. This choice is due to the fact that the case of finite dimension $n > 2$ does not seemingly require ideas different from the 2-dimensional case. Instead, axiomatizing infinite dimension does require new ideas.

This axiomatization can be compared e.g. with the "second order" solution proposed by Wehrung in [58], although he himself says (personal communication) that this solution is not very informative. That is, it turns out that a topological space X is the prime spectrum of an MV-algebra if and only if X is spectral and its lattice L of compact open sets is such that:

- either L is infinite and there is a lattice ordered group G with strong unit equipotent to L and a surjective function $f : G \to L$ such that $f(x) \leq f(y)$ if and only if every ℓ-ideal of G containing x contains y,

- or L is finite and for every $a, b \in L$ there are x, y such that $a \vee x = b \vee y = a \vee b$ and $x \wedge y = 0$.

Note that both conditions above on L are expressible in second order logic [58] proves also a very strong negative result on first order logic:

Theorem 1.7. *No class of formulas of infinitary first order logic $L_{\infty,\kappa}$ (where κ is any fixed cardinal) characterizes the lattices of the compact open sets of the spectrum of an MV-algebra.*

We also treat the problem of characterizing spectra of general lattice ordered Abelian groups, possibly without strong unit.

1.1. Related work

As a related work we can cite [12], where one finds a study of lattices of subpolyhedra of a given polyhedron, in the framework of intuitionistic logic and Tarski-style completeness theorems for this logic. They prove that the lattice of open subpolyhedra of any compact polyhedron of $\mathbb{R}^n$ is a locally finite Heyting algebra; by complementation, this implies that the lattice of closed subpolyhedra of $\mathbb{R}^n$ is a locally finite co-Heyting algebra.

Our results on lattices of rational polyhedra can be seen as a particular case of these, with the technical difference that we consider *rational* polyhedra rather than usual (real) polyhedra. Another difference is that we focus on Łukasiewicz logic rather than intuitionistic logic, so our algebraic structures of interest are MV-algebras rather than Heyting algebras.

Our lattices (which form the class of Belluce lattices of free MV-algebras) are still locally finite co-Heyting algebras, both for rational polyhedra and for the infinite dimensional generalization (cylinder rational polyhedra) considered in this paper in Section 3.

The key tool is the obvious rational variant of the Triangulation Lemma of [12]. Actually we push further in the study of these lattices, and we give an axiomatization of some of them in monadic second order logic. This makes us conjecture that the whole class of Belluce lattices of arbitrary MV-algebras is also axiomatizable in monadic second order logic.

A notion related to our notion of cylinder polyhedron is the notion of infinite dimensional polyhedron given in [17]. An infinite dimensional polyhedron is a subset of the hypercube whose finite projections are polyhedra. This notion contains properly our notion of cylinder polyhedra. For instance, every one-element subset of $[0, 1]^X$ with rational coordinates, with X infinite, is an infinite dimensional polyhedron, whereas it is not a cylinder polyhedron.

1.2. Structure of the paper

The paper is organized as follows. After a preliminary Section 2, in Section 3 we introduce a kind of geometrical object (cylinder polyhedron) which generalizes rational polyhedra in infinite dimension, and the same for rational cones (cylinder cones). In Section 4 we consider spectra of "relative subalgebras" in the sense of [8]. In Section 5, by exploiting the Belluce operation on MV-algebras defined in the introduction, we present our main theorem where we characterize the spectra of an MV-algebra (Theorem 5.11). In Section 6 we extend the notion of Belluce lattice to Abelian ℓ-groups and we characterize the spectra of an Abelian ℓ-group (Theorem 6.11). In Section 7 we axiomatize the Belluce lattice of the free MV-algebras of dimension 1, 2 and infinite (the axioms for dimension 2 and infinite are collected in the appendices A and B of the paper). In Section 8 we give a categorial equivalence between the category of Belluce lattices of finitely presented MV-algebras and the category of lattices of rational subpolyhedra of a given rational polyhedron. In Section 9 we introduce cylinder MV-algebras, the MV-algebraic counterpart of cylinder polyhedra in the Marra-Spada duality between semisimple MV-algebras and compact Hausdorff spaces. In Section 10 we generalize usual Łukasiewicz theories with finitely many axioms and variables to what we call "limit theories" (the terminology is introduced here) which have finitely many axioms but may have infinitely many variables. In Section 11 we summarize our results and outline some possible applications of them.

2. Preliminaries

Recall from [20] that *MV-algebras* are algebraic structures $(A, \oplus, \neg, 0, 1)$ where $(A, \oplus, 0)$ is a commutative monoid, $\neg\neg x = x$, $1 = \neg 0$, $x \oplus 1 = 1$, and we have the Mangani axiom $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

Since MV-algebras form a variety, there are free MV-algebras. Let X be a set, finite or infinite. The free MV-algebra on X is the MV-algebra M_X of *McNaughton functions* f from $[0,1]^X$ to $[0,1]$, which are continuous functions and are piecewise affine, in the sense that there are affine functions $g_1, \ldots, g_l$ with integer coefficients, such that for every $x \in [0,1]^X$ there is i with $f(x) = g_i(x)$.

If $P \subseteq [0,1]^X$, we denote by $M_X|_P$ the MV-algebra of the restrictions of McNaughton functions to P.

Every MV-algebra is equipped with a natural lattice order, such that $x \leq y$ if and only if there is z with $x \oplus z = y$. There is also a natural product $x \odot y = \neg(\neg x \oplus \neg y)$ (the Łukasiewicz product, also denoted xy).

Other useful notations are $x \ominus y = x \odot \neg y$ and $d(x, y) = (x \ominus y) \oplus (y \ominus x)$ (Chang distance).

An *ideal* of an MV-algebra A is a nonempty subset I of A closed under sum and closed downwards. If $B \subseteq A$, then $\mathrm{id}(B)$ is the ideal generated by B. We also write $\mathrm{id}(a)$ for $\mathrm{id}(\{a\})$. Every ideal of the form $\mathrm{id}(a)$ with $a \in A$ is called *principal*. We denote by $\mathrm{Id_c}\, A$ the set of principal ideals of A. This set is a distributive lattice under inclusion.

Two elements $a, b \in A$ are *congruent modulo an ideal* I if $d(a, b) \in I$. Congruence is an equivalence relation and its quotient set is denoted by A/I, and this quotient has a natural structure of an MV-algebra.

An ideal I is called *prime* if $x \wedge y \in I$ implies $x \in I$ or $y \in I$. I is called *principal* if it is generated by one element.

An ideal I is called *primary* if $xy \in I$ implies $x^n \in I$ or $y^n \in I$ for some n.

The *spectrum* of A, denoted $\mathrm{Spec}(A)$, is the set of all prime ideals of A equipped with the topology generated by the opens

$$U(a) = \{P \in \mathrm{Spec}(A) \mid a \notin P\}$$

where a ranges over A. This topology is also called the *Zariski topology*.

Recall that a topological space X is called *generalized spectral* if it is sober (i.e., every irreducible closed set is the closure of a unique singleton) and the collection of all compact open subsets of X forms a basis of the topology of X, closed under intersections of any two members. If, in addition, X is compact, we say that it is *spectral*.

In the literature there is no completely satisfactory characterization of the spaces $\mathrm{Spec}(A)$, and we are interested in finding such a characterization. It is known that $\mathrm{Spec}(A)$ is always a spectral space, and this implies that the compact open subsets form a lattice. This remark allows one to pass from topology to lattice theory; this has the advantage that lattices are algebraic structures, suitable to first order (or second order) logic, whereas topological spaces are higher order objects, too complicated to be studied with first order (or second order) logic.

An MV-algebra is *local* if it has a unique maximal ideal. The *radical* of an MV-algebra is the intersection of its maximal ideals. An MV-algebra is *perfect* if it is generated by its radical.

Recall that an ℓ-*group* is a group with a lattice order structure and such that $x \leq y$ implies $x + z \leq y + z$. A *strong unit* of an ℓ -group G is an element $u \in G^+$ such that for every $x \in G$ there is $n \in \mathbb{N}$ such that $x \leq nu$.

A fundamental tool in MV-algebra theory is Mundici equivalence between the category of MV-algebras and the category of Abelian ℓ-groups with strong unit, see [44].

In an ℓ-group we can consider ℓ-*ideals*, which are subgroups J such that if $x \in J$ and $|x| \leq |y|$, then $y \in J$ (where $|x| = x \vee -x$). An ℓ-ideal J is prime again if $x \wedge y \in J$ implies $x \in J$ or $y \in J$. We can equip the set of prime ℓ-ideals of an Abelian ℓ-group G which the Zariski topology, and we obtain a space $Spec(G)$ which will be generalized spectral, and it will be spectral if and only if G has a strong unit.

We need the following definition and results on (semi)lattices.

A join homomorphism $f : A \to B$ between join semilattices is called *closed* if whenever $a_0, a_1 \in A$ an $b \in B$, if $f(a_0) \leq f(a_1) \vee b$, then there is $x \in A$ such that $a_0 \leq a_1 \vee x$ and $f(x) \leq b$.

Lemma 2.1. *Let A be an MV-algebra, S a distributive lattice with zero, and $\phi : \mathrm{Id_c}\, A \to S$ be a closed surjective join homomorphism. Then*

$$I := \{x \in A \mid \phi(\mathrm{id}(x)) = 0\}$$

is an ideal of A, and there is a unique isomorphism $\psi : \mathrm{Id_c}(A/I) \to S$ such that $\psi(\mathrm{id}(x/I)) = \phi(\mathrm{id}(x))$ for every $x \in A$.

Proof. The lemma follows from the analogous lemma 2.5 of [57] up to using Mundici equivalence and replacing unital ℓ-groups with MV-algebras. $\square$

Proposition 2.2. *Let A, B be two MV-algebras and let $f : A \to B$ be a homomorphism. Then the map $\mathrm{Id_c}(f) : \mathrm{Id_c}\, A \to \mathrm{Id_c}\, B$ given by $\mathrm{id}(x) \to \mathrm{id}(f(x))$ is a closed 0-lattice homomorphism.*

Proof. The lemma follows from the analogous proposition 2.6 of [57] up to using Mundici equivalence and replacing unital ℓ-groups with MV-algebras. □

3. Cylinder (rational) polyhedra and cones

3.1. Cylinder polyhedra

Recall that a *rational simplex* in $[0, 1]^X$, where X is a finite set, is the convex envelope of finitely many, affinely independent rational points. A *rational polyhedron* is a finite union of rational simplexes.

Given a rational polyhedron P, we denote by $\mathrm{Poly}(P)$ the lattice of the rational subpolyhedra of P under inclusion.

We introduce now a new notation, although the concept is not new. Given any set X, let $[0, 1]^X$ be the set of functions from X to $[0, 1]$. We define *cylinder (rational) polyhedron* in a hypercube $[0, 1]^X$ a subset of $[0, 1]^X$ of the form

$$C_X(P_0) = \{f \in [0, 1]^X \mid f|_Y \in P_0\}$$

where Y is a finite subset of X and $P_0 \subseteq [0, 1]^Y$ is a rational polyhedron. Here $f|_Y$ denotes the function f restricted to Y, that is, $f|_Y = f \circ j$, where $j : Y \to X$ is the inclusion map.

Intuitively, a cylinder polyhedron is a kind of Cartesian product of a rational polyhedron and a hypercube, up to a permutation of the variables.

When the set X is finite, cylinder rational polyhedra coincide with ordinary rational polyhedra.

If $P \subseteq [0, 1]^X$ is a cylinder rational polyhedron, we denote by $\mathrm{Cpoly}(P)$ the lattice of all cylinder rational polyhedra $P' \subseteq [0, 1]^X$ included in P, partially ordered by inclusion.

We follow [12] but we consider the possibility of infinite dimension. It is natural to define a cylinder simplex as a set $S \subseteq [0, 1]^X$ such that $S = \{x \in [0, 1]^X \mid x|_Y \in S_0\}$, where $Y \subseteq X$ is finite and $S_0 \subseteq [0, 1]^Y$ is a rational simplex. Then we define a cylinder triangulation of a hypercube as a finite set of cylinder simplexes where the intersection of two is empty or is a face of both.

In [12] it is shown that the lattice of subpolyhedra of a polyhedron P is a locally finite co-Heyting algebra with respect to inclusion. In the same vein:

Lemma 3.1 (Cylinder Triangulation Lemma). *Given cylinder polyhedra*

$$P, P_1, \ldots, P_m \subseteq [0, 1]^X$$

with $P_1, \ldots, P_m \subseteq P$, there exists a cylinder triangulation Σ of P such that, for each i, the collection $\{\sigma \in \Sigma \mid \sigma \subseteq P_i\}$ is a triangulation of P_i.

