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On semiconic idempotent commutative residuated lattices

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Abstract. In this paper, we study semiconic idempotent commutative residuated lattices. An algebra of this kind is a semiconic generalized Sugihara monoid if it is generated by the lower bounds of the monoid identity. We establish a category equivalence between semiconic generalized Sugihara monoids and Brouwerian algebras with a strong nucleus. As an application, we show that central semiconic generalized Sugihara monoids are strongly amalgamable.

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1. Introduction

Idempotent commutative residuated lattices form a useful tool both in algebra and logic, see e.g., [5] for details. Among them, both integral ones and non-integral ones play important roles, since they include several important algebraic counterparts of substructural logics, e.g., Brouwerian algebras, i.e. the algebras of positive intuitionistic logic, and positive Sugihara monoids (see [17]); these algebras model the positive fragment of the system **R**-mingle. In general, integral residuated lattices are better understood than their non-integral counterparts, so the establishment of a *category equivalence* between a non-integral and an integral class can increase our comprehension of the former.

A commutative residuated lattice is *integral* if each of its elements lies below the monoid identity; is *conic* if each of its elements lies above or below

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the monoid identity; is *semilinear* if it is a subdirect product of totally ordered commutative residuated lattices; is *semiconic* if it is a subdirect product of conic commutative residuated lattices; is *involutive* if it possesses a compatible involution $\neg$ (see [7]); is *idempotent* if its monoid operation is idempotent. A semilinear idempotent commutative residuated lattice is called a *Sugihara monoid* if it is involutive; is called a *relative Stone algebra* if it is integral; is called a *generalized Sugihara monoid* if it is generated by the lower bounds of the monoid identity. A Sugihara monoid is called *odd* if its monoid identity e satisfies $\neg e = e$. A semiconic idempotent commutative residuated lattice is called a *Brouwerian algebra* if it is integral; is called a *semiconic generalized Sugihara monoid* if it is generated by the lower bounds of the monoid identity. In [7], the authors proved that the variety of odd Sugihara monoids and the variety of relative Stone algebras are categorically equivalent. However, a semiconic idempotent involutive commutative residuated lattice is a Sugihara monoid (see [12]). In [8], the authors established a category equivalence between generalized Sugihara monoids and relative Stone algebras with a nucleus. In this paper, we will show that semiconic generalized Sugihara monoids and Brouwerian algebras with a strong nucleus are categorically equivalent, which generalizes the main result of [8].

We proceed as follows: in Section 2, we recall some definitions and basic facts needed in later proofs. In Section 3, we discuss some properties of semiconic idempotent commutative residuated lattices. In Section 4, we study strong nuclear Brouwerian algebras and obtain some properties of such algebras. Section 5 is devoted to establishing a category equivalence between semiconic generalized Sugihara monoids and strong nuclear Brouwerian algebras, which generalizes [8, Theorem 8.7]. In Section 6, we use this category equivalence to prove that the variety of central semiconic generalized Sugihara monoids is strongly amalgamable, which generalizes [8, Theorem 13.3].

2. Preliminaries

In this section, we will recall some basic definitions and facts on commutative residuated lattices.

A *po-monoid* is defined as a monoid $(M, \cdot, e)$ which is also a poset $(M, \leq)$, and in which $\leq$ is compatible with $\cdot$, in the sense that $(\forall a, b, c \in M)\ a \leq b \Longrightarrow ca \leq cb, ac \leq bc$. Here and subsequently, xy abbreviates $x \cdot y$. By a *lattice-ordered monoid*, we mean a po-monoid $(M, \cdot, e, \leq)$ in which the poset $(M, \leq)$ is a lattice. Moreover, a lattice-ordered monoid $(M, \cdot, e, \leq)$ is called *idempotent* if for all $a \in M$, $aa = a$; is called *commutative* if the monoid reduct $(M, \cdot, e)$ is a commutative monoid; is called *conic*, if for all $a \in M$, $a \leq e$ or $a \geq e$. For detailed information on lattice-ordered monoids, the reader is referred to reference [2].

We need the following lemma.

Lemma 2.1 [20]. *Let $(M, \cdot, e, \leq)$ be an idempotent lattice-ordered monoid with identity e, and $a, b \in M$.*

(1) $a \wedge b \leq ab \leq a \vee b$.
(2) *If* $a, b \geq e$, *then* $ab = a \vee b$.
(3) *If* $a, b \leq e$, *then* $ab = a \wedge b$.
(4) *If* $a \leq e \leq ab$, *then* $ab = b$.
(5) *If* $ab \leq e \leq a$, *then* $ab = b$.

Now, let $(P, \leq)$ be a poset. A (binary) commutative operation $\cdot$ is called *residuated* if there exists a (binary) operation $\rightarrow$ on P such that

$$(\forall x, y, z \in P)\ xy \leq z \Longleftrightarrow y \leq x \rightarrow z.$$

In this case, the operation $\rightarrow$ is called a *residual* of the operation $\cdot$. It is well known that a commutative operation $\cdot$ on the poset $(P, \leq)$ is residuated if and only if $\leq$ is compatible with $\cdot$ and for all $a, b \in P$, $\{p \in P : ap \leq b\}$ contains a greatest element (denoted by $a \rightarrow b$). A *commutative residuated lattice* is defined as an algebra $\mathbf{L} = (L, \wedge, \vee, \cdot, \rightarrow, e)$ satisfying the following conditions:

(RL1) $(L, \wedge, \vee)$ is a lattice;
(RL2) $(L, \cdot, e)$ is a commutative monoid with identity e; and
(RL3) the operation $\rightarrow$ is a residual of the operation $\cdot$.

In this case, we call the lattice $(L, \wedge, \vee)$ and the monoid $(L, \cdot, e)$ the *lattice reduct* and the *monoid reduct* of $\mathbf{L}$, respectively, and we denote the lattice order by $\leq$. Commutative residuated lattices are exactly commutative lattice-ordered monoids $\mathbf{L}$ such that for all $a, b \in L$, $\{c \in L : ac \leq b\}$ contains a greatest element. Sometimes, commutative residuated lattices are also called *commutative residuated lattice-ordered monoids* and abbreviated by CRLs.

A CRL $\mathbf{L}$ is said to be *idempotent* if for all $a \in L$, $aa = a$; is said to be *integral* if for all $a \in L$, $a \leq e$; is said to be *totally ordered* if the lattice reduct is a chain, i.e., for all $a, b \in L$, $a \leq b$ or $a \geq b$; is said to be *semilinear* if it is a subdirect product of totally ordered CRLs (see [18]); is said to be *conic* if for all $a \in L$, $a \leq e$ or $a \geq e$ (see [4,6,12]). As in [12], a CRL $\mathbf{L}$ is said to be *semiconic* if it is a subdirect product of conic CRLs and the class of all semiconic CRLs is a variety. The variety of semiconic CRLs is axiomatized, relative to CRLs, by the identity $(x \wedge e) \vee ((x \rightarrow e) \wedge e) = e$. More details on semiconic residuated lattices can be found in [3,11,12].

The following well-known properties of CRLs will be needed (see [5,13]).

Lemma 2.2. *Let* $\mathbf{L} = (L, \wedge, \vee, \cdot, \rightarrow, e)$ *be a* CRL *and* $x, y, z \in L$.

(1) $x(y \vee z) = xy \vee xz$.
(2) $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$.
(3) $(y \vee z) \rightarrow x = (y \rightarrow x) \wedge (z \rightarrow x)$.
(4) $y(y \rightarrow x) \leq x$.
(5) $e \leq x \rightarrow x$.
(6) $((x \rightarrow y) \rightarrow y) \rightarrow y = x \rightarrow y$.
(7) $x \rightarrow (y \rightarrow z) = (xy) \rightarrow z$

From now on, we denote $a \rightarrow e$ and $(a \rightarrow e) \rightarrow e$ by a^* and a^{**}, respectively. Next, we shall list some known results on conic idempotent CRLs used in the sequel.

Lemma 2.3 [12]. *Let $\boldsymbol{L}$ be a conic idempotent* CRL*, and $a, b \in L$.*

(1) *If a and b are incomparable, then $a^* = b^*$.*
(2) *The elements a and b^* are comparable.*
(3) *$a \nleq b$ if and only if $a \to b < e$.*
(4) *If $a \leq e$, then $a^* = a \to a$.*
(5) *$\{a^* : a \in L\}$ is a chain in $(L, \wedge, \vee)$.*
(6) *If $a \leq b$, then $ab = a$ or $ab{=}b$.*

3. Some properties

Let $\mathbf{L} = (L, \wedge, \vee, \cdot, \to, e)$ be a conic idempotent CRL. For $a, b \in L$, $a \parallel b$ means that a and b are incomparable under $\leq$.

Proposition 3.1. *Let $\boldsymbol{L}$ be a conic idempotent* CRL*. The following statements are true for $a, b \in L$:*

(1) *If $a < e$, then $a^* \geq e$ and $aa^* = a$.*
(2) *If $a > e$, then $a^* < e$ and $aa^* = a^*$.*
(3) *If $a, b < e$ and $a \parallel b$, then $a^* = b^* = (a \wedge b)^* = (a \vee b)^*$.*
(4) *If $a \in L$, then $(a \wedge e)^* \to (a \wedge e) = a \wedge e$.*
(5) *If $a, a' \in L$ such that $a \vee a' = e$ and $a'^{**} = a'$, then $a = e$ or $a' = e$.*

Proof. (1) Let $a \in L$ such that $a < e$. Then $ae = a < e$ and so $a < e \leq a \to e = a^*$. Since $aa^* = a(a \to e) \leq e$ by Lemma 2.2(4), $aa^* = a$ by Lemma 2.1(5).

(2) Let $a \in L$ such that $a > e$. Then by Lemma 2.3(3), $a^* = a \to e < e$. Since $aa^* = a(a \to e) \leq e$ by Lemma 2.2(4), $aa^* = a^*$ by Lemma 2.1(5).

(3) Let $a, b \in L$ such that $a, b < e$ and $a \parallel b$. Then by Lemma 2.3(1), $a^* = b^*$, and so by Lemma 2.2(3), $(a \vee b)^* = (a \vee b) \to e = (a \to e) \wedge (b \to e) = a^* \wedge b^* = a^* = b^*$.

Next, we shall prove that $(a \wedge b)^* = a^*$. As $a \wedge b \leq a$, we have $a \to e \leq (a \wedge b) \to e$, i.e., $a^* \leq (a \wedge b)^*$. Also, as $a, b < e$, we have $a \wedge b = ab$, so by Lemma 2.2(7), $(a \wedge b)^* = (ab) \to e = a \to b^*$, so it suffices to show that $a \to b^* \leq a \to e$, or equivalently, that $a(a \to b^*) \leq e$. Now $a(a \to b^*) = a^2(a \to b^*) \leq ab^* = aa^* \leq e$, as required.