Proof. First we reduce to X finite. Then we divide each P_i into a finite union of rational simplexes, and we can consider the rational polyhedral complex given by all possible intersections of simplexes. Every polyhedron of this complex is convex. Every convex polyhedron can be triangulated by induction on its dimension. $\square$

Lemma 3.2. *Let $Q \subseteq P$ be two cylinder polyhedra in $[0,1]^X$. Let Σ be a common cylinder triangulation of P and Q. Let $C = cl(P\backslash Q)$ (the closure of $P\backslash Q$ in the usual topology of $[0,1]^X$). Then Σ triangulates C as well. In particular C is a cylinder polyhedron.*

Proof. We follow the proof of Lemma 3.2 of [12]. More explicitly, if X is finite we let

$$\Sigma_C = \{\sigma \in \Sigma \mid \sigma \subseteq C\}$$
$$\Sigma^* = \{\sigma \in \Sigma \mid there\ exists\ \tau \in \Sigma\backslash\Sigma_Q\ such\ that\ \sigma\ is\ a\ face\ of\ \tau\}$$

and we observe $\Sigma_C = \Sigma^*$ and the support of these triangulations is C. Hence, C is a rational polyhedron.

If X is infinite, then we can find $Y \subseteq X$ finite and rational polyhedra $P_0, Q_0 \subseteq [0,1]^Y$ such that $P = C_X(P_0)$ and $Q = C_X(Q_0)$. Now we can apply the reasoning above to P_0 and Q_0. $\square$

So we conclude:

Theorem 3.3. *The lattice* $\mathrm{Cpoly}(P)$ *is a locally finite co-Heyting algebra with respect to inclusion, for every cylinder polyhedron P.*

Proof. If $C, D \in \mathrm{Cpoly}(P)$, then the co-implication $C \leftarrow D$ in $\mathrm{Cpoly}(P)$ is given by the closure of $C\backslash D$, which belongs to $\mathrm{Cpoly}(P)$ by the previous lemma.

To show that $\mathrm{Cpoly}(P)$ is locally finite one can give, for every finite set F of cylinder polyhedra, a common cylinder triangulation Σ of F (by the triangulation lemma) so that every element generated by F is triangulated by Σ, hence the elements generated by F are finitely many. $\square$

In order to study further the structure of $\mathrm{Poly}(P)$ or $\mathrm{Cpoly}(P)$, it is useful to restrict to rational points. In fact:

Lemma 3.4. *Let $P \subseteq [0,1]^X$ be a cylinder polyhedron. Two cylinder polyhedra in* $\mathrm{Poly}(P)$ *or* $\mathrm{Cpoly}(P)$ *are equal if and only if they have the same rational points. So, there is an isomorphism between* $\mathrm{Poly}(P)$ *or* $\mathrm{Cpoly}(P)$ *and a sublattice of the powerset of $\mathbb{Q}^X$.*

Proof. Up to restricting to a finite subset of X we can suppose X finite. Then the result follows because if a finite system of linear inequalities (strict or non-strict) with rational coefficients has a real solution, then it has a rational solution. $\square$

Note a difference for $\mathrm{Poly}(P)$ in finite dimension and $\mathrm{Cpoly}(P)$ in infinite dimension:

Proposition 3.5. *If $P \subseteq [0,1]^n$ is a rational polyhedron in a hypercube of finite dimension n, then* $\mathrm{Poly}(P)$ *is an atomic lattice. If $P \subseteq [0,1]^X$ is a nonempty cylinder polyhedron where X is an infinite set, then* $\mathrm{Cpoly}(P)$ *is an atomless lattice.*

Proof. In fact, in the finite case, the atoms are the rational points of $[0,1]^n$ (atomicity follows from the previous lemma). For the infinite case, let $P \subseteq [0,1]^X$ be a cylinder polyhedron with X infinite; then $P = \{x \in [0,1]^X \mid x|_Y \in P_0\}$, where Y is finite and P_0 is a rational polyhedron. Let $z \in X \backslash Y$; then we have the polyhedron $Q = \{x \in [0,1]^X \mid x|_Y \in P_0 \wedge x(z) = 1\}$. Then Q is a nonempty proper cylinder subpolyhedron of P. □

3.2. Cylinder polyhedra as zerosets

We note that in [45], page 19, Mundici uses a particular case of cylinder polyhedra (which he calls cylindrification of a polyhedron) to prove that Łukasiewicz logic enjoys the property of the Craig interpolation. Another definition of cylindrification similar to ours is given in [46], page 522. In the latter case, Σ_l plays the role of a rational polyhedron in a finite dimension, and Σ_l^∞ plays the role of a cylinder polyhedron (derived from Σ_l by restriction) with fixed, countably infinite dimension.

The following proposition says that cylinder polyhedra are just a geometric characterization of zerosets of McNaughton functions:

Proposition 3.6.
- *If X is finite, then rational polyhedra in $[0,1]^X$ coincide with zerosets of McNaughton functions from $[0,1]^X$ to $[0,1]$.*
- *If X is infinite, then cylinder polyhedra in $[0,1]^X$ coincide with zerosets of McNaughton functions from $[0,1]^X$ to $[0,1]$.*

Proof. For the first item, see [45], Corollary 2.10.

For the second item, let $f : [0,1]^X \to [0,1]$ be a McNaughton function. By definition, there is a finite subset Y of X, and a McNaughton function $g : [0,1]^Y \to [0,1]$ such that $f(x) = g(x|_Y)$. By the first item $\mathit{Zeroset}(g) = P_0$ is a rational polyhedron. So, $x \in \mathit{Zeroset}(f)$ if and only if $x|_Y \in P_0$, and $\mathit{Zeroset}(f)$ is a cylinder polyhedron.

Conversely, consider a cylinder polyhedron $C_X(P_0)$, where $P_0 \subseteq [0,1]^Y$ is a rational polyhedron, with Y finite subset of X. Then there is a McNaughton function $g : [0,1]^Y \to [0,1]$ such that $P_0 = \mathit{Zeroset}(g)$ by the first item; so we let $f : [0,1]^X \to [0,1]$ such that $f(x) = g(x|_Y)$. Then f is a McNaughton function. Moreover $x|_Y \in P_0$ if and only if $x \in \mathit{Zeroset}(f)$, hence $C_X(P_0) = \mathit{Zeroset}(f)$. So, every cylinder polyhedron is the zeroset of a McNaughton function. □

Since the models of a formula are the zerosets of its negation, we obtain:

Corollary 3.7.
- *If X is a finite set of variables, then rational polyhedra in $[0,1]^X$ coincide with models of formulas of Łukasiewicz logic over the variables in X.*
- *If X is infinite, then cylinder polyhedra in $[0,1]^X$ coincide with models of formulas of Łukasiewicz logic over the variables in X.*

3.3. Rational cones

Recall that a *rational simplicial cone* is a set $\sigma \subseteq \mathbb{R}^m$ of the form $\sigma = \mathbb{R}_{\geq 0}d_1 + \cdots + \mathbb{R}_{\geq 0}d_t$, where $d_1, \ldots, d_t \in \mathbb{Q}^m$ are linearly independent rational vectors. A *rational cone* is a finite union of rational simplicial cones.

3.4. Cylinder rational cones

In analogy with cylinder polyhedra, we can introduce cylinder cones.

Given any set X, let $\mathbb{R}^X$ be the set of functions from X to $\mathbb{R}$. We define *cylinder (rational) cone* in a hypercube $\mathbb{R}^X$ a subset of $\mathbb{R}^X$ of the form

$$C_X(P_0) = \{f \in \mathbb{R}^X \mid f|_Y \in P_0\}$$

where Y is a finite subset of X and $P_0 \subseteq \mathbb{R}^Y$ is a rational polyhedron. Here $f|_Y$ denotes the function f restricted to Y, that is, $f|_Y = f \circ j$, where $j : Y \to X$ is the inclusion map.

If $C \subseteq \mathbb{R}^X$ is a cylinder cone, we denote by $\mathrm{Ccone}(C)$ the lattice of all cylinder cones included in C, partially ordered by inclusion.

3.5. Cylinder cones as zerosets

Let X be a set. A *piecewise linear function* from $\mathbb{R}^X$ to $\mathbb{R}$ is a continuous, piecewise linear function with integer coefficients. Since such a function is divided into finitely many pieces and every piece is a linear polynomial, piecewise linear functions will depend only on finitely many variables, even when X is infinite.

The following proposition says that cylinder cones are just a geometric characterization of zerosets of piecewise linear functions:

Proposition 3.8.
- *If X is finite, then rational cones in $\mathbb{R}^X$ coincide with zerosets of piecewise linear functions from $\mathbb{R}^X$ to $\mathbb{R}$.*
- *If X is infinite, then cylinder cones in $\mathbb{R}^X$ coincide with zerosets of piecewise linear functions from $\mathbb{R}^X$ to $\mathbb{R}$.*

Proof. For the first item, one has to repeat the proof of [45], Corollary 2.10, especially the equivalence between (ii) and (iv), which here becomes the fact that a set $Y \subseteq \mathbb{R}^n$ is a rational cone if and only if it is the zeroset of a piecewise linear function from $\mathbb{R}^n$ to $\mathbb{R}$. This can be obtained by replacing regular complexes in (i) with regular fans (see chapter 2 of [45]).

For the second item, let $f : \mathbb{R}^X \to \mathbb{R}$ be a piecewise linear function. By definition, there is a finite subset Y of X, and a piecewise linear function $g : \mathbb{R}^Y \to \mathbb{R}$ such that $f(x) = g(x|_Y)$. By the first item $\mathit{Zeroset}(g) = C_0$ is a rational cone. So, $x \in \mathit{Zeroset}(f)$ if and only if $x|_Y \in C_0$, and $\mathit{Zeroset}(f)$ is a cylinder cone.

Conversely, consider a cylinder cone $C_X(C_0)$, where $C_0 \subseteq \mathbb{R}^Y$ is a rational cone, with Y finite subset of X. Then there is a piecewise linear function $g : \mathbb{R}^Y \to \mathbb{R}$ such that $C_0 = \mathit{Zeroset}(g)$ by the first item; so we let $f : \mathbb{R}^X \to \mathbb{R}$ such that $f(x) = g(x|_Y)$. Then f is a piecewise linear function. Moreover $x|_Y \in C_0$ if and only if $x \in \mathit{Zeroset}(f)$, hence $C_X(C_0) = \mathit{Zeroset}(f)$. So, every cylinder polyhedron is the zeroset of a piecewise linear function. $\square$

4. Relative subalgebras

Let A be an MV-algebra. It is worth pointing out that for every $a \in A$, the subspace $U(a)$ of $\mathrm{Spec}(A)$ is in turn a spectrum: in fact, $U(a) = \mathrm{Spec}(A|a)$, where $A|a$ denotes the MV-algebra A relativized to a (see [8], Theorem 8.1).

Moreover, the sets of the form $U(a)$ are exactly the compact open subsets of $\mathrm{Spec}(A)$. Since $\mathrm{Spec}(A)$ is a spectral space, the compact open subsets of $\mathrm{Spec}(A)$ form a lattice under inclusion which is an important invariant of A.

Let X be a topological space such that $X = \mathrm{Spec}(A)$ for some MV-algebra A. Denote by $K(X)$ the lattice of the compact open subsets of X. We have:

Proposition 4.1. *If $C \in K(X)$, then there is a natural map $f : K(X) \to K(C)$ such that $f(O) = O \cap C$ for every $O \in K(X)$. This map f is a surjective homomorphism of bounded lattices.*

Proof. By a direct verification. □

We note that f is closed if and only if for every $a_0, a_1 \in K(X)$ and for every $b \in K(X)$ with $b \leq C$, if $a_0 \wedge c \leq (a_1 \wedge c) \vee b$, then there is $x \in K(X)$ such that $a_0 \leq a_1 \vee x$ and $x \wedge c \leq b$.

We do not know whether this property holds for all A. If X is the Spectrum of a *Boolean* algebra, then the property does hold with $x = b \vee (X \backslash C)$.

Note that the natural inclusion from $K(C)$ to $K(X)$ is an injective homomorphism of lattices, but not of bounded lattices (C goes to C itself which is not the top of $K(X)$).

We note also that the Stone dual of the topological space $U(a)$ is isomorphic to the Belluce lattice of the relativized MV-algebra $A|a$. So, every compact open subset of an MV-space is an MV-space itself.

5. The Belluce functor for MV-algebras

The Belluce operation can be extended to MV-algebra morphisms $f : A \to B$ by letting $\beta(f)(x) = \beta(f(x))$. In this way, β is a covariant functor from the category MV of MV-algebras to the category BDL of bounded distributive lattices, see [7].

Proposition 5.1. *The Belluce lattice of M_X is dually isomorphic to the lattice* $\mathrm{Cpoly}([0,1]^X)$.

Proof. If $A = M_X$, every element f of A is a McNaughton function on $[0,1]^X$, and there is a bijection ϕ sending $\beta(f)$ to $Z(f)$, the zeroset of f. In fact,

$$\beta(f) \subseteq \beta(g) \tag{5.1}$$

is equivalent to

$$I(Z(f)) \subseteq I(Z(g)) \tag{5.2}$$

by the Wójcicki Theorem (see [41], Lemma 4.5), where $I(Z(f))$ is the set of McNaughton functions which are zero on $Z(f)$. The latter implies

$$Z(I(Z(g))) \subseteq Z(I(Z(f))) \tag{5.3}$$

and since $ZIZ = Z$ this implies

$$Z(g) \subseteq Z(f) \tag{5.4}$$

and again this implies 5.2 which is equivalent to 5.1.

Summing up, 5.1 and 5.4 are equivalent and ϕ is a well defined contravariant lattice isomorphism between $\beta(M_X)$ and the lattice of zeros of McNaughton functions in the set X of variables (or rational polyhedra when X is finite). Then the thesis follows from Proposition 3.6. □

Corollary 5.2. *$\beta(M_X)$ is a locally finite co-Heyting algebra for every set X.*

Proof. It follows from Theorem 3.3 and the previous lemma. □

Now we obtain

Proposition 5.3. *If $f : \beta(M_X) \to L$ is a closed surjective lattice homomorphism, then there is an ideal I of M_X such that $L \cong \beta(M_X)/\theta(\beta(I))$, where $\theta(\beta(I))$ is the congruence on the lattice $\beta(M_X)$ induced by the ideal $\beta(I)$.*

Proof. This follows from Lemma 2.1. □

Corollary 5.4. *A lattice is the Belluce lattice of an MV-algebra if and only if it is a closed surjective image of the Belluce lattice of a free MV-algebra.*

Proof. Let L be a lattice. Assume $L = \beta(A)$. Then there is a set X and a surjection $f : M_X \to A$. By Proposition 2.2, $\beta(f) : \beta(M_X) \to \beta(A)$ is a closed surjective lattice homomorphism. So L is a closed surjective image of the Belluce lattice of a free MV-algebra.

Conversely, assume $f : \beta(M_X) \to L$ be a closed surjective lattice homomorphism. By Lemma 2.1, L is the Belluce lattice of some MV-algebra.

□

Note in particular what happens when A is semisimple. Then we have:

Corollary 5.5. *If A is a semisimple MV-algebra, $A = M_X/I(C)$ where C is a closed subset of $[0,1]^X$, and X is any set, then $\beta(A) \cong \beta(M_X)/\equiv_C$, where $\beta(f) \equiv_C \beta(g)$ holds if and only if there is a cylinder polyhedron $R \supseteq C$ such that $Z(f) \cap R = Z(g) \cap R$.*

When $I(C)$ is principal, then C itself is a cylinder polyhedron, and $\beta(f) \equiv_C \beta(g)$ if and only if $Z(f) \cap C = Z(g) \cap C$.

We can also be more precise and consider lattices L such that $|L| \leq \kappa$, where κ is any infinite cardinal. In fact:

Lemma 5.6. *Let κ be an infinite cardinal. Suppose $|L| \leq \kappa$ and $L = \beta(A)$ for some MV-algebra A. Then there is an MV-algebra A' such that $|A'| \leq \kappa$ and $L = \beta(A')$.*

Proof. $\beta(A)$ is the lattice of principal ideals of A. For every element i of $\beta(A)$ let us choose a generator g_i of i in A. Let A' be the subalgebra of A generated by the g_i's. Note that $|A'| \leq \kappa$. Let us show that the lattices $\beta(A)$ and $\beta(A')$ are isomorphic.

For every $i \in \beta(A)$ let $\gamma(i) = i \cap A'$. Note that $\gamma(i)$ is the ideal generated by g_i in A', so $\gamma(i) \in \beta(A')$ and $\gamma : \beta(A) \to \beta(A')$. Let us show that the function γ is an isomorphism of lattices.

Clearly if $i \subseteq j$ then $\gamma(i) \subseteq \gamma(j)$. Conversely if $\gamma(i) \subseteq \gamma(j)$ then $i \cap A' \subseteq j$, so $g_i \in j$ and $i \subseteq j$. In particular γ is injective.

Finally let us show that γ is surjective. Let i' be a principal ideal of A'. We must find $i \in \beta(A)$ such that $i' = i \cap A'$. Now suppose i' is generated by g'. Let i be the principal ideal generated by g' in A. Then i is generated by g_i, which is in A'. So, $g' \leq ng_i$ for some n. Moreover $g_i \in i$, so $g_i \leq ng'$ for some n. Suppose $h \in i \cap A'$; then $h \leq ng_i$ for some n, hence $h \leq ng'$ for some n and $h \in i'$. So, $i \cap A' \subseteq i'$. Conversely, $i' \subseteq A'$ and $g' \leq ng_i$ for some n, so $g' \in i$ and $i' \subseteq i$. Summing up, $i' \subseteq i \cap A'$ and $i' = i \cap A'$. This means that γ is surjective. □

Corollary 5.7. *Let X be an infinite set. A lattice L such that $|L| \leq |X|$ is the Belluce lattice of an MV-algebra if and only if there is a closed surjective lattice homomorphism $\pi : \beta(M_X) \to L$.*

Proof. Suppose $|L| \leq |X|$ and L is the Belluce lattice of an MV-algebra A. By the previous lemma, there is an MV-algebra A' such that $L = \beta(A')$ and $|A'| \leq |X|$. So $A' = M_X/I$ where I is an ideal of M_X, and the lattice $\beta(A') = \beta(M_X)/\beta(I)$ is a closed epimorphic image of $\beta(M_X)$.

Conversely, if there is a closed surjective lattice homomorphism $\pi : \beta(M_X) \to L$, then L is the Belluce lattice of an MV-algebra by Corollary 5.4. □

For $|X| = \omega$ in particular we obtain:

Corollary 5.8. *A countable lattice L is the Belluce lattice of an MV-algebra if and only if there is a closed surjective lattice homomorphism $\pi : \beta(M_\omega) \to L$.*

This characterization of countable Belluce lattices can be compared with the characterization of countable, ℓ-representable lattices given in [57].

By Proposition 5.1 we obtain also:

Corollary 5.9. *A lattice L is the Belluce lattice of some MV-algebra if and only if for some X there is a closed surjective lattice homomorphism π : $\mathrm{Cpoly}([0,1]^X)^{op} \to L$.*

Hence, given two cylinder polyhedra $p, q \in \mathrm{Cpoly}([0,1]^X)$, $\pi(p) = \pi(q)$ if and only if there is a rational polyhedron $r \in \mathrm{Ker}(\pi)$ such that $p \cap r = q \cap r$. Note that $\mathrm{Ker}(\pi)$ is a filter.

In particular, if L is the Belluce lattice of some finitely presented MV-algebra, then X can be chosen to be finite, and the filter $Ker(\pi)$ can be taken principal, say generated by a single polyhedron r, and then $\pi(p) = \pi(q)$ if and only if $p \cap r = q \cap r$.

The following theorem is essentially Lemma 2.1 in [57] (up to replacing ℓ-groups with MV-algebras, which is possible by Mundici equivalence):

Theorem 5.10 [57]. *A topological space X is the spectrum of an MV-algebra if and only if X is spectral and the lattice $K(X)$ of its compact open sets is the Belluce lattice of an MV-algebra.*

This theorem allows one to move from the spectrum problem for MV-algebras to the problem of characterizing those lattices which are the Belluce lattices of some MV-algebra.

By combining Corollary 5.9 and Theorem 5.10 we obtain:

Theorem 5.11. *A topological space X is the spectrum of an MV-algebra if and only if X is spectral and for some set Y there is a closed surjective lattice homomorphism $\pi : \mathrm{Cpoly}([0,1]^Y)^{op} \to K(X)$. If $X = \mathrm{Spec}(A)$, then $K(X) = \beta(A)$.*

Actually we can suppose Y infinite:

Corollary 5.12. *A topological space X is the spectrum of an MV-algebra if and only if X is spectral and for some infinite set Y there is a closed surjective lattice homomorphism $\pi : \mathrm{Cpoly}([0,1]^Y)^{op} \to K(X)$. If $X = \mathrm{Spec}(A)$, then $K(X) = \beta(A)$.*

6. The case of Abelian ℓ-groups and perfect MV-algebras

6.1. The Belluce lattice of ℓ-groups

In this section we turn to Abelian ℓ-groups and we want to carry over the theory of Belluce lattices from MV-algebras to Abelian ℓ-groups. It turns out that there is a close relationship between spectra of Abelian ℓ-groups and spectra of perfect MV-algebras or local MV-algebras. We begin with setting up the theory of Belluce lattice for Abelian ℓ-groups.

Lemma 6.1. *If I is an ℓ-ideal of an ℓ-group G and $a \notin I$, then there is a prime ℓ-ideal P such that $I \subseteq P$ and $a \notin P$.*

Proof. The argument is similar to the one for MV-algebras. We provide some details.

Suppose $I \subseteq G$ is an ℓ-ideal and $a \notin I$. Let P be a maximal ℓ-ideal among those which contain I and do not contain a (P exists by Zorn Lemma). Suppose for an absurdity that P is not prime. Then there are $x, y \in P$ such that $x \wedge y \in P$, $x \notin P$, $y \notin P$. By choice of P we have $a \in id(P \cup \{x\})$ and $a \in id(P \cup \{y\})$. So for some $p \in P$ and some integer k we have $|a| \leq p + k|x|$ and $a \leq p + k|y|$. Taking the infimum we have

$$\begin{aligned} a &\leq (p + k|x|) \wedge (p + k|y|) \leq 2(p \vee k|x|) \wedge 2(p \vee k|y|) \\ &= 2(p \vee k|x|) \wedge (p \vee k|y|) = 2(p \vee k(|x| \wedge |y|)) \leq 2(p \vee k|x \wedge y|) \end{aligned}$$

and the latter element belongs to P, so $a \in P$, which is a contradiction. □

Corollary 6.2. *Given two elements a, b of an Abelian ℓ-group G the following are equivalent:*

- *a, b belong to the same prime ℓ-ideals;*
- *a, b belong to the same ℓ-ideals;*
- *a, b generate the same ℓ-ideal.*

By the previous corollary, Belluce lattices can be defined for Abelian ℓ-groups exactly as we did for MV-algebras (either as lattices of principal ℓ-ideals, or as quotients modulo lying in the same prime ℓ-ideals).

In [7], Prop. 25, there is a relation between the spectrum of an Abelian ℓ-group and the spectrum of its Di Nola-Lettieri equivalent MV-algebra. In this section we make something similar for Belluce lattices.

We can say that an MV-algebra or lattice is *local* if it has a unique maximal ideal. Further an MV-algebra or bounded lattice is called *perfect* if it is generated by the intersection of its maximal ideals (although this notion seems to be more common in MV-algebras than in lattices). The perfect skeleton of an MV-algebra A is the largest perfect subalgebra of A, denoted $\mathrm{Perf}(A)$.

Lemma 6.3. *If A is a local MV-algebra then $\beta(A) \cong \beta(\mathrm{Perf}(A))$.*

Proof. Let $A' = \mathrm{Perf}(A)$. Let $\phi : \beta(A) \to \beta(A')$ such that

$$\phi(\beta(x)) = \begin{cases} \beta(x), & x \in A' \\ \beta(1), & \text{otherwise.} \end{cases}$$

Then ϕ is a lattice isomorphism. □

Corollary 6.4. *If A is a local MV-algebra then* $\mathrm{Spec}(A) \cong \mathrm{Spec}(\mathrm{Perf}(A))$.

Lemma 6.5. *A is a local MV-algebra if and only if $\beta(A)$ is a local lattice.*

Proof. If A is local, then

- $\mathrm{Rad}(A)$ is the only maximal ideal of A, and
- $\beta(\mathrm{Rad}(A))$ is the maximal ideal of $\beta(A)$ (see [6], Theorem 13).