(4) Let $a \in L$. If $a \geq e$, then $(a \wedge e)^* \to (a \wedge e) = e^* \to e = e \to e = e = a \wedge e$. If $a < e$, then by (1), $a^* \geq e$ and $aa^* = a$, so $a \leq a^* \to a$. On the other hand, since $a^* \geq e > a$, $a^* \to a < e$ by Lemma 2.3(3) and so $a^*(a^* \to a) \in \{a^*, a^* \to a\}$ by Lemma 2.3(6), which together with $a^*(a^* \to a) \leq a < e$, derives that $a^* \to a = a^*(a^* \to a) \leq a$. Thus $(a \wedge e)^* \to (a \wedge e) = a^* \to a = a = a \wedge e$.

(5) By assumption, $a' = a'^{**} \nparallel a$ by Lemma 2.3(2). Then $a' \leq a$ or $a \leq a'$. Because $a \vee a' = e$, $a = a \vee a' = e$ or $a' = a \vee a' = e$. □

Proposition 3.2 [3]. *Let $\boldsymbol{L}$ be a conic idempotent* CRL*, and let $a, b \in L$ such that $a \leq e$. If $b < a$ or $a \parallel b$, then $a \to b = b$ or $a \to b \parallel a$.*

Let $\mathbf{L}$ be a conic idempotent CRL and $A \subseteq L$. The subalgebra $Sg^{\mathbf{L}}A$ of $\mathbf{L}$ generated by A is defined as the intersection of all subalgebras $\mathbf{B}$ of $\mathbf{L}$ such that $A \subseteq B$, but it is well known that $Sg^{\mathbf{L}}A$ is also the set of all $t^{\mathbf{L}}(a_1, \ldots, a_n)$

such that $n \in \omega$ and t is an n-ary term in the signature $\cdot, \to, \wedge, \vee, e$ and $a_1, \ldots, a_n \in A$. Let $L^+ = \{a \in L : a \geq e\}$ and $L^- = \{b \in L : b \leq e\}$.

Theorem 3.3. *Let $\mathbf{L}$ be a conic idempotent* CRL. *If $a \in Sg^{\mathbf{L}}L^-$ such that $a \geq e$, then $a = b^*$ for some $b \in L^-$.*

Proof. In $\mathbf{L}$, the product of two comparable factors is always one of the factors, by Lemma 2.3 (6). Also, $\{x^* : x \in L\}$ is a chain , by Lemma 2.3(5), so $x^*y^*, x^* \wedge y^*, x^* \vee y^* \in \{x^*, y^*\}$ for all $x, y \in L$. (These facts are used repeatedly without comment below.)

Now $a = t^{\mathbf{L}}(a_1, \ldots, a_n)$ for suitable $n \in \omega$, $\vec{a} = a_1, \ldots, a_n \in L^-$ and a term t in the signature $\cdot, \to, \wedge, \vee, e$. We prove the desired conclusion by induction on the complexity of t.

If t is e or a variable, then since $a_i \leq e \leq a$ for all i, we have $a = e = e^*$.

Now assume that t is neither e nor a variable, and that, whenever $e \leq s^{\mathbf{L}}(\vec{c})$ for a less complex term s than t and some $\vec{c} \in L^-$, then $s^{\mathbf{L}}(\vec{c}) = b^*$ for some $b \in L^-$. This is the induction hypothesis (IH). We may assume that $e < a$, again because $e = e^*$.

Now t is $r \,\Box\, s$ for some $\Box \in \{\cdot, \to, \wedge, \vee\}$ and some terms r, s in $\cdot, \to, \wedge, \vee, e$, both of which are therefore less complex than t.

Suppose first that $\Box$ is $\cdot$, so $a = r^{\mathbf{L}}(\vec{a})s^{\mathbf{L}}(\vec{a})$. As $a > e$ and $\mathbf{L}$ is conic, at least one of $r^{\mathbf{L}}(\vec{a})$, $s^{\mathbf{L}}(\vec{a})$ belongs to L^+. If both belong to L^+, then by the IH, $a = b^*c^* \in \{b^*, c^*\}$ for some $b, c \in L^-$. In the opposite case, we may assume (by symmetry) that $r^{\mathbf{L}}(\vec{a}) < e \leq s^{\mathbf{L}}(\vec{a})$, so $a \in \{r^{\mathbf{L}}(\vec{a}), s^{\mathbf{L}}(\vec{a})\}$, but $r^{\mathbf{L}}(\vec{a}) < e < a$, so $a = s^{\mathbf{L}}(\vec{a}) = b^*$ for some $b \in L^-$, by the IH.

Next, suppose $\Box$ is $\to$. Since $e < a = r^{\mathbf{L}}(\vec{a}) \to s^{\mathbf{L}}(\vec{a})$, we have $r^{\mathbf{L}}(\vec{a}) \leq s^{\mathbf{L}}(\vec{a})$. Every idempotent CRL satisfies $e \leq x \leq y \Longrightarrow x \to y = y$, so if $e \leq r^{\mathbf{L}}(\vec{a})$ then $a = s^{\mathbf{L}}(\vec{a}) = b^*$ for some $b \in L^-$, by the IH. Also, every idempotent CRL satisfies $x \leq y \leq e \Longrightarrow x \to y = x^*$, so if $s^{\mathbf{L}}(\vec{a}) \leq e$ then $a = r^{\mathbf{L}}(\vec{a})^*$ (with $r^{\mathbf{L}}(\vec{a}) \in L^-$). We may therefore assume that $r^{\mathbf{L}}(\vec{a}) < e < s^{\mathbf{L}}(\vec{a})$, as $\mathbf{L}$ is conic. Every conic idempotent CRL satisfies $x \leq e \leq y \Longrightarrow x \to y = x^*y$, by [12, Lemma 2.7(iv)], so $a = r^{\mathbf{L}}(\vec{a})^*s^{\mathbf{L}}(\vec{a}) = b^*c^* \in \{b^*, c^*\}$ for $b{:=}r^{\mathbf{L}}(\vec{a}) \in L^-$ and some $c \in L^-$, by the IH.

If $\Box$ is $\wedge$, then since $e < a = r^{\mathbf{L}}(\vec{a}) \wedge s^{\mathbf{L}}(\vec{a})$, we have $r^{\mathbf{L}}(\vec{a}), s^{\mathbf{L}}(\vec{a}) \in L^+$, so by the IH, $a = b^* \wedge c^* \in \{b^*, c^*\}$ for some $b^*, c^* \in L^-$.

Finally, suppose $\Box$ is $\vee$. As $\mathbf{L}$ is conic and $e < a = r^{\mathbf{L}}(\vec{a}) \vee s^{\mathbf{L}}(\vec{a})$, at least one of $r^{\mathbf{L}}(\vec{a}), s^{\mathbf{L}}(\vec{a})$ belongs to L^+. If both belong to L^+, then $a = b^* \vee c^* \in \{b^*, c^*\}$ for some $b, c \in L^-$, by the IH. In the opposite case, we may assume (by symmetry) that $r^{\mathbf{L}}(\vec{a}) < e \leq s^{\mathbf{L}}(\vec{a})$, whence $a = s^{\mathbf{L}}(\vec{a}) = b^*$ for some $b \in L^-$, by the IH. □

Definition 3.4. The variety **SGSM** of semiconic generalized Sugihara monoids consists of the semiconic idempotent CRLs that satisfy $(x \vee e)^{**} = x \vee e$.

Finally, we obtain some properties of semiconic generalized Sugihara monoids.

Lemma 3.5. *Let $\mathbf{L}$ be a semiconic generalized Sugihara monoid. Then $\mathbf{L}$ satisfies $x = (x \wedge e)(x^* \wedge e)^*$.*

Proof. Because **L** is semiconic, it suffices to prove this under the assumption that **L** is conic.

Let $a \in L$. If $a \leq e$, then by Proposition 3.1(1), $a^* \geq e$, so $(a^* \wedge e)^* = e^* = e$. Hence $a = a \wedge e = (a \wedge e)e = (a \wedge e)(a^* \wedge e)^*$. If $a > e$, then $a^* < e$ by Proposition 3.1(2). Since **L** is a semiconic generalized Sugihara monoid, $a = a \vee e = (a \vee e)^{**} = a^{**} = (a^*)^* = (a^* \wedge e)^*$. Hence $a = ea = (a \wedge e)(a^* \wedge e)^*$. □

An element a of a CRL will be called *negative* if $a \leq e$. We have the following results.

Corollary 3.6. *A semiconic idempotent* CRL ***L*** *is a semiconic generalized Sugihara monoid if and only if it is generated by its negative elements.*

Proof. The forward implication follows from Lemma 3.5. In the implication from right to left, **L** can be assumed conic, because of three facts: (1) **L** is semiconic; (2) a surjective homomorphism always maps a generating set onto a generating set; (3) a homomorphism h from **L** onto a conic algebra **M** sends L^- onto M^-. In (3), $h[L^-] \subseteq M^-$, because h is isotone and preserves e. Conversely, if $m \in M^-$, then $m = h(\ell)$ for some $\ell \in L$, in which case $\ell \wedge e \in L^-$ and $h(\ell \wedge e) = m \wedge e = m$. Let **L** be a conic idempotent CRL which is generated by its negative elements. Let $a \in L$ such that $a \geq e$. Then by Theorem 3.3, there exists $c \in L^-$ such that $a = c^*$, so by Lemma 2.2(6), $a = c^* = c^{***} = a^{**}$. It follows that for all $b \in L$, $(b \vee e)^{**} = b \vee e$, which implies that **L** is a semiconic generalized Sugihara monoid. □

Corollary 3.7. *Let* ***L*** *be a conic idempotent* CRL *that is generated by its negative elements. Then L^+ is a chain in* ***L***.

Proof. By the previous corollary, **L** satisfies $x \vee e = (x \vee e)^{**}$, so L^+ is contained in $\{a^* : a \in L\}$, which is a chain. □

Lemma 3.8. *Let* ***L*** *be a semiconic generalized Sugihara monoid. Then* ***L*** *satisfies $(x \wedge y)^* = x^* \vee y^*$.*

Proof. Because **L** is semiconic, it suffices to prove this under the assumption that **L** is conic.

Let $a, b \in L$. If $a \leq b$, then $b^* = b \to e \leq a \to e = a^*$ by Lemma 2.2(3) and so $(a \wedge b)^* = a^* = a^* \vee b^*$. Similarly, if $b \leq a$, then $(a \wedge b)^* = a^* \vee b^*$. If $a \parallel b$, then by Corollary 3.7, $a, b > e$ is impossible, so $a, b < e$, whence $(a \wedge b)^* = a^* \vee b^*$ by Proposition 3.1(3). □

4. Strong nuclei

A *nucleus* of a CRL **L** is a function $N : L \longrightarrow L$ such that, for all $a, b \in L$,

(N1) $a \leq Na = NNa$,
(N2) $N(a \wedge b) \leq Na$,
(N3) $NaNb \leq N(ab)$.

A *nuclear* CRL is the expansion of a CRL **L** by a nucleus N. For detailed information on nuclear CRLs, the reader is referred to references [5,8].

For our purpose, we introduce the following concept.

Definition 4.1. Let **L** be a CRL and let $\Diamond$ be a nucleus of **L**. $\Diamond$ is a *strong nucleus* if it satisfies the following conditions: for all $a, b \in L$,

(SN1) $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$,
(SN2) $e \leq (\Diamond a \to b) \vee (b \to \Diamond a)$.

The expansion of a CRL by a strong nucleus $\Diamond$ is called a strong nuclear CRL.