Conversely, if $\beta(A)$ is local and M is the largest ideal of $\beta(A)$, then $\beta^{-1}(M)$ is the largest ideal of A and A is local (see [6], Theorem 15). □

Theorem 6.6. *Let L be a local Belluce lattice. Then $L\backslash\{1\}$ is isomorphic to the Belluce lattice of an Abelian ℓ-group G. Conversely, if L is the Belluce lattice of an Abelian ℓ-group G, then the lattice $L \cup \{\infty\}$ (where ∞ is an element above all elements of L) is a local Belluce lattice.*

Proof. Let L be the Belluce lattice of a local MV-algebra. Then $L = \beta(A)$ where A is perfect. Let $G = \Delta(A)$ be the Di Nola-Lettieri ℓ-group corresponding to A (see [26]). Then $\mathrm{Spec}(G) \cong \mathrm{Spec}(A)\backslash\{M\}$, where M is the unique maximal ideal of A. Then $K(\mathrm{Spec}(G)) \cong K(\mathrm{Spec}(A))\backslash\{1\}$ (see [7]). But $K(\mathrm{Spec}(A) \cong \beta(A) \cong L$ so $K(\mathrm{Spec}(G)) \cong L\backslash\{1\}$.

Conversely, let $L = \beta_l(G)$. Then $\mathrm{Spec}(L) \cong \mathrm{Spec}(G)$. Again by [7], $\Delta^{-1}(G)$ is a perfect MV-algebra with largest ideal M and we have $\mathrm{Spec}(G) \cup \{M\} \cong \mathrm{Spec}(\Delta^{-1}(G))$, so $\beta(\Delta^{-1}(G))$ is the Belluce lattice of a local MV-algebra and $L \cup \{M\} = \beta_l(\Delta^{-1}(G))$. □

Corollary 6.7. *Let X be a spectral space and $K(X)$ the lattice of its compact open sets. The following are equivalent:*

(1) X is the spectrum of some Abelian ℓ-group;

(2) X is homeomorphic to $\mathrm{Spec}(K(X))$ *and* $K(X) \cup \{\infty\}$ *is the Belluce lattice of a local MV-algebra A, where $\infty > x$ for every $x \in K(X)$.*

Moreover $A = M_Y/J$ for some set Y, where J is a primary ideal of M_Y, and $\beta(A) = K(X) \cup \{\infty\} = \beta(M_Y)/\beta(J)$.

6.2. A characterization of spectra of Abelian ℓ-groups

If G is an Abelian ℓ-group, then $\mathrm{Spec}(G)$ is a generalized spectral topological space (see [13]). In particular, it has a basis of compact open sets of the form $U(g) = \{P \in \mathrm{Spec}(G) \mid g \notin P\}$ and these sets form a lattice L (with minimum, but the maximum exists only if G has a strong unit).

As in MV-algebras, two elements in G belong to the same prime ℓ-ideals if and only if they generate the same ℓ-ideal. Hence, if we send $U(g)$ to the ℓ-ideal generated by g, we have a well defined lattice isomorphism between L and the lattice of principal ℓ-ideals of G. Moreover L is isomorphic to the quotient of G modulo lying in the same prime ideals, and we call such a quotient the Belluce lattice of G, in analogy with MV-algebras.

With a proof similar to [6], Theorem 20, for the Belluce lattice of MV-algebras, it can be shown:

Theorem 6.8. *The spectrum of the Belluce lattice of an Abelian ℓ-group G is isomorphic to the spectrum of G.*

Theorem 6.9. *A lattice L is the lattice of principal ideals of an Abelian ℓ-group if and only if L is a closed epimorphic image of the lattice of principal ideals of a free Abelian ℓ-group.*

Proof. The proof is analogous to the proof of Corollary 5.4 up to replacing MV-algebras with Abelian ℓ-groups. □

Theorem 6.10. *The lattice of principal ideals of the free Abelian ℓ-group over a set X is isomorphic to the lattice of cylinder rational cones in $\mathbb{R}^X$.*

Proof. Let $F_G(X)$ be the group of piecewise linear functions from $\mathbb{R}^X$ to $\mathbb{R}$. By [3], $F_G(X)$ is the free Abelian ℓ-group on the set X. By Proposition 3.8, the zerosets of the functions $f \in F_G(X)$ are the cylinder rational cones in $\mathbb{R}^X$. Like in MV-algebras, if we associate the principal ideal of f with the zeroset of f, we obtain a lattice isomorphism between the lattice of principal ideals of $F_G(X)$ and the lattice of cylinder rational cones in $\mathbb{R}^X$. □

By putting together the previous results, we obtain a characterization of spectra of Abelian ℓ-groups:

Theorem 6.11. *A topological space X is the spectrum of an Abelian ℓ-group if and only if:*

- *X is generalized spectral;*
- *the lattice of compact open subsets of X is a closed epimorphic image of the dual lattice of cylinder rational cones in $\mathbb{R}^Y$ for some set Y.*

Now we conjecture that an axiomatic description of lattices of cylinder cones in $\mathbb{R}^X$ is possible along the lines of the description of the lattices of cylinder polyhedra in $[0,1]^X$ given in the next section.

We conclude with a link between lattice of polyhedra and lattices of cones:

Corollary 6.12. *Let X be an infinite set. There is an epimorphism from the lattice* $\mathrm{Cpoly}([0,1]^X)$ *to the lattice* $\mathrm{Ccone}(\mathbb{R}^X) \cup \{\infty\}$.

Proof. Let $P = \Gamma(Z\ lex\ F_G(X), (1,0))$. P is a perfect MV-algebra of the same cardinality as X, so there is a MV-algebra epimorphism from $Free_{MV}(X)$ to P and a lattice epimorphism from $\beta(Free_{MV}(X))$ to $\beta(P)$. We know the lattice $\beta(Free_{MV}(X))$ is isomorphic to $\mathrm{Cpoly}([0,1]^X)$, and $\beta(P)\backslash\{max(\beta(P)\}$ is isomorphic to $\beta(F_G(X))$. Adding a point at infinity, $\beta(P)$ is isomorphic to $\beta(F_G(X)) \cup \{\infty\})$, that is to $\mathrm{Ccone}(\mathbb{R}^X) \cup \{\infty\}$. By composition we have a lattice epimorphism from $\beta(Free_{MV}(X))$ to $\mathrm{Ccone}(\mathbb{R}^X) \cup \{\infty\}$. Finally $\beta(Free_{MV}(X))$ is isomorphic to $\mathrm{Cpoly}([0,1]^X)$. □

It would be interesting to find a direct geometric proof of the previous corollary.

7. Logical characterizations

7.1. Logical characterization of the lattice of one dimensional rational polyhedra

The logical characterization by Wehrung of Belluce lattices (see Introduction) is given in second order logic, and Wehrung himself showed that there is no such characterization in first order logic (see introduction). Now, as an intermediate logic, we can consider *monadic* second order logic, which is both a considerably weaker variant of second order logic and a considerably weaker variant of first order logic. For this reason, as an application of Theorem 5.11, we are interested in describing Belluce lattices with monadic second order logic.

In this and the following subsections we would like to axiomatize, in monadic second order logic, the lattices $L_X = \mathrm{Poly}([0,1]^X)$ when X has one, two or infinitely many elements. The case of a finite number of elements greater than 2 is analogous to the case of 2 elements and will be omitted. Instead, the case of one dimension is different from the case of dimension two, essentially because we do not know how to prove, or postulate in monadic second logic, the Thales theorem of plane geometry, which is a key step in constructing points with given (rational) coordinates. On the other hand, points in one dimension have the advantage of being totally ordered by the betweenness relation and being countably many, so that we can apply Cantor back and forth theorem on the categoricity of the ordered rational line.

So in this section we consider the lattice $L_1 = \mathrm{Poly}([0,1])$.

Let us call an element x of L_1 *connected* if and only if it is not the disjoint union of two nonzero elements (note that x is connected in our sense if and only if the closure of x in $\mathbb{R}$ is connected). The connected sets are exactly the atoms and the intervals.

Note that connectedness is first order definable in L_1. So, also the ternary relation of *betweenness* is definable:

Definition. The atom x is between y and z if and only if every connected element which contains y, z contains x as well.

We note that the lattice L_1 satisfies the following axioms:

Definition. An atom is *extremal* if it is not the intersection of two elements which are nonatomic and connected. An atom is *inner* if it is not extremal.

Axiom 7.1.1. There are exactly two extremal atoms (denoted 0 and 1).

Axiom 7.1.2. The binary relation "x is between 0 and y" is a total order on the atoms, dense in itself, with maximum and minimum.

Definition. The *interval* AB is the smallest connected set containing A and B.

Axiom 7.1.3. For every pair of atoms A, B the interval AB exists. Connected elements coincide with atoms or intervals.

We note that *finiteness* of sets of atoms and intervals can be formulated in monadic second order logic. In fact:

- a set S of atoms is finite if and only if every nonempty subset of S has a minimum and a maximum in the total betweenness ordering $0 < x < y$;
- a set S of intervals is finite if and only if the set of the extremes of the elements of S is a finite set of atoms.

Axiom 7.1.4. Every element is a finite disjoint supremum of connected elements.

Axiom 7.1.5. Let F, F' be finite sets of connected elements. In L_1, we have $sup\ F \leq sup\ F'$ if and only if every atom below some element of F is below some element of F'.

The next axiom is used to prove that (inner) atoms are countably many. Note that we cannot axiomatize directly the countability of the rational order, because the monadic second order theory of the rational order is not categorical, see [51]. However, in the lattice we can encode pairs of segments, and in this way we can in some sense mimic a full second order quantifier. Our choice of encoding of pairs is somewhat arbitrary, as it happens often with encodings.

Definition. An *encoded pair of atoms* is an element of L_1 which is a supremum of $\{0, A, B\}$ or $\{1, A, B\}$ or $\{0, A\}$ where A, B are different inner atoms. The projections are defined as follows:

- $\pi_1(sup\{0, A, B\}) = min\{A, B\}$,
- $\pi_2(sup\{0, A, B\}) = max\{A, B\}$,
- $\pi_1(sup\{1, A, B\}) = max\{A, B\}$,
- $\pi_2(sup\{1, A, B\}) = min\{A, B\}$,
- $\pi_1(sup\{0, A\}) = \pi_2(sup\{0, A\}) = A$.

Now the idea is to postulate that there is a Peano triple (a model of Peano's axioms) whose domain is the set of inner atoms.

Axiom 7.1.6. There is an inner atom z and a relation S between inner atoms (that is a set of encoded pairs) such that:

- S encodes an injective function on inner atoms (the successor function);
- z (the initial element) is not in the range of S;
- (induction principle) if P is a property of inner atoms such that $P(z)$ holds and $P(A)$ implies $P(SA)$ for every inner atom A, then P is true in all inner atoms.

The previous axiom implies that the inner atoms are countably many, so by adding 0 and 1, we have that the atoms are countably many.

Let us call 1-*polyhedral lattice* any lattice satisfying the axioms 7.1.1)-6).

Theorem 7.1. *Every* 1*-polyhedral lattice* L' *is isomorphic to* L_1*.*

Proof. Suppose a lattice L' satisfies the axioms 7.1.1)-6). Since the atoms of L' are countably many, by Cantor back and forth theorem on the rationals, there is a bijection β_1 between the atoms of L_1 and L' which respects the betweenness relation: $A < B < C$ if and only if $\beta_1(A) < \beta_1(B) < \beta_1(C)$.

Let us extend β_1 to intervals by letting $\beta_1(AB) = \beta_1(A)\beta_1(B)$. Since betweenness among atoms is preserved, an atom A is below an interval J if and only if $\beta_1(A)$ is below $\beta_1(J)$. Hence, two finite sets F, F' verify the condition in axiom 5 if and only if $\beta_1(F), \beta_1(F')$ verify the same condition.

So, by axiom 5, we can extend β_1 to L', sending *sup* F to *sup* $\beta_1(F)$ for every finite set F of atoms and intervals, and β_1 is an isomorphism between L' and L_1. $\square$

7.2. Logical characterization of the lattice of rational subpolyhedra of $[0, 1]^2$

Let $L_2 = \mathrm{Poly}([0, 1]^2)$. We want to characterize the lattice L_2 in monadic second order logic. To this aim we postulate the existence of a subset K_2 of L_2 (the lattice of the rational convex polygons) satisfying the following axioms (plus the ones listed in "Appendix A").

Axiom 7.2.1. L_2 is a distributive lattice. Every element of L_2 is the supremum of a finite subset of K_2.