Since a strong nuclear CRL **L** is nuclear, by [8, Theorem 7.1], we have the following result.

Theorem 4.2. *A strong nuclear* CRL $\boldsymbol{B}$ *and its* CRL*-reduct* $\boldsymbol{A}$ *always have the same congruences. In particular,* $\boldsymbol{B}$ *is [finitely] subdirectly irreducible if and only if* $\boldsymbol{A}$ *is.*

Notation 4.3. For a class **C** of CRLs, we use **SNC** to denote the class of all strong nuclear CRLs $(\mathbf{A}, \Diamond)$ such that $\mathbf{A} \in \mathbf{C}$.

Because the definition of a strong nucleus can be made purely equational, we have:

Theorem 4.4. *If* $\boldsymbol{V}$ *is a variety of* CRL*s, then* $\boldsymbol{SNV}$ *is also a variety.*

Theorem 4.5. *Let* $\boldsymbol{L}$ *be a semilinear* CRL*. If* $(\boldsymbol{L}, \Diamond)$ *is nuclear, then* $(\boldsymbol{L}, \Diamond)$ *is strong nuclear.*

Proof. Assume $(\mathbf{L}, \Diamond)$ is nuclear. Then $(N1-3)$ hold. Because **L** is semilinear, it suffices to prove $(SN1-2)$ under the assumption that **L** is a chain. Let $a, b \in L$. If $a \leq b$, then by $(N2)$, $\Diamond a \leq \Diamond b$, so $\Diamond(a \vee b) = \Diamond b = \Diamond a \vee \Diamond b$. Similarly, if $b \leq a$, then $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$. Thus $(SN1)$ holds. Since **L** is a chain, $\Diamond a \leq b$ or $b < \Diamond a$. Then $e \leq \Diamond a \to b$ or $e \leq b \to \Diamond a$, which implies that $(SN2)$ holds. □

A *Brouwerian algebra* is an integral idempotent CRL, i.e., a CRL in which $ab = a \wedge b$ for all elements a, b. The variety of Brouwerian algebras is denoted by **BrA**. A *relative Stone algebra* is a semilinear Brouwerian algebra, i.e., a subdirect product of totally ordered Brouwerian algebras. The variety of relative Stone algebras is denoted by **RSA**.

Notation 4.6. From now on, **SNBrA** shall denote the variety of strong nuclear Brouwerian algebras, and **SNRSA** the variety of strong nuclear relative Stone algebras.

Definition 4.7. Let **L** be a strong nuclear CRL. **L** is called a $\Diamond$*-chain* if for all $a, b \in L$, $\Diamond a \not\parallel b$.

It is well known that a CRL **L** is finitely subdirectly irreducible if and only if its identity element e is join-irreducible in the lattice reduct $(L, \vee, \wedge)$ and an idempotent CRL **L** is subdirectly irreducible if and only if the set $\{a \in L : a < e\}$ has a greatest element (see [8]).

Proposition 4.8. *Let* $\boldsymbol{L} \in \boldsymbol{SNBrA}$.

(1) *If* $\boldsymbol{L}$ *is finitely subdirectly irreducible, then* $\boldsymbol{L}$ *is a* $\Diamond$*-chain.*

(2) *If* $\boldsymbol{L}$ *is subdirectly irreducible, then* $\boldsymbol{L}$ *is a* $\Diamond$*-chain in which the set* $\{a \in L : a < e\}$ *has a greatest element.*

Proof. (1) If $\mathbf{L}$ is finitely subdirectly irreducible, then by Theorem 4.2, the CRL-reduct of $\mathbf{L}$ is finitely subdirectly irreducible, so e is join-irreducible. Let $a, b \in L$. Since $\mathbf{L}$ is integral, by $(SN2)$, $e = (\Diamond a \to b) \vee (b \to \Diamond a)$, so $\Diamond a \to b = e$ or $b \to \Diamond a = e$. It follows that $\Diamond a \leq b$ or $b \leq \Diamond a$. Thus $\Diamond a \not\parallel b$, which implies that $\mathbf{L}$ is a $\Diamond$-chain.

(2) If $\mathbf{L}$ is subdirectly irreducible, then by Theorem 4.2, the CRL-reduct of $\mathbf{L}$ is subdirectly irreducible, so the set $\{a \in L : a < e\}$ has a greatest element, which implies that e is join-irreducible. Thus $\mathbf{L}$ is finitely subdirectly irreducible. By (1), $\mathbf{L}$ is a $\Diamond$-chain. □

Corollary 4.9. *If* $\boldsymbol{B} \in \boldsymbol{SNBrA}$, *then* $\boldsymbol{B}$ *is a subdirect product of strong nuclear Brouwerian algebras that are* $\Diamond$*-chains.*

Definition 4.10. For every CRL $\mathbf{L}$, the negative cone of $(L, \wedge, \vee, \cdot, \to, e)$ is the algebra $\mathbf{L}^- = (L^-, \wedge, \vee, \cdot, \to^-, e)$, where $L^- = \{x \in L : x \leq e\}$, and $x \to^- y = (x \to y) \wedge e$ for $x, y \in L^-$. If $a \in L^-$, then $a^{**} \in L^-$. When $\mathbf{L}$ is semiconic and idempotent, then $\mathbf{L}^- \in \mathbf{BrA}$ and a strong nucleus of $\mathbf{L}^-$ is defined by $\Diamond a = a^{**}$ (i.e., $\Diamond a = (a \to^{\mathbf{L}} e) \to^{\mathbf{L}} e$ for all $a \in L^-$). We use $\mathbf{L}^-_{\Diamond}$ to denote the resulting algebra $(\mathbf{L}^-, \Diamond) \in \mathbf{SNBrA}$, which we call the strong nuclear negative cone of $\mathbf{L}$.

Example 4.11. Let $L = \{a_1, e, b_1, b_2, b_3, b_4, b_5\}$. We define an order relation $\leq$ on L by $b_5 < b_4, b_3 < b_2 < b_1 < e < a_1$ and $b_4 \parallel b_3$, see Figure 1. We can define a multiplication operation on L by $xy = x \wedge y$ for all $x, y \in L^- \backslash \{e\}$ and $ce = c$ for all $c \in L$. We define a binary operation on L by $a \to b = \max\{p \in L : ap \leq b\}$ for all $a, b \in L$. It is clear that $\mathbf{L} = (L, \wedge, \vee, \cdot, \to, e) \in \mathbf{SGSM}$ and $\mathbf{L}^- \in \mathbf{BrA}$. Since the set $\{a \in L : a < e\}$ has a greatest element, $\mathbf{L}$ is subdirectly irreducible. So $\mathbf{L}^- \notin \mathbf{RSA}$. We can define a strong nucleus on $\mathbf{L}^-$ by $\Diamond b_1 = \Diamond b_2 = \Diamond b_3 = \Diamond b_4 = \Diamond b_5 = b_1$ and $\Diamond e = e$. Then $(\mathbf{L}^-, \Diamond) \in \mathbf{SNBrA}$. But $(\mathbf{L}^-, \Diamond) \notin \mathbf{SNRSA}$.

5. A functor from SNBrA To SGSM

Recall that two categories C and D are said to be *equivalent* if there exist functors $F : C \longrightarrow D$ and $G : D \longrightarrow C$ such that $G \circ F \cong 1_C$ and $F \circ G \cong 1_D$, where $\cong$ denotes the natural isomorphism of functors. In the *concrete category* associated with a class of similar algebras, the objects are the members of the class, and the morphisms are all the algebraic homomorphisms between pairs of objects. We denote the set of homomorphisms of $\mathbf{L}$ into $\mathbf{M}$ by $\mathrm{Hom}(\mathbf{L}, \mathbf{M})$. Two isomorphically-closed classes of similar algebras, C and D, are said to be *categorically equivalent* if the corresponding concrete categories are equivalent. For this, it is sufficient (and necessary) that some functor $F : C \longrightarrow D$ should have the following properties:

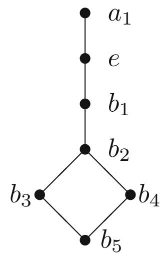

Figure 1. **L**

(i) for each $\mathbf{U} \in D$, there exists $\mathbf{L} \in C$ with $F(\mathbf{L}) \cong \mathbf{U}$, and
(ii) the map $h \mapsto F(h)$ from $\text{Hom}(\mathbf{L}, \mathbf{M})$ to $\text{Hom}(F(\mathbf{L}), F(\mathbf{M}))$ is bijective, for all $\mathbf{L}, \mathbf{M} \in C$.

In this case, F and some functor G from D to C witness the equivalence of these concrete categories. We call G a *reverse* functor for F, and vice versa. Note that C and D are not assumed to have the same algebraic similarity type.

In this section we will prove that **SGSM** and **SNBrA** are categorically equivalent. Definition 4.10 gives us a way to associate a strong nuclear Brouwerian algebra $\mathbf{L}_{\Diamond}^{-}$ with a given semiconic generalized Sugihara monoid $\mathbf{L}$. The construction becomes a functor from **SGSM** to **SNBrA** if we also restrict **SGSM**-morphisms to the negative cones of their domains. We call this the *strong nuclear negative cone functor*. Next we will construct a reverse functor from **SNBrA** To **SGSM**.

To begin with, we introduce some concepts.

Let $\mathbf{L} = (L, \wedge, \vee, \cdot, \rightarrow, e, \Diamond)$ be a strong nuclear Brouwerian algebra, where, as usual, $\leq$ denotes the lattice order of $\mathbf{L}$. We define $S(\mathbf{L}) = \{(a, a') \in L \times L : a \vee a' = e \text{ and } \Diamond a' = a'\}$.

Define an order $\leq$ on $S(\mathbf{L})$ as follows: for $(a, a'), (b, b') \in S(\mathbf{L})$,

$$(a, a') \leq (b, b') \text{ if and only if both } a \leq b \text{ and } b' \leq a'.$$

Lemma 5.1. *$(S(\boldsymbol{L}), \leq)$ is a lattice.*

Proof. Since $\mathbf{L}$ is a lattice, we have $(a, a') \vee (b, b') = (a \vee b, a' \wedge b')$ and $(a, a') \wedge (b, b') = (a \wedge b, a' \vee b')$. We only need to prove that $(a \vee b, a' \wedge b'), (a \wedge b, a' \vee b') \in S(\mathbf{L})$. Since $\mathbf{L}$ is a Brouwerian algebra, $(L, \wedge, \vee)$ is a distributive lattice. We have $(a \vee b) \vee (a' \wedge b') = (a \vee b \vee a') \wedge (a \vee b \vee b') = (e \vee b) \wedge (e \vee a) = e \wedge e = e$ and $(a \wedge b) \vee (a' \vee b') = (a \vee a' \vee b') \wedge (b \vee a' \vee b') = (e \vee b') \wedge (e \vee a') = e \wedge e = e$. By $(N2)$, we have $\Diamond(a' \wedge b') \leq \Diamond a'$ and $\Diamond(a' \wedge b') \leq \Diamond b'$, which imply $\Diamond(a' \wedge b') \leq \Diamond a' \wedge \Diamond b'$. On the other hand, by $(N3)$, $\Diamond a' \wedge \Diamond b' = \Diamond a' \Diamond b' \leq \Diamond(a'b') = \Diamond(a' \wedge b')$. Thus $\Diamond(a' \wedge b') = \Diamond a' \wedge \Diamond b' = a' \wedge b'$. By $(SN1)$, we have $\Diamond(a' \vee b') = \Diamond a' \vee \Diamond b' = a' \vee b'$. This implies that $(a \vee b, a' \wedge b'), (a \wedge b, a' \vee b') \in S(\mathbf{L})$. □

Lemma 5.2. *Let $\boldsymbol{L} \in$ **SNBrA** such that $\boldsymbol{L}$ is a $\Diamond$-chain and $(a, a') \in S(\boldsymbol{L})$. Then*

(1) *$a = e$ or $a' = e$;*
(2) *$a \to a' = a'$ and $a' \to a = a$;*
(3) *$\Diamond a = \Diamond b = \Diamond(a \wedge b) = \Diamond(a \vee b)$ for all $b \in L$ such that $b \parallel a$;*
(4) *$\Diamond a < b \wedge c$ for all $b, c \in L$ such that $\Diamond a < b$ and $\Diamond a < c$;*
(5) *$b \to \Diamond a = \Diamond a$ for all $b \in L$ such that $b > \Diamond a$.*

Proof. (1) If $a \parallel a'$, then since $(a, a') \in S(\mathbf{L})$, $\Diamond a' = a' \parallel a$. But since $\mathbf{L}$ is a $\Diamond$-chain, $\Diamond a' \nparallel a$, a contradiction. Thus $a \nparallel a'$. We have

$$e = a \vee a' = \begin{cases} a & if\ a' \leq a; \\ a' & if\ a' > a. \end{cases}$$

This implies that $a = e$ or $a' = e$.

(2) Let $(a, a') \in S(\mathbf{L})$. By (1), $a = e$ or $a' = e$. If $a = e$, then $a \to a' = e \to a' = a'$ and $a' \to a = a' \to e = e = a$. If $a' = e$, then $a \to a' = a \to e = e = a'$ and $a' \to a = e \to a = a$.

(3) Let $b \in L$ such that $b \parallel a$. Since $\mathbf{L}$ is a $\Diamond$-chain, $a \nparallel \Diamond b$. Because $a \parallel b$ and $b \leq \Diamond b$ by $(N1)$, $a \leq \Diamond b$. This implies that $\Diamond a \leq \Diamond(\Diamond b) = \Diamond b$ by $(N1-2)$. Similarly, $\Diamond b \leq \Diamond a$. Thus $\Diamond a = \Diamond b$. It follows that $\Diamond(a \wedge b) = \Diamond a \wedge \Diamond b = \Diamond a$ by $(N2-3)$ and $\Diamond(a \vee b) = \Diamond a \vee \Diamond b = \Diamond a$ by $(SN1)$.

(4) Let $b, c \in L$ such that $\Diamond a < b$ and $\Diamond a < c$. Suppose that $\Diamond a = b \wedge c$. Then by $(N2-3)$, $\Diamond a = \Diamond\Diamond a = \Diamond(b \wedge c) = \Diamond b \wedge \Diamond c$. Since $\mathbf{L}$ is a $\Diamond$-chain, $\Diamond b \wedge \Diamond c = \Diamond b$ or $\Diamond b \wedge \Diamond c = \Diamond c$. Hence $\Diamond a = \Diamond b$ or $\Diamond a = \Diamond c$. But $\Diamond a < b \leq \Diamond b$ and $\Diamond a < c \leq \Diamond c$ by $(N1)$, a contradiction. Consequently, $\Diamond a < b \wedge c$.

(5) Let $b \in L$ such that $b > \Diamond a$. Then $b \to \Diamond a = \Diamond a$ or $b \to \Diamond a \parallel b$ by Proposition 3.2. Suppose that $b \to \Diamond a \parallel b$. Then we have $b\Diamond a = b \wedge \Diamond a = \Diamond a \Longrightarrow \Diamond a \leq b \to \Diamond a \Longrightarrow \Diamond a = b\Diamond a \leq b(b \to \Diamond a) \leq \Diamond a \Longrightarrow b(b \to \Diamond a) = \Diamond a$. Thus by $(N1)$ and (3), we have $\Diamond a = \Diamond\Diamond a = \Diamond(b(b \to \Diamond a)) = \Diamond(b \wedge (b \to \Diamond a)) = \Diamond b \geq b > \Diamond a$, a contradiction. Consequently, $b \to \Diamond a = \Diamond a$. □

We define a multiplication $\circ$ on $S(\mathbf{L})$ in the following way: for (a, a'), $(b, b') \in S(\mathbf{L})$,

$$(a, a') \circ (b, b') = (((a \to b') \wedge (b \to a')) \to (a \wedge b), (a \to b') \wedge (b \to a')).$$

Lemma 5.3. *$(S(\boldsymbol{L}), \circ, (e, e))$ is an idempotent commutative monoid with an identity (e, e).*

Proof. Let $(a, a'), (b, b') \in S(\mathbf{L})$. We need to prove that $(a, a') \circ (b, b') \in S(\mathbf{L})$. Let $m = (a \to b') \wedge (b \to a')$ and so $(a, a') \circ (b, b') = (m \to (a \wedge b), m)$. We need to show that $\Diamond m = m$ and $(m \to (a \wedge b)) \vee m = e$. By Corollary 4.9, $\mathbf{L}$ is a subdirect product of strong nuclear Brouwerian algebras that are $\Diamond$-chains, so it suffices to prove the equalities under the assumption that $\mathbf{L}$ is a $\Diamond$-chain. Then by Lemma 5.2(1), we have $a = e$ or $a' = e$ and $b = e$ or $b' = e$.

We consider the following cases:

- If $a = e$ and $b = e$, then $m = (e \to b') \wedge (e \to a') = b' \wedge a'$. Thus by $(N2-3)$, $\Diamond m = \Diamond(b' \wedge a') = \Diamond b' \wedge \Diamond a' = b' \wedge a' = m$ and $(m \to (a \wedge b)) \vee m = ((m \to e) \vee m = e \vee m = e$.

- If $a' = e$ and $b' = e$, then $m = (a \to e) \wedge (b \to e) = e$. Thus $\Diamond m = \Diamond e = e = m$ and $(m \to (a \wedge b)) \vee m = (e \to (a \wedge b)) \vee e = e$.
- If $a = e$ and $b' = e$, then $m = (e \to e) \wedge (b \to a') = b \to a'$. Since $\mathbf{L}$ is a $\Diamond$-chain, $a' \not\parallel b$. This implies that $b \leq a'$ or $a' < b$. If $b \leq a'$, then $m = b \to a' = e$. Thus $\Diamond m = \Diamond e = e$ and $(m \to (a \wedge b)) \vee m = (e \to (a \wedge b)) \vee e = e$. If $a' < b$, then by Lemma 5.2(5), $m = b \to a' = a'$. Thus $\Diamond m = \Diamond a' = a' = m$ and $(m \to (a \wedge b)) \vee m = (a' \to (e \wedge b)) \vee a' = (a' \to b) \vee a' = e \vee a' = e$.
- If $a' = e$ and $b = e$, then the result follows from the previous case, by symmetry.

Thus, $S(\mathbf{L})$ is closed under $\circ$.

By symmetry, $(a, a') \circ (b, b') = (b, b') \circ (a, a')$ for $(a, a'), (b, b') \in S(\mathbf{L})$. We will show that for all $(a, a') \in S(\mathbf{L})$, $(a, a') \circ (a, a') = (a, a')$ and $(e, e) \circ (a, a') = (a, a')$. By the subdirect decomposition, it suffices to prove that $(a, a') \circ (a, a') = (a, a')$ and $(e, e) \circ (a, a') = (a, a')$ under the assumption that $\mathbf{L}$ is a $\Diamond$-chain. By definition of $\circ$ and Lemma 5.2(2), $(a, a') \circ (a, a') = (((a \to a') \wedge (a \to a')) \to (a \wedge a), (a \to a') \wedge (a \to a')) = (a, a')$ and $(e, e) \circ (a, a') = (((e \to a') \wedge (a \to e)) \to (e \wedge a), (e \to a') \wedge (a \to e)) = (a' \to a, a') = (a, a')$, for any $(a, a') \in S(\mathbf{L})$. Now, it remains to show that $\circ$ satisfies the associative law. Let $(a, a'), (b, b'), (c, c') \in S(\mathbf{L})$. We only need to show that $((a, a') \circ (b, b')) \circ (c, c') = (a, a') \circ ((b, b') \circ (c, c'))$ under the assumption that $\mathbf{L}$ is a $\Diamond$-chain. Because $\circ$ is commutative, we only need to consider the following cases:

- If $a = e, b = e, c = e$, then

$$\begin{aligned}((e, a') \circ (e, b')) \circ (e, c') &= (((e \to b') \wedge (e \to a')) \to (e \wedge e), \\ &\qquad (e \to b') \wedge (e \to a')) \circ (e, c') \\ &= ((a' \wedge b') \to e, a' \wedge b') \circ (e, c') \\ &= (e, a' \wedge b') \circ (e, c') \\ &= (e, (e \to c') \wedge (e \to (a' \wedge b'))) \\ &= (e, c' \wedge (a' \wedge b')) \\ &= (e, a' \wedge b' \wedge c')\end{aligned}$$

and

$$\begin{aligned}(e, a') \circ ((e, b') \circ (e, c')) &= (e, a') \circ (e, b' \wedge c') \\ &= (e, (b' \wedge c') \wedge a') \\ &= (e, a' \wedge b' \wedge c').\end{aligned}$$

Thus $((a, a') \circ (b, b')) \circ (c, c') = (a, a') \circ ((b, b') \circ (c, c'))$.
- If $a = e, b = e, c' = e$, then

$$\begin{aligned}((e, a') \circ (e, b')) \circ (c, e) &= (e, a' \wedge b') \circ (c, e) \\ &= ((c \to (a' \wedge b')) \to c, c \to (a' \wedge b'))\end{aligned}$$

$$= \begin{cases} (c,e) & \text{if} \quad c \leq a', b', \\ (e, a' \wedge b') & \text{if } c > a' \text{or } c > b' \text{ (by} \\ & \text{Lemma 5.2(5));} \end{cases}$$

and

$$\begin{aligned}(e,a') \circ ((e,b') \circ (c,e)) &= (e,a') \circ ((c \to b') \to c, c \to b') \\ &= (((c \to b') \wedge (((c \to b') \to c) \to a')) \to \\ &\quad ((c \to b') \to c), (c \to b') \wedge (((c \to b') \to c) \to a')) \\ &= \begin{cases} (c,e) & \text{if } \ c \leq a', b', \\ (e, a' \wedge b') & \text{if } \ c > b' \quad \text{(by} \\ & \text{Lemma 5.2(5)),} \\ (e,a') = (e, a' \wedge b') & \text{if } \ a' < c \leq b' \text{ (by} \\ & \text{Lemma 5.2(5));} \end{cases} \\ &= \begin{cases} (c,e) & \text{if } \ c \leq a', b', \\ (e, a' \wedge b') & \text{if } \ c > a' \text{ or } c > b'. \end{cases}\end{aligned}$$

It follows that $((a,a') \circ (b,b')) \circ (c,c') = (a,a') \circ ((b,b') \circ (c,c'))$.