The monadic second order definability of finiteness for subsets of K_2 follows from the axioms in "Appendix A".

Axiom 7.2.2. Given two finite sets $F, F' \subseteq K_2$, we have *sup* $F \leq$ *sup* F' if and only if every atom below some element of F is below some element of F'.

Now we have the main result of this subsection.

Let us call 2-*polyhedral lattice* any lattice satisfying the axioms of "Appendix A" and the axioms 7.2.1 and 7.2.2.

Theorem 7.2. *Every* 2*-polyhedral lattice* L' *is isomorphic to* L_2*.*

Proof. L_2 is isomorphic to the lattice of suprema of finite subsets of K_2 with the order defined in Axiom 2. Any other polyhedral lattice L' is isomorphic to the lattice of suprema of finite subsets of some lattice K' which verifies the axioms of "Appendix A". By Theorem 12.4 of "Appendix A", there is an isomorphism β_2 between K' and K_2, sending bijectively atoms to atoms, such that an atom a is below an element s if and only if $\beta_2(a)$ is below $\beta_2(s)$. Hence, two finite sets F, F' verify the condition in axiom 2 if and only if $\beta_2(F), \beta_2(F')$ verify the same condition. So we can extend β_2 to L' by letting $\beta_2(sup\ F) = sup\ \beta_2(F)$, and β_2 is an isomorphism between L' and L_2. □

7.3. Logical characterization of the lattice of cylinder rational subpolyhedra of $[0, 1]^X$ when X is infinite

We expect that the axiomatization of L_2 given in the previous subsection can be modified to an axiomatization of L_n for every single natural number n. However, we prefer to pass directly to the monadic second order axiomatization of the infinite dimensional lattices L_X when X is an infinite set. In fact, the infinite dimensional case is particularly interesting because of Corollary 5.12.

In this case there are no atoms, but every nonzero element of L_X has a finite codimension, in the sense that it is described by a finite set of inequalities.

We want to axiomatize the class of lattices $L_X = \text{Cpoly}([0, 1]^X)$, where X is infinite.

Let K_X be the set of convex elements of L_X. We give in "Appendix B" an axiomatization of K_X (in monadic second order logic). To pass from K_X to L_X we add two postulates:

Axiom 7.3.1. L_X is a lattice. Every element of L_X is the supremum of a finite subset of K_X.

The definability of finite subsets in K_X in monadic second order logic follows from the axioms in "Appendix B". Other notions used in the next axiom (pseudoatoms and compatibility) are explained in "Appendix B". Suffice it to say that a compatibility class of pseudoatoms corresponds to a "good" class of atoms for an n-dimensional geometry.

Axiom 7.3.2. Let F, F' be finite subsets of K_X . Assume every element of $F \cup F'$ is supremum of a finite set of pseudoatoms compatible with a pseudoatom A. Then in L_X, $sup\ F \leq sup\ F'$ if and only if every pseudoatom compatible with A below some element of F is below some element of F'. (see "Appendix B" for pseudoatoms and compatibility).

Now we have the main result of this subsection.

Let us call *X-polyhedral lattice* any lattice satisfying the axioms of "Appendix B" and the axioms 7.3.1,7.3.2 above.

Theorem 7.3. *Any X-polyhedral lattice L' is isomorphic to L_λ, where λ is the cardinality of X in L'.*

Proof. L_X is isomorphic to the lattice of finite subsets of K_X with the order defined in Axiom 2. Any other polyhedral lattice L' is isomorphic to the lattice of finite subsets of some lattice K' which verifies the axioms of "Appendix B". By Theorem 12.8 of "Appendix B", if X in L' has an infinite size λ, there is an isomorphism β_λ between K' and K_λ with the properties listed in Theorem 12.8. In particular, if A is a pseudoatom compatible with all the elements of a set G of pseudoatoms, then $A \leq sup\ G$ if and only if $\beta_\lambda(A) \leq sup\ \beta_\lambda(G)$.

Hence, F, F' verify the condition in Axiom 2 if and only if $\beta_\lambda(F), \beta_\lambda(F')$ verify the same condition.

So we can extend the map β_λ to finite suprema of elements of K' by letting $\beta_\lambda(sup\ F) = sup\ \beta_\lambda(F)$. Now β_λ is an isomorphism between L' and L_X. □

7.4. A lower bound and an upper bound for ℓ-representability

The previous sections were aimed at using monadic second order logic to describe Belluce lattices. In this section we come back to full second order logic.

We assume the usual Zermelo-Fraenkel set theory plus the axiom of choice. Moreover, our second order quantifiers will range always over *all* possible subsets, relations, etc. of the domain.

Call ESO existential second order logic, that is, the class of formulas of second order logic given by a finite sequence of existential second order quantifiers followed by a first order formula. In the (Kleene) analytical hierarchy, $ESO = \Sigma^1_1$.

Call $ESOW$ (a new fragment of second order logic) the closure of first order logic under first order quantifiers, existential second order quantifiers, and weak monadic second order quantifiers (the latters range over *finite* subsets of the domain).

Note that $ESOW$ is a slight syntactic extension of ESO but, unlike ESO, lacks invariance under ultraproduct. Moreover $ESOW$ is a subset of the fragment Σ^1_2 of the Kleene analytical hierarchy and coincides with ESO on the structure $\mathbb{N}$ of natural numbers with addition and multiplication, since in $\mathbb{N}$ finite sets of natural numbers can be coded by single natural numbers via binary coding.

The next theorem is a lower bound:

Theorem 7.4. *The class of Belluce lattices of Abelian ℓ-groups is not definable by any class of ESO formulas.*

Proof. It is enough to show that the class of Belluce lattices of ℓ-groups is not closed under ultrapowers.

We define a lattice D, called D_ω in [57] (here we prefer to drop the ω subscript in the notation).

Let $P_{cf}(\omega)$ be the set of all finite or cofinite subsets of ω, ordered by inclusion, and let $\{0, 1, 2\}$ be a chain with three elements where $0 < 1 < 2$.

Let D be the set of all pairs $(x, y) \in P_{cf}(\omega) \times \{0, 1, 2\}$ such that either x is finite and $y = 0$, or x is cofinite and $y \neq 0$.

In [57] it is shown that D is the Belluce lattice of an ℓ-group.

Now let U be any nonprincipal ultrafilter over ω and let $D' = D^\omega/U$ be the ultrapower of D modulo U. We say that a subset of ω is *large w.r.t. U* if it belongs to U, and *small w.r.t. U* otherwise. Given $c \in D^\omega$, let $(c)_U$ be the equivalence class of c modulo U in D'.

Clearly D' is a lattice. We have to show that D' is not a Belluce lattice. By [57] it is enough to show that D' does not have countably based differences, that is, that there are $a, b \in D'$ such that $[b \ominus a]$ has no coinitial countable subset, where $[b \ominus a] = \{x \in D' \mid b \leq a \vee x\}$.

As a, b we take the classes modulo U of the ω-sequences constantly equal to $(\omega, 1)$ and $(\omega, 2)$ respectively.

We note that in D we have

$$\begin{aligned}[(\omega,2) \ominus (\omega,1)] &= \{(x,y) \in D \mid (\omega,2) \leq (\omega,1) \vee (x,y)\}\\ &= \{(x,y) \in D \mid 2 \leq 1 \vee y\}\\ &= \{(x,y) \in D \mid y = 2\}\\ &= \{(x,2) \mid x \subseteq \omega \text{ cofinite}\}.\end{aligned}$$

Hence in D', $[b \ominus a]$ is the set of all sequences in D' which have a large set of components (w.r.t. U) of the form $(x, 2)$ where x is a cofinite subset of ω.

We must show that $[b \ominus a]$ has no countable coinitial subset.

To this aim, let S be a countable subset of $[b \ominus a]$. Let us enumerate S, possibly with repetitions, as $S = \{(c_1)_U, (c_2)_U, \ldots, (c_n)_U, \ldots\}$ where $c_1, c_2, \dots \in D^\omega$ (repetitions allow us to include the case when S is *finite*).

Then for every i and for a large set (w.r.t U) of $j \in \omega$, we have $(c_i)_j = (x_{ij}, 2)$, where x_{ij} is a cofinite subset of ω. Up to changing a small set (w.r.t U) of components of $c_1, c_2, \ldots$, with $(\omega, 2)$, we can suppose that for *every* $i, j \in \omega$ we have $(c_i)_j = (x_{ij}, 2)$, where x_{ij} is a cofinite subset of ω.

For every $j \in \omega$ let

$$y_j = x_{1j} \cap x_{2j} \cap \cdots \cap x_{jj} \backslash \{min(x_{1j} \cap x_{2j} \cap \cdots \cap x_{jj})\}.$$

Then for every $j \in \omega$, y_j is a cofinite subset of ω, and $y_j \subset x_{ij}$ for every $i \leq j$ (note that we have a proper inclusion).

Take the sequence $d \in D^\omega$ such that $d_j = (y_j, 2)$ for every $j \in \omega$. Since y_j is a cofinite subset of ω for every $j \in \omega$, we have $(d)_U \in [b \ominus a]$. Moreover, $d_j < (c_i)_j$ for every $j \geq i$, and since the ultrafilter U is not principal, we have $\{j \in \omega \mid j \geq i\} \in U$ for every $i \in \omega$. So $\{j \in \omega \mid d_j < (c_i)_j\} \in U$ for every $j \in \omega$, hence $(d)_U < (c_i)_U$ for every $i \in \omega$, and $(d)_U$ is strictly below every element of S.

Summing up, S is not coinitial in $[b \ominus a]$. □

As an upper bound we have:

Theorem 7.5. *The class of Belluce lattices of ℓ-groups is definable by an ESOW formula.*

Proof. A lattice L is ℓ-representable if and only if either L is finite and completely normal (by [57]) or there is a group structure G on the set L and there is a surjective function $f : G^+ \to L$ such that:

- if $f(x) \leq f(y)$, then x is below some multiple z of y;
- if not $f(x) \leq f(y)$, then there is an ℓ-ideal that contains x but not y.

To complete the proof, we note that the relation "z is multiple of y" is expressible in weak monadic second order logic:

Lemma 7.6. *If y, z are elements of an Abelian l-group G, then z is a multiple of y if and only if there is a finite set $F \subseteq G$ such that:*

- *$y \in F$;*
- *for every $w \in F$, if $w \neq z$, then $w + y \in F$.*

This concludes the proof of Theorem 7.5. □

We note that the previous lemma holds more generally in torsion-free groups.

8. Finitely presented MV-algebras and polyhedral lattices

In this section we describe a categorial equivalence between the range of the Belluce functor restricted to finitely presented MV-algebras and a category of lattices associated to rational polyhedra.

A finitely presented MV algebra has the form $A = M_X / \operatorname{id}(f)$, where X is finite and $\operatorname{id}(f)$ is the ideal generated by a function f. Then

$$\begin{aligned}\beta(A) &= \beta(M_X / \operatorname{id}(f))) \\ &= \beta(M_X)/\{\beta(g) \mid g \in \operatorname{id}(f)\} \\ &= \beta(M_X)/\{\beta(g) \mid Zf \subseteq Zg\},\end{aligned}$$

so the lattice $\beta(A)$ is $\beta(M_X)$ modulo the ideal $J(Zf)$, where $J(P)$ is the filter of the rational polyhedra including P.

From this we obtain:

Proposition 8.1. *Given a rational polyhedron $P \subseteq [0,1]^X$, with X finite, we have the isomorphism of lattices*

$$\beta(M_X|_P) \cong \operatorname{Poly}(P)^{op},$$

where $\operatorname{Poly}(P)$ *is the lattice of rational polyhedra included in P.*

Proof. Note that, for every rational polyhedron P, the congruence associated to $J(P)$ is the relation θ_P such that $Q\theta_P Q'$ if and only if $Q \cap P = Q' \cap P$. The congruence classes of θ_P are in bijection with the rational polyhedra included in P. □

It would be interesting to characterize explicitly the lattices of the form $\operatorname{Poly}(P)$. For instance, they are never totally ordered, unless P is a point. So, no linearly ordered lattice $L \neq \{0,1\}$ can be the Belluce lattice of a finitely presented MV-algebra A, because otherwise A should be both semisimple and linearly ordered, hence A would be simple and $\beta(A) = \{0,1\}$. Moreover, the lattices $\operatorname{Poly}(P)$ tend to have a kind of "fractal behavior": for instance, suppose $P \subseteq [0,1]$ is simply a rational segment in one dimension (whose subpolyhedra are finite unions of rational points and rational segments); then P contains properly another rational segment Q, and if we take a linear isomorphism from P onto Q, every subpolyhedron of P will correspond to a subpolyhedron of Q and conversely. So, Poly(P) contains a copy of itself.

Finitely presented MV-algebras play a useful rôle because every MV-algebra is a colimit of finitely presented MV-algebras, and the functor β preserves colimits, see [25], Theorem 10.

Let us call $BLATT$ the range of the functor β, and $FP - BLATT$ the range of the functor β restricted to the category of finitely presented MV-algebras. Note that:

- we can suppose that the functor β is injective on objects, and
- the range of a functor which is injective on the objects of a category is still a category (in particular, morphisms are closed under composition; the other axioms of categories are easily satisfied).

Let us call $POLY$ the category where the objects are the Poly(P), where P is a rational polyhedron, and where a morphism f from Poly(P) to Poly(Q) is obtained from a definable map $\phi : Q \to P$ by taking the inverse image. Namely, given a subpolyhedron $P' \subseteq P$, we let $f_\phi(P') = \phi^{-1}(P')$.

Finally, let us call $FP-Q$ the category of quotients of the Belluce lattices of free finitely generated MV-algebras modulo principal ideals, and Q the category of arbitrary quotients of the Belluce lattices of free MV-algebras. There is a natural bijection between the objects of $FP-Q$ and $FP-BLATT$, sending $\beta(F)/\beta(I)$ to $\beta(F/I)$, and a similar one between the objects of Q and $BLATT$. In $FP-Q$ the morphisms are induced from morphisms in $FP-BLATT$, and in Q the morphisms are induced by morphisms in $BLATT$. So we have:

Lemma 8.2. *The categories $FP - Q$ and $FP - BLATT$ are equivalent. The categories Q and $BLATT$ are equivalent.*

Theorem 8.3. *The categories $FP - BLATT$ and $POLY$ are equivalent.*

Proof. We describe an equivalence η. On objects, the equivalence η is given by the canonical lattice isomorphism between $\beta(M_X|_P)$ and Poly(P). On morphisms, if $h : M_X|_P \to M_Y|_Q$ is an MV-algebra morphism between finitely presented MV-algebras, then h is the image in the Marra-Spada functor MS (see [41]) of a Z-map $\phi_h : Q \to P$, and the equivalence η sends $\beta(h)$ to f_{ϕ_h}.

The operation η is really an equivalence. In fact, on arrows it is well defined, injective and surjective. Surjectivity follows directly from the definition of the category $POLY$. For well-definedness and injectivity, it is enough to show that for every pair of Z-maps $\phi, \psi : Q \to P$, we have $\beta(MS(\phi)) = \beta(MS(\psi))$ if and only if $f_\phi = f_\psi$.

Now suppose $\beta(MS(\phi)) = \beta(MS(\psi))$. Then

$$\beta(MS(\phi))(a) = \beta(MS(\psi))(a)$$

for every McNaughton function $a : P \to [0, 1]$. By definition of β we have $\beta(MS(\phi)(a)) = \beta(MS(\psi)(a))$. By definition of the Marra-Spada functor, $\beta(a \circ \phi) = \beta(a \circ \psi)$. By the Wójcicki Theorem, $Z(a \circ \phi) = Z(a \circ \psi)$, where Z denotes the zeroset. So $\phi^{-1}(Z(a)) = \psi^{-1}(Z(a))$ for every a. Since every subpolyhedron P' of P has the form $Z(a)$, this means $\phi^{-1}(P') = \psi^{-1}(P')$ for every P', that is, $f_\phi = f_\psi$. The converse is analogous. □

9. Cylinder MV-algebras and cylinder polyhedral lattices

In this section we somewhat generalize the results of the previous section from finitely presented MV-algebras to a class of MV-algebras, which we call cylinder MV-algebras. This class, it seems, has not received much attention in the literature. It is an intermediate class between finitely presented MV-algebras and semisimple MV-algebras.

Recall from [41] that a definable map between two sets $C \subseteq [0,1]^X, D \subseteq [0,1]^Y$ is an Y-tuple of McNaughton functions from $[0,1]^X$ to $[0,1]$ sending C to D.

Let us call *cylinder MV-algebra* an MV-algebra $M_X/\operatorname{id}(f)$, where X can be finite or infinite.

Note the following:

Lemma 9.1.
- *Every cylinder MV-algebra is semisimple.*
- *In the Marra-Spada duality, cylinder MV-algebras correspond exactly to cylinder polyhedra.*

Proof. For the first point, by the Wójcicki Theorem we have $M_X/\operatorname{id}(f) = M_X/I(Z(f)) = M_X|_{Z(f)}$ and the latter is an MV-algebra of McNaughton functions over the set $Z(f)$, hence it is semisimple.

For the second point, the duality of [41] sends $Z(f)$ to $M_X/\operatorname{id}(f) = M_X/I(Z(f)) = M_X|_{Z(f)}$ and the latter is a cylinder MV-algebra by definition. □

We note that Proposition 8.1 generalizes to the infinite dimensional case as follows:

Proposition 9.2. *Given a cylinder polyhedron $P \subseteq [0,1]^X$, with X infinite, we have the isomorphism of lattices $\beta(M_X|_P) \cong \operatorname{Cpoly}(P)^{op}$, where $\operatorname{Cpoly}(P)$ is the lattice of rational polyhedra included in P.*

Proof. Note that, for every cylinder polyhedron P, the congruence associated to $J(P)$ is the relation θ_P such that $Q\theta_P Q'$ if and only if $Q \cap P = Q' \cap P$. The congruence classes of θ_P are in bijection with the cylinder polyhedra included in P. □

Let us call $CFP - BLATT$ the range of the Belluce functor restricted to cylinder MV-algebras. Let us call $CPOLY$ the category whose objects are the lattices $\operatorname{Cpoly}(P)$, where P is a cylinder polyhedron, and where a morphism from $\operatorname{Cpoly}(P)$ to $\operatorname{Cpoly}(Q)$ is obtained by a definable map via the inverse image. In fact, note the following lemma:

Lemma 9.3. *The inverse image of a cylinder polyhedron under a definable map is a cylinder polyhedron.*

Proof. Let $d : P \to Q$ be a definable map between cylinder polyhedra $P \subseteq [0,1]^J$ and $Q \subseteq [0,1]^I$. Then $d(x) = (d_i(x))_{i \in I}$, where I is a possibly infinite set and each d_i is a McNaughton function.

Let $Q' \subseteq Q$ be a cylinder polyhedron. By Proposition 3.6 we have $Q' = Z(f)$ where $f : [0,1]^I \to [0,1]$ is a McNaughton function. Hence $d^{-1}(Q') =$

$Z(f \circ d)$. Since f depends only on a finite set F of coordinates, we have $f(x) = g(x|_F)$, where g is a McNaughton function on a finite dimensional cube. Hence $f \circ d = g \circ (d|_F)$. But a finite composition of McNaughton functions is still a McNaughton function. So $d^{-1}(Q')$ is the zeroset of a McNaughton function and, by Proposition 3.6, it is a cylinder polyhedron. □

Note that the previous lemma does not extend to direct images. For instance, the image of the definable map $d = (d_i)$ where $d_i(x) = 1$ for every $i \in I$ is the singleton of the point $(1, 1, 1, \dots) \in [0, 1]^I$, which is not a cylinder polyhedron when I is infinite (it is, however, an infinite dimensional polyhedron in the sense of [17]).

By the previous lemma, every definable map $\phi : Q \to P$, where Q, P are cylinder polyhedra, gives a lattice homomorphism $f_\phi : \mathrm{Cpoly}(P) \to \mathrm{Cpoly}(Q)$ by $f_\phi(P') = \phi^{-1}(P')$.

Theorem 9.4. *The categories $CFP - BLATT$ and $CPOLY$ are equivalent.*

Proof. There is an equivalence η' which is a natural generalization of the equivalence η of Theorem 8.3.

On objects, the equivalence η' is given by the canonical lattice isomorphism between $\beta(M_X|_P)$ and $CPoly(P)$ for every cylinder polyhedron P. On morphisms, if $h : M_X|_P \to M_Y|_Q$ is an MV-algebra morphism between cylinder MV-algebras, then h is the image in the Marra-Spada functor MS (see [41]) of a Z-map $\phi_h : Q \to P$, and the equivalence η' sends $\beta(h)$ to f_{ϕ_h}.

The operation η' is really an equivalence, and the proof is analogous to the one for η in Theorem 8.3. □

10. Cylinder polyhedra and diagrams of theories

Recall the definition of cylinder polyhedron. That is, given any set X, any finite subset $Y \subseteq X$ and any rational polyhedron $P \subseteq [0, 1]^Y$, we can consider the X-cylindrification of P:

$$C_X(P) = \{a \in [0, 1]^X \mid a|_Y \in P\}.$$

We know from [45] that rational polyhedra $P \subseteq [0, 1]^Y$, with Y finite, correspond to finite theories of Łukasiewicz logic in the set Y of variables. We associate to each cylinder polyhedron a diagram of finite theories as follows.

Consider X, Y, P as above. If $Y \subseteq F \subseteq G \subseteq X$, where F, G are finite, we have a natural, definable map $\gamma_{FG}(P) : C_G(P) \to C_F(P)$ given by restriction to F.

Moreover this map is an injective Z-map; we can consider the category CD of closed sets and definable maps, and in this category, the limit of the diagram $(C_F(P), \gamma_{FG}(P))$ is $C_X(P)$.

We recall the correspondence between finite theories and models in Łukasiewicz logic described in [45]. Finite theories depend always on the choice of a finite number of propositional variables (although this dependence is usually "tacitly understood", see e.g. [45], page 171). The natural theory associated to a cylinder polyhedron, however, depends on an infinite number of variables, in general. So we have an extension of the usual $Mod - Theor$ duality (see [45]) between rational polyhedra and finite theories to a duality between cylinder polyhedra and certain theories in an infinite number of variables.

Since $C_F(P)$ and $C_G(P)$ are rational polyhedra, the definable map $\gamma_{FG}(P)$ gives also a corresponding map between their theories. The theory of $C_G(P)$ may be called the G-expansion of the theory of $C_F(P)$, where a G-tuple t verifies the G-expansion of the theory of $C_F(P)$ if and only if $t|_F$ models the theory of $C_F(P)$.

In order to formalize the ideas above in a result, we call *limit theory* (the terminology is ours, not to be confused with other notions in the literature) a theory with a finite number of axioms but over infinitely many variables. Then we have:

Theorem 10.1. *The correspondence between rational polyhedra and finite theories in Łukasiewicz logic extends to a correspondence between cylinder polyhedra and limit theories. If $P \subseteq [0,1]^X$ is a cylinder polyhedron, then $P =$ Zeroset(f) where f is a McNaughton function from $[0,1]^X$ to $[0,1]$, and the limit theory of P is the theory in the language X axiomatized by single axiom $1 - f$.*

11. Conclusions

The results presented in this paper can be useful in many situations. For instance, in [29,32] we have two sheaf representations of MV-algebras. The former is via sheaves of MV-chains over the prime spectrum, the second is via sheaves of local MV-algebras over the maximal spectrum. Both approaches have their advantages: MV-chains are relatively simple MV-algebras but prime spectra are not completely understood, whereas local MV-algebras are relatively complicated MV-algebras and maximal spectra are well known to be compact Hausdorff spaces, see [28] (see also [41] for a categorial presentation). We believe that our results could help in understanding sheaf representations of MV-algebras and their associated Abelian unital ℓ-groups. In fact, now we know an intrinsic description of Belluce lattices of MV-algebras, and this, via duality, may shed light on the topological properties of prime spectra of MV-algebras: if we achieve an intrinsic topological description of these spectra, then the result of [29] can be more sharply stated as a representation result of MV-algebras as global sections of sheaves on a well defined kind of topological spaces.

For a possible logical characterization of Belluce lattices we have seen that Belluce lattices of free MV-algebras can be axiomatized in monadic second order logic. We do not know if there is such an axiomatization for arbitrary Belluce lattices. Also, we think it possible to describe the Stone duals of the lattices of free MV-algebras, which are topological spaces (hence the description should be topological, rather than logical as in the case of lattices).

We conjecture that one can take the results of [48] on endomorphisms of free MV-algebras and lift them to (closed) endomorphisms of the corresponding Belluce lattices.

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12. Appendices

Appendix A: Axioms for K_2

We propose the following list of axioms.

We did not check whether the list is redundant.

We postulate the existence of a set $\Pi \subseteq K_2$ and an atom O (the origin) with the following properties.

Intuitively, Π is a set of parallelograms and is used to define the relations of congruence and parallelism between segments.