- If $a = e, b' = e, c' = e$, then

$$\begin{aligned}((e,a') \circ (b,e)) \circ (c,e) &= ((b \to a') \to b, b \to a') \circ (c,e) \\ &= \begin{cases} (b,e) \circ (c,e) = (b \wedge c, e) & \text{if } \ b \leq a', \\ (e,a') \circ (c,e) = (c,e) & \text{if } \ c \leq a' < b \text{ (by} \\ & \text{Lemma 5.2(5)),} \\ (e,a') \circ (c,e) = (e,a') & \text{if } \ a' < b, c \quad \text{(by} \\ & \text{Lemma 5.2(5));} \end{cases}\end{aligned}$$

and

$$\begin{aligned}(e,a') \circ ((b,e) \circ (c,e)) &= (e,a') \circ (e \to (b \wedge c), e) \\ &= (e,a') \circ (b \wedge c, e) \\ &= (((b \wedge c) \to a') \to (b \wedge c), (b \wedge c) \to a') \\ &= \begin{cases} (b \wedge c, e) & \text{if } \ b \leq a', \\ ((c \to a') \to c, c \to a') = (c,e) & \text{if } \ c \leq a' < b, \\ (a' \to (b \wedge c), a') = (e,a') & \text{if } \ a' < b, c \text{ (by} \\ & \text{Lemma 5.2(4,5)).} \end{cases}\end{aligned}$$

However, $((a,a') \circ (b,b')) \circ (c,c') = (a,a') \circ ((b,b') \circ (c,c'))$.

- If $a' = e, b' = e, c' = e$, then $((a,e) \circ (b,e)) \circ (c,e) = (a \wedge b, e) \circ (c,e) = (a \wedge b \wedge c, e)$ and $(a,e) \circ ((b,e) \circ (c,e)) = (a,e) \circ (b \wedge c, e) = (a \wedge b \wedge c, e)$. Hence $((a,a') \circ (b,b')) \circ (c,c') = (a,a') \circ ((b,b') \circ (c,c'))$.

$\square$

Moreover, we may prove the following result.

Lemma 5.4. *Let* $\boldsymbol{L}$ *be a strong nuclear Brouwerian algebra. Then* $(S(\boldsymbol{L}), \circ, (e,e), \leq)$ *is an idempotent commutative lattice-ordered monoid.*

Proof. Let $(a, a'), (b, b') \in S(\mathbf{L})$ such that $(a, a') \leq (b, b')$. We only need to prove that $(a, a') \circ (c, c') \leq (b, b') \circ (c, c')$ for every $(c, c') \in S(\mathbf{L})$.

Since $(a, a') \leq (b, b')$, $a \leq b$ and $b' \leq a'$. Then by Lemma 2.2(2,3), $b \to c' \leq a \to c'$ and $c \to b' \leq c \to a'$. Thus we have $(b \to c') \wedge (c \to b') \leq (a \to c') \wedge (c \to a') \Longrightarrow ((b \to c') \wedge (c \to b'))(((a \to c') \wedge (c \to a')) \to (a \wedge c)) \leq ((a \to c') \wedge (c \to a'))(((a \to c') \wedge (c \to a')) \to (a \wedge c)) \leq a \wedge c \leq b \wedge c$, whence $((a \to c') \wedge (c \to a')) \to (a \wedge c) \leq ((b \to c') \wedge (c \to b')) \to (b \wedge c)$. Consequently, $(a, a') \circ (c, c') = (((a \to c') \wedge (c \to a')) \to (a \wedge c), (a \to c') \wedge (c \to a')) \leq (((b \to c') \wedge (c \to b')) \to (b \wedge c), (b \to c') \wedge (c \to b')) = (b, b') \circ (c, c')$. □

We may define a binary operation $\to$ on $S(\mathbf{L})$ in the following way: for $(a, a'), (b, b') \in S(\mathbf{L})$,

$$(a, a') \to (b, b') = ((a \to b) \wedge (b' \to a'), ((a \to b) \wedge (b' \to a')) \to \Diamond(a \wedge b \wedge b')).$$

Theorem 5.5. *Let $\boldsymbol{L} = (L, \wedge, \vee, \circ, \to, e)$ be a strong nuclear Brouwerian algebra. Then $\boldsymbol{S(L)} = (S(\boldsymbol{L}), \wedge, \vee, \circ, \to, (e, e))$ is a semiconic generalized Sugihara monoid.*

Proof. Firstly, we shall prove that $(a, a') \to (b, b') \in S(\mathbf{L})$ for all $(a, a'), (b, b') \in S(\mathbf{L})$. We need to verify that $(((a \to b) \wedge (b' \to a'))) \vee (((a \to b) \wedge (b' \to a')) \to \Diamond(a \wedge b \wedge b')) = e$ and $\Diamond(((a \to b) \wedge (b' \to a')) \to \Diamond(a \wedge b \wedge b')) = ((a \to b) \wedge (b' \to a')) \to \Diamond(a \wedge b \wedge b')$. We may assume without loss of generality that $\mathbf{L}$ is a $\Diamond$-chain. We consider the following cases:

Case 1 $a = e, b = e$. We only need to check the following subcases:

(1) If $b' \leq a'$, then $((a \to b) \wedge (b' \to a')) \vee (((a \to b) \wedge (b' \to a')) \to \Diamond(a \wedge b \wedge b')) = (b' \to a') \vee ((b' \to a') \to \Diamond b') = e \vee (e \to b') = e$ and $\Diamond(((a \to b) \wedge (b' \to a')) \to \Diamond(a \wedge b \wedge b')) = \Diamond(e \to \Diamond b') = \Diamond\Diamond b' = b' = ((a \to b) \wedge (b' \to a')) \to \Diamond(a \wedge b \wedge b')$.

(2) If $b' > a'$, then by Lemma 5.2(5), $((a \to b) \wedge (b' \to a')) \vee (((a \to b) \wedge (b' \to a')) \to \Diamond(a \wedge b \wedge b')) = (b' \to a') \vee ((b' \to a') \to \Diamond b') = a' \vee (a' \to b') = a' \vee e = e$ and $\Diamond(((a \to b) \wedge (b' \to a')) \to \Diamond(a \wedge b \wedge b')) = \Diamond(a' \to \Diamond b') = \Diamond(a' \to b') = \Diamond e = e = ((a \to b) \wedge (b' \to a')) \to \Diamond(a \wedge b \wedge b')$.

Case2 $a = e, b' = e$. Then $((a \to b) \wedge (b' \to a')) \vee ((a \to b) \wedge (b' \to a')) \to \Diamond(a \wedge b \wedge b') = (b \wedge a') \vee ((b \wedge a') \to \Diamond b) = (b \wedge a') \vee e = e$ and $\Diamond(((a \to b) \wedge (b' \to a')) \to \Diamond(a \wedge b \wedge b')) = \Diamond((b \wedge a') \to \Diamond b) = \Diamond e = e = ((a \to b) \wedge (b' \to a')) \to \Diamond(a \wedge b \wedge b')$.

Case3 $a' = e, b = e$. Then $((a \to b) \wedge (b' \to a')) \vee (((a \to b) \wedge (b' \to a')) \to \Diamond(a \wedge b \wedge b')) = e \vee (e \to \Diamond(a \wedge b')) = e$ and $\Diamond(((a \to b) \wedge (b' \to a')) \to \Diamond(a \wedge b \wedge b')) = \Diamond(e \to \Diamond(a \wedge b')) = \Diamond\Diamond(a \wedge b') = \Diamond(a \wedge b') = ((a \to b) \wedge (b' \to a')) \to \Diamond(a \wedge b \wedge b')$.

Case4 $a' = e, b' = e$. We only need to check the following subcases:

(1) If $a \leq b$, then $((a \to b) \wedge (b' \to a')) \vee ((a \to b) \wedge (b' \to a')) \to \Diamond(a \wedge b \wedge b') = e \vee ((a \to b) \to \Diamond(a \wedge b)) = e$ and $\Diamond(((a \to b) \wedge (b' \to a')) \to \Diamond(a \wedge b \wedge b')) = \Diamond((a \to b) \to \Diamond(a \wedge b)) = \Diamond(e \to \Diamond a) = \Diamond\Diamond a = \Diamond a = ((a \to b) \wedge (b' \to a')) \to \Diamond(a \wedge b \wedge b')$.

(2) If $a > b$, then by Proposition 3.2, $a \to b = b$ or $a \to b \parallel a$. If $a \to b = b$, then $((a \to b) \wedge (b' \to a')) \vee (((a \to b) \wedge (b' \to a')) \to \Diamond(a \wedge b \wedge b')) = b \vee (b \to \Diamond b) = b \vee e = e$ and $\Diamond(((a \to b) \wedge (b' \to a')) \to \Diamond(a \wedge b \wedge b')) = \Diamond e = e = ((a \to b) \wedge (b' \to a')) \to \Diamond(a \wedge b \wedge b')$. If $a \to b \parallel a$, then $\Diamond a = \Diamond(a \to b) = \Diamond(a \wedge (a \to b))$ by Lemma 5.2(3). Since $ab = a \wedge b = b$, $b \leq a \to b$, and so $b = ab \leq a(a \to b) \leq b$, which implies that $a(a \to b) = b$. Thus $a \to b \leq \Diamond(a \to b) = \Diamond(a \wedge (a \to b)) = \Diamond(a(a \to b)) = \Diamond b$, whence $((a \to b) \wedge (b' \to a')) \vee (((a \to b) \wedge (b' \to a')) \to \Diamond(a \wedge b \wedge b')) = (a \to b) \vee ((a \to b) \to \Diamond b) = (a \to b) \vee e = e$ and $\Diamond(((a \to b) \wedge (b' \to a')) \to \Diamond(a \wedge b \wedge b')) = \Diamond e = e = ((a \to b) \wedge (b' \to a')) \to \Diamond(a \wedge b \wedge b')$.

(3) If $a \parallel b$, then $\Diamond a = \Diamond b = \Diamond(a \wedge b)$ by Lemma 5.2(3) and $a \to b = b$ or $a \to b \parallel a$ by Proposition 3.2. If $a \to b = b$, then $((a \to b) \wedge (b' \to a')) \vee (((a \to b) \wedge (b' \to a')) \to \Diamond(a \wedge b \wedge b')) = b \vee (b \to \Diamond(a \wedge b)) = b \vee (b \to \Diamond b) = b \vee e = e$ and $\Diamond(((a \to b) \wedge (b' \to a')) \to \Diamond(a \wedge b \wedge b')) = \Diamond e = e = ((a \to b) \wedge (b' \to a')) \to \Diamond(a \wedge b \wedge b')$. If $a \to b \parallel a$, then $\Diamond(a \to b) = \Diamond a = \Diamond b = \Diamond(a \wedge b)$ by Lemma 5.2(3). Hence $((a \to b) \wedge (b' \to a')) \vee (((a \to b) \wedge (b' \to a')) \to \Diamond(a \wedge b \wedge b')) = (a \to b) \vee ((a \to b) \to \Diamond(a \wedge b)) = (a \to b) \vee ((a \to b) \to \Diamond(a \to b)) = (a \to b) \vee e = e$ and $\Diamond(((a \to b) \wedge (b' \to a')) \to \Diamond(a \wedge b \wedge b')) = \Diamond e = e = ((a \to b) \wedge (b' \to a')) \to \Diamond(a \wedge b \wedge b')$.