Basics

Axiom A1. K_2 is a bounded lattice and contains the minimum and the maximum of L_2. The binary infimum of K_2 coincides with the binary infimum of L_2.

Note that the binary suprema in K_2 and L_2 differ in general. Intuitively the infimum in K_2 is intersection, the supremum is the convex hull, and the atoms are the rational points of the unit square.

In this appendix the capital letters $A, B, C, \ldots$ denote atoms unless otherwise specified.

Definition. A *segment* is the supremum of two atoms. A *triangle* is a supremum of three atoms. A *line* is a maximal segment.

For convenience we denote the segment (i.e. supremum) of A and B by AB rather than $A \vee B$. Similarly for triangles ABC, etc.

Our segments are not oriented, so we identify AB with BA.

Definition. A *zero segment* is simply an atom.

Axiom A2. For every segment there is an atom disjoint with it. Every nonzero segment is contained in a unique line. If $AB = CD$ then $\{A, B\} = \{C, D\}$.

Betweenness

Definition (betweenness). We let $A < B < C$ if $B \leq AC$ and $B \neq A, C$.

Axiom A3. For every two atoms A, B, The binary relation $A < X < Y < B$ is a strict total order between the atoms of AB different from A and B.

Congruence, parallelism and comparison of segments

Because we are in two dimensions, we can use parallelograms; then two segments are congruent when they are two opposite sides of a parallelogram, unless the two segments are contained in a same line (in which case we need two parallelograms to establish that they are congruent). So we give the following definitions:

Definition. AB is *strictly congruent* to CD if the supremum of A, B, C, D lies in Π and $AB \wedge CD = 0$.

Definition. The segment AB is *congruent* to the segment CD (written $AB = cCD$) if and only if there are E, F such that AB, EF and CD, EF are strictly congruent. (this is a kind of congruence relation inspired by the axiomatization of real vector spaces in [36]).

Definition. AB is *parallel* to CD if two nonzero subsegments of them are congruent or either AB or CD is zero.

Axiom A4. Congruence and parallelism between nonzero segments are equivalence relations. Any two segments contained in the same segment are parallel. All zero segments are congruent to each other.

Definition (segment comparison). We let $AB > CD$ (as segments) if and only if AB and CD are parallel and there is E such that $A < E < B$ and $AE = cCD$. This notation should not be confused with the order of the lattice K_2.

Axiom A5. Any two parallel segments are either comparable or congruent. Zero segments are smaller than nonzero segments.

Axiom A6. parallel segment comparison is irreflexive and transitive.

Parallel sum of segments

Definition (parallel sum). We write $AB + pCD = A'C'$ if AB and CD are parallel, and there is an atom B' such that $A'B' = cAB$, $B'C' = cCD$ and $A' < B' < C'$.

Axiom A7. The parallel sum, when it exists, is unique up to congruence. The parallel segment sum, as a partial operation on parallel segments, is commutative and associative, and preserves the relations of comparison and congruence. Zero segments are neutral for parallel sum.

Partiality of parallel sum holds due to truncation problems (for instance we cannot make the parallel sum of two lines because this sum should be longer than both, whereas lines have maximal length). However, note that if $A < C < B$ then $AB = AC + pCB$. More generally, by induction on n, we can prove that if $A < C_1 < \cdots < C_n < B$, then $AB = AC_1 + pC_1C_2 + p \cdots + pC_nB$.

Division of segments

In this subsection we want to put sufficiently strong axioms so to have the divisibility of a segment into any finite number of congruent parts. Again we use the fact that we are in two dimensions, and we can state an axiom analogous to Thales Theorem of plane euclidean geometry.

Axiom A8. Let AB, CD be two parallel segments. Then $AB \wedge CD$ is zero, or an atom, or a segment.

Axiom A9. Let AB, CD be non parallel segments. Then $AB \wedge CD$ is zero or an atom.

Axiom A10 (Playfair axiom, inspired by the parallel axiom of elementary geometry in the form of Playfair). Let TUV be a triangle and let $A \in VU$. There is a segment AA' parallel to TV and such that $A' \in TU$.

Axiom A11 (Thales axiom, inspired by the Thales Theorem of elementary geometry). Let TUV be a triangle and $V \leq A \leq B \leq C \leq D \leq U$. Consider the segments AA', BB', CC', DD' parallel to TV and ending in TU. Then $T \leq A' \leq B' \leq C' \leq D' \leq U$. If $AB = cCD$ then $A'B' = cC'D'$. If $AB \wedge CD = 0$ then $A'B' \wedge C'D' = 0$. If $AB \wedge CD$ is an atom then $A'B' \wedge C'D'$ is an atom.

Axiom A12. Every segment AB can be divided in two congruent parts, that is, there is $C \leq AB$ such that $AC = cCB$.

The axiom implies divisibility in 2^n congruent parts for every integer $n \geq 2$, by induction.

More generally, we can prove:

Lemma 12.1. *We can divide any (nonzero) segment AB in n congruent parts for every integer $n \geq 3$.*

Proof. Consider a triangle ABC, and subdivide BC in 2^n congruent segments; let $B = P_1 < P_2 \cdots < P_{2^n+1} = C$ the extremes of these segments. Then one can draw n lines parallel to P_nA and passing through $P_1, \ldots, P_n$ respectively; the intersection of these n lines with AB give n congruent segments by Thales axiom. □

The rationality axiom

In this subsection we want to establish that the ratio between two parallel segments (where the second is nonzero) is always a rational number.

Definition. AB is a *submultiple* of CD if there is a finite set F of points of CD such that every segment with extremes in two consecutive points of F is congruent to AB.

Note that the definition is given in monadic second order logic, because the set F is finite if and only if every nonempty subset of F has a minimum and a maximum in the betweenness order of CD.

Axiom A13 (rationality axiom). Any two parallel segments have a common submultiple.

By the previous axioms, every segment is a rational multiple of any other segment parallel to it, and the ratio of two parallel segments (where the second is nonzero) is always well defined. We denote the ratio between AB and CD by $AB : CD$.

Assigning coordinates to atoms and coordinate lines

Intuitively, the elements of K_2 live in the unit square of the (rational) Cartesian plane xy. In particular, atoms should have two rational coordinates between 0 and 1, and lines parallel to an axis x or y should have a rational coordinate.

Definition. An atom is *extreme* if it is not interior to any segment.

Axiom A14. There are exactly four extreme atoms in K_2. One of them is O.

Definition. Two extreme atoms A, B are *adjacent* if no atom of the segment AB is internal to a triangle.

Axiom A15. There are two extremes O_x, O_y adjacent to O; we call OO_x the x axis and OO_y the y axis. The axes are lines.

Intuitively, the previous two lines are the axes x and y of the Cartesian plane.

Axiom A16. Two lines, each of them parallel to some different coordinate line, meet in a single atom.

Axiom A17. For every coordinate line l and every atom A outside l there is a unique line l' parallel to l and passing through A.

Definition. The x-th *projection* of an atom A, written A_x, is the intersection of the x axis with the line parallel to the y axis and passing through A, which is an atom and is unique by the previous axioms. Likewise we define the y-th projection.

Definition. The *x-coordinate* of an atom A is the ratio between the segment OA_x and the x axis. Likewise we define the y coordinate.

Atoms and coordinates

Lemma 12.2. *The coordinate assignment gives a bijection between atoms and pairs of rational numbers between* 0 *and* 1*.*

Proof. Any atom has a pair of coordinates associated to its projections on the axes x and y. Conversely, given two rational numbers $p, q \in [0, 1]$, a point with coordinates (p, q) is obtained (as usual) by intersection of two lines parallel to the x and y axis, the first line passing through the point of the x axis with coordinate p, the second line through the point of the y axis with coordinate q. □

Lemma 12.3.
- *Finite sets of atoms contained in any segment AB are definable in monadic second order logic in K_2.*
- *Finite sets of atoms are definable in monadic second order logic in K_2.*

Proof. The first point holds because atoms in a segment AB are totally ordered by the relation $A < x < y < B$.

The second point holds because a set F of atoms is finite if and only if the projections of the elements of F range over a finite set. □

Segments and coordinates

Definition. Two segments AB, CD are *equioriented* if $O < A_x < B_x$ if and only if $O < C_x < D_x$, and the same holds for y.

Axiom A18. Let AB, CD be parallel, equioriented segments. Then AB is a multiple of CD if and only if and there is a positive integer n such that $A_xB_x = nC_xD_x$ and $A_yB_y = nC_yD_y$.

Note that the previous axiom is monadically expressible. In fact, in order to express the existence of n above in monadic second order logic, we can say that there are finite sets F_1, F_2 of atoms in A_xB_x and A_yB_y and a bijection γ between F_1 and F_2, such that the subsegments between consecutive points are congruent to C_xD_x and C_yD_y respectively. The finiteness of F_1, F_2 can be imposed by saying that every nonempty subset of them has a maximum and a minimum in the betweenness order, and the bijection γ can be realized with a set of segments with one extreme in F_1 and the other in F_2.

Axiom A19. $B \leq AC$ (in the order of K_2) if and only if $A_xC_x \geq A_xB_x$, $A_yC_y \geq A_yB_y$, and there is D such that AB, AD, AC are equioriented, and AB and AC are multiples of AD.

By the previous two axioms, the relation $B \leq AC$ depends only on the coordinates of A, B, C.

Reduction to atoms and segments

Axiom A20. Every element of K_2 is the supremum of a finite set of atoms.

Axiom A21 (from convex polygons to segments). Let F, F' be finite sets of atoms. Then *sup* $F \leq$ *sup* F' if and only if every set $T \subseteq K_2$ containing F' and closed under segment (that is if $A, B \in T$, then every atom in AB is in T) contains F.

The previous axiom in a sense reduces the calculation of the convex hull of n points to an iterated calculation of the segment between two points. Note that the axiom is expressible in monadic second order logic, and indeed, it seems crucial for the description of K_2 in monadic second order logic.

The final theorem for K_2

Theorem 12.4. *Let K' be a lattice satisfying the axioms of this appendix. By Lemma 12.2 there is a function β_2 which maps each atom of K' to the unique atom of K_2 with the same pair of coordinates. Let us extend β_2 to K' by letting $\beta_2(sup\ F) = sup\ \beta_2(F)$, where F is any finite set of atoms. Then β_2 is a well defined lattice isomorphism from K' to K_2.*

Proof. Recall that the relation $B \leq AC$ between three atoms depends only on the coordinates of A, B, C. Hence, by induction on n, also the fact that an atom A is below the supremum of n atoms $B_1, \ldots, B_n$ depends only on the coordinates of $A, B_1, \ldots, B_n$. And by a further induction on m, the fact that the supremum of $A_1, \ldots, A_m$ is below the supremum of $B_1, \ldots, B_n$ depends only on the coordinates of $A_1, \ldots, A_m, B_1, \ldots, B_n$. Since β_2 respects the coordinates of the atoms, we have $sup\ F \leq sup\ G$ if and only if $sup\ \beta_2(F) \leq sup\ \beta_2(G)$, and $sup\ F \leq sup\ G$ if and only if $\beta_2(sup\ F) \leq \beta_2(sup\ G)$. So β_2 is monotonic and injective. Moreover β_2 is surjective on atoms, and since every element of K_2 is a finite supremum of atoms, β_2 is surjective on K_2. So β_2 is bijective and its inverse is monotonic, so β_2 is an isomorphism. □

Appendix B: Axioms for K_X

We note that the axioms for K_2 can be modified so to axiomatize every single K_n for every integer $n > 2$. We avoid details for simplicity. So we pass directly to the infinite dimensional case, and we axiomatize the lattice K_X when X is infinite.

We propose the following list of axioms.

We did not check whether the list is redundant.

The main difference with K_2 is that the lattice K_2 is atomic, whereas K_X is atomless. However, in K_X we have a kind of substitute for atoms, which we call pseudoatoms, which can be appropriately axiomatized.

Basics

We postulate the existence of sets $X, Par, \Pi \subseteq K_X \subseteq L_X$ satisfying the following axioms. Intuitively:

- X is the set of hyperplanes of the form $\{f \in [0,1]^X \mid f(i) = 0\}$ for some $i \in X$;
- Par is the set of hyperplanes (parallel to elements of X) of the form $\{f \in [0,1]^X \mid f(i) = a\}$ for some $i \in X$ and for some rational a between 0 and 1;
- a *pseudoatom* is a finite nontrivial intersection of elements of Par, so it has the form

$$A = \{f \in [0,1]^X \mid f(i_1) = a_1, \ldots, f(i_n) = a_n\}$$

  where $\{i_1, \ldots, i_n\} \subseteq X$ has size $n \geq 2$;
- Π is a set of parallelograms whose vertices are pseudoatoms, and these parallelograms are used, as in "Appendix A", to define congruence and parallelism of segments.