Secondly, we establish the residuation axiom

$$(c, c') \leq (a, a') \to (b, b') \Longleftrightarrow (a, a') \circ (c, c') \leq (b, b').$$

We need to verify $(a, a') \to (b, b') = \max\{(c, c') \in S(\mathbf{L}) : (a, a') \circ (c, c') \leq (b, b')\}$ for all $(a, a'), (b, b') \in S(\mathbf{L})$. Once again, we may assume without loss of generality that $\mathbf{L}$ is a $\Diamond$-chain.

Let $(a, a'), (b, b'), (c, c') \in S(\mathbf{L})$ such that $(a, a') \circ (c, c') \leq (b, b')$. Then $(((a \to c') \wedge (c \to a')) \to (a \wedge c), ((a \to c') \wedge (c \to a'))) \leq (b, b')$. So $((a \to c') \wedge (c \to a')) \to (a \wedge c) \leq b$ and $b' \leq (a \to c') \wedge (c \to a')$. To show that $(a, a') \to (b, b') = \max\{(c, c') \in S(\mathbf{L}) : (a, a') \circ (c, c') \leq (b, b')\}$, we need to consider the following cases:

Case1 $a = e, b = e$. We only need to check the following subcases:

(1) If $b' \leq a'$, then by the definition of $\to$, $(a, a') \to (b, b') = (e, a') \to (e, b') = ((e \to e) \wedge (b' \to a'), ((e \to e) \wedge (b' \to a')) \to \Diamond(e \wedge b')) = (b' \to a', (b' \to a') \to \Diamond b') = (e, e \to b') = (e, b')$. So by the definition of $\circ$, $(a, a') \circ ((a, a') \to (b, b')) = (e, a') \circ (e, b') = (e, a' \wedge b') = (e, b') \leq (b, b')$. On the other hand, since $(a, a') \circ (c, c') = (e, a') \circ (c, c') \leq (b, b') = (e, b')$, $((c' \wedge (c \to a')) \to c, c' \wedge (c \to a')) \leq (e, b')$. It follows that $b' \leq c' \wedge (c \to a') \leq c'$. This implies that $(c, c') \leq (e, b') = (a, a') \to (b, b')$. Thus $(a, a') \to (b, b') = \max\{(c, c') \in S(\mathbf{L}) : (a, a') \circ (c, c') \leq (b, b')\}$.

(2) If $b' > a'$, then by the definition of $\to$ and Lemma 5.2(5), $(a, a') \to (b, b') = (e, a') \to (e, b') = ((e \to e) \wedge (b' \to a'), ((e \to e) \wedge (b' \to a')) \to \Diamond(e \wedge b')) = (b' \to a', (b' \to a') \to \Diamond b') = (a', a' \to b') = (a', e)$. So by the definition of $\circ$, $(a, a') \circ ((a, a') \to (b, b')) = (e, a') \circ$

$(a', e) = (a', e) \le (b, b')$. On the other hand, since $(a, a') \circ (c, c') = (e, a') \circ (c, c') \le (b, b') = (e, b')$, $((c' \wedge (c \to a')) \to c, c' \wedge (c \to a')) \le (e, b')$. Thus $b' \le c' \wedge (c \to a')$. Suppose that $c > a'$. Then by Lemma 5.2(5), $b' \le c' \wedge (c \to a') = c' \wedge a' \le a'$, which is contrary to $b > a'$. Thus $c \le a' < b' \le e$, whence $c' = e$. It follows that $(c, c') = (c, e) \le (a', e) = (a, a') \to (b, b')$ and so $(a, a') \to (b, b') = \max\{(c, c') \in S(\mathbf{L}) : (a, a') \circ (c, c') \le (b, b')\}$.

Case2 $a = e, b' = e$. Then by the definition of $\to$, $(a, a') \to (b, b') = (e, a') \to (b, e) = (b \wedge a', (b \wedge a') \to \Diamond b) = (b \wedge a', e)$. So by the definition of $\circ$, $(a, a') \circ ((a, a') \to (b, b')) = (e, a') \circ (b \wedge a', e) = (((b \wedge a') \to a') \to (b \wedge a'), (b \wedge a') \to a') = (b \wedge a', e) \le (b, e) = (b, b')$. On the other hand, since $(a, a') \circ (c, c') = (e, a') \circ (c, c') \le (b, b') = (b, e)$, $((c' \wedge (c \to a')) \to c, c' \wedge (c \to a')) \le (b, e)$. Thus $(c' \wedge (c \to a')) \to c \le b$ and $c' \wedge (c \to a') = e$. The latter formula derives $c' = e$ and $c \to a' = e$. Since $a' = \Diamond a'$ and $\mathbf{L}$ is a $\Diamond$-chain, $a' \nparallel c$. Then $c \le a'$ or $c > a'$. Suppose that $c > a'$. Then by Lemma 5.2(5), $c \to a' = a'$, which derives that $a' = c \to a' = e$. But $e \ge c > a'$, a contradiction. Consequently, $c \le a'$. Thus $(c' \wedge (c \to a')) \to c = (e \wedge e) \to c = c \le b$, which together $c \le a'$, derives $c \le b \wedge a'$. It follows that $(c, c') = (c, e) \le (b \wedge a', e) = (a, a') \to (b, b')$, whence $(a, a') \to (b, b') = \max\{(c, c') \in S(\mathbf{L}) : (a, a') \circ (c, c') \le (b, b')\}$.

Case3 $a' = e, b = e$. Then by the definition of $\to$, $(a, a') \to (b, b') = (a, e) \to (e, b') = (e, \Diamond(a \wedge b'))$. So by the definition of $\circ$, $(a, a') \circ ((a, a') \to (b, b')) = (a, e) \circ (e, \Diamond(a \wedge b')) = ((a \to \Diamond(a \wedge b')) \to a, a \to \Diamond(a \wedge b'))$. Since $ab' = a \wedge b' \le \Diamond(a \wedge b')$, $b' \le a \to \Diamond(a \wedge b')$ and so $(a, a') \circ ((a, a') \to (b, b')) = ((a \to \Diamond(a \wedge b')) \to a, a \to \Diamond(a \wedge b')) \le (e, b') = (b, b')$. On the other hand, we have $(a, e) \circ (c, c') = (a, a') \circ (c, c') \le (b, b') = (e, b') \Longrightarrow ((a \to c') \to (a \wedge c), a \to c') \le (e, b')$. Thus $b' \le a \to c' \Longrightarrow a \wedge b' = ab' \le c' \Longrightarrow \Diamond(a \wedge b') \le \Diamond c' = c' \Longrightarrow (c, c') \le (e, \Diamond(a \wedge b')) = (a, a') \to (b, b')$. Consequently, $(a, a') \to (b, b') = \max\{(c, c') \in S(\mathbf{L}) : (a, a') \circ (c, c') \le (b, b')\}$.

Case4 $a' = e, b' = e$. Then by the definition of $\to$, $(a, a') \to (b, b') = (a, e) \to (b, e) = (a \to b, (a \to b) \to \Diamond(a \wedge b))$. So by the definition of $\circ$, $(a, a') \circ ((a, a') \to (b, b')) = (a, e) \circ (a \to b, (a \to b) \to \Diamond(a \wedge b)) = ((a \to ((a \to b) \to \Diamond(a \wedge b))) \to (a \wedge (a \to b)), a \to ((a \to b) \to \Diamond(a \wedge b)))$. Since $a(a \to b) \le b$ and $a(a \to b) = a \wedge (a \to b) \le a$, we have $a(a \to b) \le a \wedge b \Longrightarrow a \le (a \to b) \to (a \wedge b) \le (a \to b) \to \Diamond(a \wedge b) \Longrightarrow a \to ((a \to b) \to \Diamond(a \wedge b)) = e$. Thus $(a, a') \circ ((a, a') \to (b, b')) = ((a \to ((a \to b) \to \Diamond(a \wedge b))) \to (a \wedge (a \to b)), a \to ((a \to b) \to \Diamond(a \wedge b))) = (e \to (a \wedge (a \to b)), e) = (a \wedge (a \to b), e) = (a(a \to b), e) \le (b, e) = (b, b')$. On the other hand, we have $(a, e) \circ (c, c') = (a, a') \circ (c, c') \le (b, b') = (b, e) \Longrightarrow ((a \to c') \to (a \wedge c), a \to c') \le (b, e)$. Hence $(a \to c') \to (a \wedge c) \le b$ and $a \to c' = e$. So $ac = a \wedge c = e \to (a \wedge c) = (a \to c') \to (a \wedge c) \le b$ and $a = ae = a(a \to c') \le c'$. It follows that $c \le a \to b$ and $\Diamond a \le \Diamond c' = c'$. Thus we have $\Diamond(a \wedge b) \le \Diamond a \Longrightarrow (a \to b) \to \Diamond(a \wedge b) \le (a \to b) \to \Diamond a \le (a \to b) \to c' \le c \to c' = c'$. Therefore

$(c, c') \leq (a \to b, (a \to b) \to \Diamond(a \wedge b)) = (a, a') \to (b, b')$. Consequently, $(a, a') \to (b, b') = \max\{(c, c') \in S(\mathbf{L}) : (a, a') \circ (c, c') \leq (b, b')\}$.

We conclude that $\mathbf{S(L)} = (S(\mathbf{L}), \wedge, \vee, \circ, \to, (e, e))$ is an idempotent CRL. Finally, we will show that $\mathbf{S(L)}$ is a semiconic generalized Sugihara monoid. We only need to prove that $((a, a') \wedge (e, e)) \vee (((a, a') \to (e, e)) \wedge (e, e)) = (e, e)$ and $((a, a') \vee (e, e))^{**} = (a, a') \vee (e, e)$ for all $(a, a') \in S(\mathbf{L})$. Again, we may assume that $\mathbf{L}$ is a $\Diamond$-chain. Let $(a, a') \in S(\mathbf{L})$. Then we have $((a, a') \wedge (e, e)) \vee (((a, a') \to (e, e)) \wedge (e, e)) = (a, e) \vee ((a', a' \to \Diamond a) \wedge (e, e)) = (a, e) \vee (a', e) = (a \vee a', e) = (e, e)$ and $((a, a') \vee (e, e))^{**} = (((a, a') \vee (e, e)) \to (e, e)) \to (e, e) = ((e, a') \to (e, e)) \to (e, e) = (a', a' \to \Diamond e) \to (e, e) = (a', e) \to (e, e) = (e, e \to \Diamond a') = (e, \Diamond a') = (e, a') = (a, a') \vee (e, e)$. □

Theorem 5.6. *If $\boldsymbol{L} = (L, \wedge, \vee, \cdot, \to, e, \Diamond)$ is a strong nuclear Brouwerian algebra, then $\boldsymbol{L} \cong \boldsymbol{S(L)}^{-}_{\Diamond}$.*

Proof. We can define a mapping θ as follows:

$$\theta : \mathbf{L} \longrightarrow \mathbf{S(L)}^{-}_{\Diamond}; \quad a \mapsto (a, e).$$

Obviously, θ is a bijection, which preserves $\wedge, \vee$ and e. It remains to show that θ preservers $\to$ and $\Diamond$. Let $a, b \in L$. Then $\theta(a \to b) = (a \to b, e) = ((a \to b) \wedge e, ((a \to b) \to \Diamond(a \wedge b)) \vee e) = (a \to b, (a \to b) \to \Diamond(a \wedge b)) \wedge (e, e) = ((a, e) \to (b, e)) \wedge (e, e) = (a, e) \to^{-} (b, e) = \theta(a) \to^{-} \theta(b)$ and $\Diamond\theta(a) = \Diamond(a, e) = ((a, e) \to (e, e)) \to (e, e)) = (e, \Diamond a) \to (e, e) = (\Diamond a, e) = \theta(\Diamond a)$. □

We shall prove that any semiconic generalized Sugihara monoid $\mathbf{L}$ is isomorphic to $\mathbf{S}(\mathbf{L}^{-}_{\Diamond})$.

Theorem 5.7. *If $\boldsymbol{L} = (L, \wedge, \vee, \cdot, \to, e)$ is a semiconic generalized Sugihara monoid, then $\boldsymbol{L} \cong \boldsymbol{S}(\boldsymbol{L}^{-}_{\Diamond})$.*

Proof. We can define a mapping h as follows:

$$h : \mathbf{L} \longrightarrow \mathbf{S}(\mathbf{L}^{-}_{\Diamond}); \quad a \mapsto (a \wedge e, a^* \wedge e).$$

Let $a \in L$. Then by Lemma 2.2(6), $a^{***} = ((a \to e) \to e) \to e = a \to e = a^*$. Since $\mathbf{L}$ is a semiconic generalized Sugihara monoid, $(a \wedge e) \vee (a^* \wedge e) = (a \wedge e) \vee ((a \to e) \wedge e) = e$ and $\Diamond(a^* \wedge e) = (a^* \wedge e)^{**} = (a^{**} \vee e^*)^* = (a^{**} \vee e)^* = a^{***} \wedge e^* = a^* \wedge e$ by Lemmas 3.8 and 2.2(3). So $h(a) = (a \wedge e, a^* \wedge e) \in S(\mathbf{L}^{-}_{\Diamond})$. This implies that the definition of h is logical.

It follows from Lemma 3.5 that h is one-to-one. To see that it is onto, let $(a, a') \in S(\mathbf{L}^{-}_{\Diamond})$, so $a, a' \leq e$ in $\mathbf{L}$, $a \vee a' = e$ and $a'^{**} = \Diamond a' = a'$ by Definition 4.10. Let $b = (a \to a') \to a$. We claim that $h(b) = (a, a')$, i.e., that

$$(((a \to a') \to a) \wedge e, ((a \to a') \to a)^* \wedge e) = (a, a').$$

Because $\mathbf{L}$ is semiconic, it suffices to prove this under the assumption that $\mathbf{L}$ is conic. Since $a'^{**} = \Diamond a' = a'$, by Proposition 3.1(5), $a = e$ or $a' = e$. If $a = e$, then since $a' \leq e$, we have $b = (a \to a') \to a = (e \to a') \to e = a' \to e = (a')^* \geq e$ by Proposition 3.1(1), and $b^* = (a')^{**} = \Diamond a' = a'$. Hence $h(b) = (b \wedge e, b^* \wedge e) = ((a')^* \wedge e, a' \wedge e) = (e, a') = (a, a')$. If $a' = e > a$, then $a^* \geq e$, whence $a^* > a$, so by Proposition 3.1(4), $b = (a \to a') \to a = (a \to e) \to a =$

$a^* \to a = a$. In this case, $h(b) = (b \wedge e, b^* \wedge e) = (a \wedge e, a^* \wedge e) = (a, e) = (a, a')$. This implies that h is onto.

Obviously, $h(e) = (e, e)$. By Lemmas 2.2(3) and 3.8, h preserves $\vee$ and $\wedge$.

Next, we show that h preserves $\cdot$. Let $a, b \in L$. The desired result $h(a) \circ h(b) = h(ab)$ boils down to

$$(u \to^- (a \wedge b \wedge e), u) = ((ab) \wedge e, (ab)^* \wedge e),$$

where $u = (((a \wedge e) \to^- (b^* \wedge e)) \wedge ((b \wedge e) \to^- (a^* \wedge e)) = ((a \wedge e) \to (b^* \wedge e)) \wedge ((b \wedge e) \to (a^* \wedge e)) \wedge e$.

Again we may assume that **L** is conic. We consider the following cases:

Case1 $a \le e, b \le e$. Then by Lemma 2.1(3), $ab = a \wedge b \le e$ and by Proposition 3.1(1), $a^* \ge e, b^* \ge e, (ab)^* \ge e$. So $u = (a \to e) \wedge (b \to e) \wedge e = a^* \wedge b^* \wedge e = e$. Hence $h(a) \circ h(b) = (e \to^- (a \wedge b \wedge e), e) = ((e \to (a \wedge b)) \wedge e, e) = (a \wedge b, e) = ((ab) \wedge e, (ab)^* \wedge e) = h(ab)$.

Case2 $a \ge e, b \ge e$. Then by Lemma 2.1(2), $ab = a \vee b \ge e$ and by Proposition 3.1(2), $a^* \le e, b^* \le e, (a \vee b)^* \le e$. So $(a \vee b)^{**} = a \vee b$ by the definition of semiconic generalized Sugihara monoid and $u = (e \to b^*) \wedge (e \to a^*) \wedge e = b^* \wedge a^* \wedge e = a^* \wedge b^* = (a \vee b)^* \le e$ by Lemma 2.2(3). Hence $h(a) \circ h(b) = ((a^* \wedge b^*) \to^- (a \wedge b \wedge e), a^* \wedge b^*) = ((a^* \wedge b^*) \to e) \wedge e, a^* \wedge b^*) = ((a^* \wedge b^*)^* \wedge e, (a \vee b)^*) = ((a \vee b)^{**} \wedge e, (a \vee b)^*) = ((a \vee b) \wedge e, (a \vee b)^* \wedge e) = ((ab) \wedge e, (ab)^* \wedge e) = h(ab)$.

Case3 $a < e < b$. Then $b^* < e \le a^*$ by Proposition 3.1(1,2). We only need to check the following subcases:

(1) If $ab = a$, then $ab < e \Longrightarrow ae = a \le b \to e = b^* \Longrightarrow e \le a \to b^*$. Hence $u = (a \to b^*) \wedge (e \to e) \wedge e = e$, whence $h(a) \circ h(b) = ((e \to (a \wedge b \wedge e)) \wedge e, e) = (a, e) = (a \wedge e, a^* \wedge e) = ((ab) \wedge e, (ab)^* \wedge e) = h(ab)$.

(2) If $ab = b$, then since $b^* b = b^*$ by Proposition 3.1(2), $b^* \wedge a = b^* a = b^* ba = b^* b = b^*$, which implies that $b^* < a$ and $b^* \le a \to b^*$. On the other hand, by Lemma 2.2(7), $b(a \to b^*) = b(a \to (b \to e) = b(ab \to e) = ab(ab \to e) \le e \Longrightarrow a \to b^* \le b \to e = b^*$. Thus $a \to b^* = b^*$. It follows that $u = (a \to b^*) \wedge (e \to e) \wedge e = b^* \wedge e = b^*$. Thus by Proposition 2.3(3), $h(a) \circ h(b) = ((b^* \to (a \wedge b \wedge e)) \wedge e, b^*) = ((b^* \to a) \wedge e, b^*) = (e, b^*) = (b \wedge e, b^* \wedge e) = ((ab) \wedge e, (ab)^* \wedge e) = h(ab)$.

Case4 $b < e < a$. Similar to Case 3, $h(a) \circ h(b) = h(ab)$.

We have proved that h is an isomorphism between the lattice-ordered monoid reducts of **L** and $\mathbf{S}(\mathbf{L}^-_\Diamond)$. Since $x \to y = \max\{z \in L : xz \le y\}$ for all $x, y \in L$, $\to$ is first-order definable in terms of $\cdot, \wedge$. Therefore $h(a \to b) = h(a) \to h(b)$ for all $a, b \in L$. It follows that h is an isomorphism between **L** and $\mathbf{S}(\mathbf{L}^-_\Diamond)$. □

Lemma 5.8. *Let $\boldsymbol{L}$ be a strong nuclear Brouwerian algebra, and $(a, a') \in S(\boldsymbol{L})$. Then $(a, a') = (a, e) \circ (e, a')$.*

Proof. It suffices to prove the equality under the assumption that **L** is a $\Diamond$-chain. By Lemma 5.2(2), we have $(a, e) \circ (e, a') = (((a \to a') \to a), a \to a') = (a' \to a, a') = (a, a')$. □

Theorem 5.9. *Let $\boldsymbol{L}$ and $\boldsymbol{M}$ be strong nuclear Brouwerian algebras.*

(1) *If* $h : \boldsymbol{L} \longrightarrow \boldsymbol{M}$ *is a homomorphism, then* $S(h) : (a, a') \mapsto (h(a), h(a'))$ *is a homomorphism from* $\boldsymbol{S(L)}$ *into* $\boldsymbol{S(M)}$*.*
(2) *The map* $h \mapsto S(h)$ *is a bijection from* $\mathrm{Hom}(\boldsymbol{L}, \boldsymbol{M})$ *to* $\mathrm{Hom}(\boldsymbol{S(L)}, \boldsymbol{S(M)})$.

Proof. (1) follows straightforwardly from the definitions of the operations.

(2) Let $h_1, h_2 \in \mathrm{Hom}(\mathbf{L}, \mathbf{M})$ such that $S(h_1) = S(h_2)$. Let $a \in L$. Then $(a, e) \in S(\mathbf{L})$. We have $S(h_1)(a, e) = S(h_2)(a, e)$. It follows that $(h_1(a), h_1(e)) = (h_2(a), h_2(e))$ and so $(h_1(a), e) = (h_2(a), e)$. Hence $h_1(a) = h_2(a)$, which implies that $h_1 = h_2$. Thus the map $h \mapsto S(h)$ is injective on $\mathrm{Hom}(\mathbf{L}, \mathbf{M})$. Finally, we will show the map $h \mapsto S(h)$ is surjective.