First of all we postulate:

Axiom B1. K_X with the order induced by L_X is a bounded lattice and contains the minimum and the maximum of L_X. The infimum of K_X coincides with the infimum of L_X.

Note that the suprema in K_X and L_X differ in general.

On Par and pseudoatoms

Axiom B2. $X \subseteq Par$. Zero (the minimum of K_X) does not belong to Par and is not a pseudoatom. the infimum of two elements of Par is zero or a pseudoatom. the infimum of two pseudoatoms is zero or a pseudoatom.

Axiom B3. Every set containing the binary infima of elements of Par and closed under binary infimum contains all pseudoatoms.

The previous two axioms imply that pseudoatoms coincide with finite infima of at least two elements of Par. Note that the second axiom is expressed in monadic second order logic.

Axiom B4. For every pseudoatom A there is a unique subset $T(A)$ of Par such that A is the infimum of $T(A)$.

Note that the set $T(A)$ is finite and has size at least 2.

Definition. We say that $H, H' \in Par$ are *parallel* (as hyperplanes) if they are equal or disjoint.

Axiom B5. Parallelism in Par is an equivalence relation. Every element of Par is parallel to a unique element of X.

Axiom B6. Suppose that A, B are pseudoatoms and no element of $T(A)$ is disjoint from any element of $T(B)$. Then $A \wedge B$ is a pseudoatom.

The axiom implies that every finite subset of Par containing no pair of disjoint elements has an infimum which is a pseudoatom.

Definition. $T_X(A)$ is the set of elements of X parallel to some element of $T(A)$.

Definition. Two pseudoatoms A, B are called *compatible* if $T_X(A) = T_X(B)$.

Now we can replace with pseudoatoms (taken in a fixed compatibility class) the atoms occurring in finite dimensional lattices $\mathrm{Poly}([0,1]^n)$, and replace the axioms in dimension 2 of the previous appendix with axioms in codimension n where, intuitively, the "atoms in codimension n" are the pseudoatoms A such that $T(A)$ has n elements.

The segment of two compatible pseudoatoms A, B is denoted by AB.

Axiom B7. Let A, B, C, D be compatible pseudoatoms. If $AB = CD$ then $\{A, B\} = \{C, D\}$.

Betweenness

Same as the homonymous subsection of "Appendix A", up to replacing atoms with compatible pseudoatoms.

Congruence, parallelism and comparison of segments

Same as the homonymous subsection of "Appendix A", up to replacing atoms with compatible pseudoatoms.

Parallel sum of segments

Same as the homonymous subsection of "Appendix A", up to replacing atoms with compatible pseudoatoms.

Division of segments

Same as the homonymous subsection of "Appendix A", up to replacing atoms with compatible pseudoatoms.

The rationality axiom

Same as the homonymous subsection of "Appendix A", up to replacing atoms with compatible pseudoatoms.

In particular, we can define the ratio of two parallel segments AB, CD, where A, B, C, D are compatible pseudoatoms, assuming $C \neq D$.

Assigning coordinates to elements of $\boldsymbol{Par}$ and pseudoatoms

The assignment of coordinates changes and is more complicate with respect to "Appendix A". Parallel sum, congruence, comparison and parallelism are defined for segments with extremes pseudoatoms, but not for "segments with extremes in Par". The problem is that we do not have "big enough" parallelograms to define these notions. So we choose an indirect way of defining coordinates of elements of Par.

Definition. We define an element $H \in Par$ *extremal* if it does not disconnect the space, that is, for every $U, V \in K_X$ disjoint from H, UV is also disjoint from H.

Axiom B8. Every element of X is extremal.

Axiom B9. For every $H \in X$ there is only another extremal $H' \in Par$ parallel to H, which will be called an anti-coordinate hyperplane.

Intuitively, $H = \{f \in [0,1]^X \mid f(i) = 0\}$ for some $i \in X$, and $H' = \{f \in [0,1]^X \mid f(i) = 1\}$.

Axiom B10. For every $H \in Par$ and A pseudoatom disjoint from H there is a unique $H' \in Par$ parallel to H and containing A.

Definition. For every pseudoatom A, let O_A (the *origin relative to* A) be the intersection of $T_X(A)$. Note that O_A is a pseudoatom compatible with A.

Axiom B11. Let A be a pseudoatom and $H \in T_X(A)$. The AH-axis is a segment O_AB, where B is the infimum of $T'(A)$, and $T'(A)$ is $T(A)$ where the element parallel to H is replaced by the unique extremal hyperplane disjoint from H. Note that B is a pseudoatom compatible with A.

Definition. If $H \in T_X(A)$, $A \in H'$ and H' is parallel to H, we define A_H the intersection of H' with the AH-axis. A_H is also called the H-th *projection* of A.

We would like to define parallel sum, congruence, comparison and parallelism between "segments with extremes in the elements of Par parallel to the same $H \in X$", so to assign "coordinates" to elements of Par. The problem is that segments with hyperplane extremes are "too big". So we follow an indirect way, by considering the intersection of the elements of Par with coordinate axes relative to any pseudoatom A. To do it, since we have many possibilities for A, we add the following independence axiom:

Axiom B12 (independence of parallel sum of hyperplane segments). Let $H_i \in Par, 1 \leq i \leq 6$ parallel to the same $H \in X$. Let A be a pseudoatom such that $H \in T_X(A)$. Let $A_i = H_i \wedge O_A B$. If $A_1 A_2 = A_3 A_4 + p A_5 A_6$, then the same holds for any other A' such that $H \in T_X(A')$.

The same independence from A holds for congruence of coordinate segments associated to elements of Par:

Axiom B13. (independence of comparison of hyperplane segments). Let $H_i \in Par, 1 \leq i \leq 4$ parallel to the same $H \in X$. Let A such that $H \in T_X(A)$. Let $A_i = H_i \wedge O_A B$. If $A_1 A_2 > A_3 A_4$, then the same holds for any other A' such that $H \in T_X(A')$.

We add also an axiom on independence of betweenness:

Axiom B14. Let $H_i \in Par, 1 \leq i \leq 3$ parallel to the same $H \in X$. Let A such that $H \in T_X(A)$. Let $A_i = H_i \wedge O_A B$. If $A_1 > A_2 > A_3$, then the same holds for any other A' such that $H \in T_X(A')$.

Definition. If A is any pseudoatom and $H \in T_X(A)$, the H-*coordinate* of A is the ratio between the segments $O_A A_H$ and $O_A B$, and the *coordinate* of $H' \in Par$ parallel to H is the H-th coordinate of the point $H' \wedge O_A B$.

By the previous axioms, the coordinate of any element $H' \in Par$ is a unique, well defined rational number between 0 and 1. For every $H \in X$, every rational number between 0 and 1 is the coordinate of some element $H' \in Par$ parallel to H.

Pseudoatoms and coordinates

Lemma 12.5. *The coordinate assignment gives a bijective function between pseudoatoms A and functions from finite subsets of X to the set of the rational numbers between* 0 *and* 1.

Proof. Any pseudoatom A has a finite set of coordinates, one for each $H \in T_X(A)$. Conversely, let f be a function from a finite subset G of X to the rational numbers between 0 and 1. For every $g \in G$ there is a hyperplane H_g parallel to X with coordinate $f(g)$. Then the intersection of all H_g is a pseudoatom whose g-th coordinate is $f(g)$. $\square$

Definability of finiteness

Lemma 12.6.
- *Finite sets of compatible pseudoatoms contained in a segment AB are definable in monadic second order logic in K_X.*
- *Finite sets of compatible pseudoatoms are definable in monadic second order logic in K_X.*

Proof. The first point holds because compatible pseudoatoms in a segment AB are totally ordered by the relation $A < x < y < B$.

The second point holds because a set of compatible pseudoatoms is finite if and only if the projections of its elements range over a finite set.

□

Segments and coordinates

This subsection is analogous to the homonymous subsection of "Appendix A". The difference is that a pseudoatom can have any finite number of coordinates rather than two.

As usual, $A, B, C, D, \ldots$ are assumed to be compatible pseudoatoms.

Definition. Two segments AB, CD are *equioriented* if $O < A_H < B_H$ is equivalent to $O < C_H < D_H$ for every $H \in T_X(A)$.

Axiom B15. Let AB, CD be equioriented segments. Then AB is a multiple of CD if and only if and there is a positive integer n such that $A_H B_H = nC_H D_H$ for every $H \in T_X(A)$.

In order to express the existence of n as above in monadic second order logic, we can say that there is a set F (necessarily finite) which intersects every segment $A_H B_H$ (for $H \in T_X(A)$) in finitely many points, the number of these points is independent from H (via bijections given by suitable sets of segments with extremes in F), and every two consecutive members of F on each axis H span a segment congruent to $C_H D_H$.

Axiom B16. $B \leq AC$ if and only if $A_H C_H \geq A_H B_H$ for every $H \in T_X(A)$ and there is D such that AB, AD, AC are equioriented and AC, AB are multiples of AD.

By the previous two axioms, the relation $B \leq AC$ among compatible pseudoatoms depends only on the coordinates of A, B, C.

The dimension reduction axiom

The following axiom in a sense reduces the dimension (and increases the codimension) of pseudoatoms:

Axiom B17. Let A be a pseudoatom. Then $T_X(A)$ is a proper subset of X. Let $H \in X \backslash T_X(A)$ and let H' be the extremal element of Par parallel to H and different from H. Then A is the supremum of $A \wedge H$ and $A \wedge H'$.

From convex bodies to segments

Axiom B18. For every nonzero element e of K_X, there is a pseudoatom A such that e is the supremum of a finite set of pseudoatoms compatible with A.

By the previous axioms we have:

Lemma 12.7. *Finite subsets of K_X are definable in monadic second order logic.*

Proof. By the previous axiom and the dimension reduction axiom, a subset F of K_X is finite if and only if there is a finite set G of compatible pseudoatoms such that every element of F is supremum of a subset of G.

□

Axiom B19 (from convex bodies to segments). Let F, F' be finite sets of compatible pseudoatoms. Then sup $F \leq sup\ F'$ if and only if every set T of pseudoatoms containing F' and closed under compatible segment (that is if $A, B \in T$ are compatible with the elements of F', then every element below AB compatible with A is in T) contains F.

The previous axiom reduces the calculation of the convex hull of a compatible finite set of pseudoatoms to a finite iteration of the betweenness relation $A \leq BC$. Note that the axiom is expressible in monadic second order logic.

The final theorem for K_X

Theorem 12.8. *Let K' be a lattice satisfying the axioms of this appendix, where X is a set of given infinite cardinality λ. By Lemma 12.5 there is a function β_λ which maps each pseudoatom in K' to the unique pseudoatom of K_X with the same coordinates. Let us extend β_λ to K' by letting $\beta_\lambda(sup\ F) = sup\ \beta_\lambda(F)$, where F is any finite set of compatible pseudoatoms. Then β_X is a well defined isomorphism from K' to K_X.*

Proof. Recall that the relation $B \leq AC$ between three compatible pseudoatoms depends only on the coordinates of A, B, C. Hence, by induction on n, also the fact that a pseudoatom A is below the supremum of n pseudoatoms $B_1, \ldots, B_n$ depends only on the coordinates of $A, B_1, \ldots, B_n$. And by a further induction on m, the fact that the supremum of $A_1, \ldots, A_m$ is below the supremum of $B_1, \ldots, B_n$ depends only on the coordinates of $A_1, \ldots, A_m, B_1, \ldots, B_n$, assuming all of them are compatible. If A_i and B_j are not compatible, we can use the dimension reduction axiom and we can express each of them as a supremum of a single finite subset E of compatible pseudoatoms, and the coordinates of the elements of E depend only on the coordinates of the A_i and B_j, so we can assume A_i and B_j compatible.

Since β_λ respects the coordinates of the pseudoatoms, for every two finite sets F, G of compatible pseudoatoms we have $sup\ F \leq sup\ G$ if and only if $\beta_\lambda(sup\ F) \leq \beta_\lambda(sup\ G)$. So β_λ is monotonic and injective. Moreover β_λ is surjective on pseudoatoms, and since every element of K_λ is a finite supremum of pseudoatoms, β_λ is surjective on K_λ. So β_λ is bijective and its inverse is monotonic; summing up, β_λ is an isomorphism. □

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