Let $g \in \mathrm{Hom}(\mathbf{S(L)}, \mathbf{S(M)})$ and $a \in L$. Suppose that $g(a, e) = (\tilde{a}, \bar{e})$. Then since $(a, e) \le (e, e)$ and $g(e, e) = (e, e)$, $g(a, e) = (\tilde{a}, \bar{e}) \le (e, e)$, and so $\bar{e} \ge e$, which implies that $\bar{e} = e$. Hence $g(a, e) = (\tilde{a}, e)$. We can define a mapping $\tilde{g}$ as follows:

$$\tilde{g} : \mathbf{L} \longrightarrow \mathbf{M}; a \mapsto \tilde{a} \text{ such that } g(a, e) = (\tilde{a}, e).$$

We will prove that $\tilde{g} \in \mathrm{Hom}(\mathbf{L}, \mathbf{M})$. Let $a, b \in L$. Then since $g \in \mathrm{Hom}(\mathbf{S(L)}, \mathbf{S(M)})$, $(\tilde{g}(ab), e) = (\tilde{g}(a \wedge b), e) = g(a \wedge b, e) = g(a \wedge b, e \vee e) = g((a, e) \wedge (b, e)) = g(a, e) \wedge g(b, e) = (\tilde{a}, e) \wedge (\tilde{b}, e) = (\tilde{a} \wedge \tilde{b}, e \vee e) = (\tilde{a}\tilde{b}, e)$, which implies that $\tilde{g}(ab) = \tilde{a}\tilde{b} = \tilde{g}(a)\tilde{g}(b)$ and $\tilde{g}(a \wedge b) = \tilde{a} \wedge \tilde{b} = \tilde{g}(a) \wedge \tilde{g}(b)$. Similarly, $\tilde{g}(a \vee b) = \tilde{g}(a) \vee \tilde{g}(b)$. By Theorem 5.6, $\mathbf{L} \cong \mathbf{S(L)}^-_\Diamond$. It follows that $(a \to b, e) = (a, e) \to^- (b, e)$ and $(\Diamond a, e) = \Diamond(a, e)$. Hence $(\tilde{g}(a \to b), e) = g(a \to b, e) = g((a, e) \to^- (b, e)) = g(a, e) \to^- g(b, e) = (\tilde{g}(a), e) \to^- (\tilde{g}(b), e) = (\tilde{g}(a) \to \tilde{g}(b), e)$ and $(\tilde{g}(\Diamond a), e) = g(\Diamond a, e) = g(\Diamond(a, e)) = \Diamond g(a, e) = \Diamond(\tilde{g}(a), e) = (\Diamond \tilde{g}(a), e)$. Thus $\tilde{g}(a \to b) = \tilde{g}(a) \to \tilde{g}(b)$ and $\tilde{g}(\Diamond a) = \Diamond \tilde{g}(a)$. Consequently, $\tilde{g} \in \mathrm{Hom}(\mathbf{L}, \mathbf{M})$.

Let $(a, a') \in S(\mathbf{L})$. Then $\Diamond a' = a'$ and so $\Diamond \tilde{g}(a') = \tilde{g}(\Diamond a') = \tilde{g}(a')$. Thus $(a', e)^* = (a', e) \to (e, e) = ((a' \to e) \wedge (e \to e), ((a' \to e) \wedge (e \to e)) \to \Diamond(a' \wedge e)) = (e, \Diamond a') = (e, a')$. Similarly, $(\tilde{a'}, e)^* = (e, \tilde{a'}) = (e, \tilde{g}(a'))$. By Lemma 5.8, we have $(a, a') = (a, e) \circ (e, a')$. Thus $g(a, a') = g((a, e) \circ (e, a')) = g((a, e) \circ (a', e)^*) = g(a, e) \circ (g(a', e))^* = (\tilde{g}(a), e) \circ (\tilde{g}(a'), e)^* = (\tilde{g}(a), e) \circ (e, \tilde{g}(a')) = (\tilde{g}(a), \tilde{g}(a')) = S(\tilde{g})(a, a')$. Therefore, $g = S(\tilde{g})$. □

By Theorems 5.6–5.9, we have the following result, which generalizes [8, Theorem 8.7].

Theorem 5.10. *The variety of semiconic generalized Sugihara monoids and the variety of strong nuclear Brouwerian algebras are categorically equivalent. In particular, a category equivalence from* ***SNBrA*** *to* ***SGSM*** *is witnessed by the functor that sends* $\boldsymbol{L}$ *to* $\boldsymbol{S(L)}$ *and* h *to* $S(h)$ *for all* $\boldsymbol{L}, \boldsymbol{M} \in$ ***SNBrA*** *and all* $h \in \mathrm{Hom}(\boldsymbol{L}, \boldsymbol{M})$*. The strong nuclear negative cone functor sending* $\boldsymbol{L}$ *to* $\boldsymbol{L}^-_\Diamond$ *and* g *to* $g|_{L^-}$ *for all* $\boldsymbol{L}, \boldsymbol{M} \in$ ***SGSM*** *and* $g \in \mathrm{Hom}(\boldsymbol{L}, \boldsymbol{M})$ *is a reverse functor for* S*.*

6. Applications

In this section we will use the present category equivalences to give some new results about non-integral varieties, which generalize the main results of [8].

Let $\mathbf{K}$ be a quasivariety of algebras. A homomorphism h between algebras in $\mathbf{K}$ is called a ($\mathbf{K}$-)*epimorphism* provided that, for any two homomorphisms f, g from the co-domain of h to a single member of $\mathbf{K}$, if $f \circ h = g \circ h$, then $f = g$. Clearly, every surjective homomorphism between algebras in $\mathbf{K}$ is an epimorphism. If the converse holds, then $\mathbf{K}$ is said to have the *ES property.* The *strong ES property* for $\mathbf{K}$ is the demand that, whenever $\mathbf{A}$ is a subalgebra of some $\mathbf{B} \in \mathbf{K}$ and $b \in B \setminus A$, then there are two homomorphisms from $\mathbf{B}$ to a single member of $\mathbf{K}$ that agree on $\mathbf{A}$ but not at b. The *weak ES property* for $\mathbf{K}$ forbids non-surjective $\mathbf{K}$-epimorphisms $h : \mathbf{A} \longrightarrow \mathbf{B}$ in all cases where $\mathbf{B}$ is generated (as an algebra) by $X \cup h[A]$ for some finite $X \subseteq B$.

The *amalgamation property* for a class $\mathbf{K}$ of similar algebras is the demand that for any $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbf{K}$ and any two embeddings $g_B : \mathbf{A} \longrightarrow \mathbf{B}$ and $g_C : \mathbf{A} \longrightarrow \mathbf{C}$, there exist an algebra $\mathbf{D} \in \mathbf{K}$ and embeddings $f_B : \mathbf{B} \longrightarrow \mathbf{D}$ and $f_C : \mathbf{C} \longrightarrow \mathbf{D}$ such that $f_B \circ g_B = f_C \circ g_C$. The *strong amalgamation property* for $\mathbf{K}$ asks, in addition, that $\mathbf{D}$, f_B and f_C can be chosen so that $(f_B \circ g_B)[A] = f_B[B] \cap f_C[C]$.

These conditions are linked as follows (see [10, Sec.2.5.3] and [14,15,19]).

Theorem 6.1. *A quasivariety has the strong amalgamation property if and only if it has the amalgamation and weak ES properties. In that case, it also has the strong ES property.*

Given a class $\mathbf{X}$ of similar algebras, we use $\mathbf{V(X)}$ to denote the variety generated by $\mathbf{X}$. A variety is *finitely generated* if it has the form $\mathbf{V(X)}$ for some finite set $\mathbf{X}$ consisting of finite algebras.

Theorem 6.2 [1]. *Let $\boldsymbol{K}$ be a variety consisting of expansions of Brouwerian algebras, where every member of $\boldsymbol{K}$ has the same congruences as its Brouwerian algebra reduct. Then* (i) *$\boldsymbol{K}$ has the weak ES property, and* (ii) *if $\boldsymbol{K}$ is finitely generated, it has the ES property.*

Recall that a category equivalence F between two varieties $\mathbf{V}$ and $\mathbf{W}$ induces an isomorphism $\overline{F} : \mathbf{E} \mapsto \{\mathbf{B} \in \mathbf{W} : \mathbf{B} \cong F(\mathbf{A}) \text{ for some } \mathbf{A} \in \mathbf{E}\}$ between their subvariety lattices. Moreover, F restricts to a category equivalence from $\mathbf{E}$ to $\overline{F}(\mathbf{E})$ for each subvariety $\mathbf{E}$ of $\mathbf{V}$. In this situation, $\mathbf{E}$ is finitely generated if and only if $\overline{F}(\mathbf{E})$ is. It is well known that a category equivalence functor F between quasivarieties preserves the amalgamation, ES, weak ES and strong ES properties. By the remarks above, Theorem 6.2 (ii) and the category equivalence in Theorem 5.10, we have the following result, which generalizes [1, Theorem 7.3].

Theorem 6.3. *Every finitely generated subvariety of the variety* ***SGSM*** *has the ES property.*

The following result is essentially due to Maksimova (see [9, Chapter 6]).

Theorem 6.4 (Maksimova). *The variety all Brouwerian algebras has the amalgamation property.*

Definition 6.5. The variety **CSGSM** of central semiconic generalized Sugihara monoids consists of the algebras in **SGSM** that satisfy $e \leq x^{**} \vee (x^{**} \rightarrow x)$.

The subdirectly irreducible algebras in **CSGSM** are exactly the conic idempotent CRLs $\mathbf{L}$ such that the set $\{a \in L : a < e\}$ has a greatest element and for all $a \in L$, $a^{**} = a$ or $a^* = e$.

Let $\mathbf{U}$ be the subvariety of **SNBrA** axiomatized by $\Diamond(\Diamond x \to x) = e$.

Let $\mathbf{A}$ be a totally ordered Brouwerian algebra and $\mathbf{B}$ be a Brouwerian algebra such that $A \cap B = \{e\}$. The *ordinal sum* $\mathbf{A} \oplus \mathbf{B}$ is the unique Brouwerian algebra with universe $A \cup B$ such that the order $\leq$ of $A \cup B$ restricts to the original order of $\mathbf{A}$ and to that of $\mathbf{B}$, while $a < b$ whenever $e \neq a \in A$ and $b \in B$.

Lemma 6.6. *Let $\mathbf{L} \in \mathbf{U}$ such that $\mathbf{L}$ is [finitely] subdirectly irreducible. Let $L_1 = \{a \in L : \Diamond a = a\}$ and $L_2 = \{a \in L : \Diamond a = e\}$. Then*
1. *$\mathbf{L}_1$ is a chain, $L_1 \cap L_2 = \{e\}$ and $L = L_1 \cup L_2$.*
2. *$\mathbf{L}_1$ and $\mathbf{L}_2$ are subalgebras of $\mathbf{L}$ and $\mathbf{L} = \mathbf{L}_1 \oplus \mathbf{L}_2$.*
3. *If $\mathbf{L}_2$ is nontrivial, then $\mathbf{L}_2$ is [finitely] subdirectly irreducible.*

Proof. (1) Because $\mathbf{L}$ is [finitely] subdirectly irreducible, $\mathbf{L}$ is a $\Diamond$-chain and e is join-irreducible by Theorem 4.2 and Proposition 4.8. Then $\Diamond a \nparallel b$ for all $a, b \in L$. This implies that $\mathbf{L}_1$ is a chain. Let $a \in L_1 \cap L_2$. Then $\Diamond a = a$ and $\Diamond a = e$. Hence $a = e$. Thus $L_1 \cap L_2 = \{e\}$. Let $a \in L$ such that $a \notin L_1$. Then $a < \Diamond a$ by $(N1)$. Hence $\Diamond a \to a = a$ or $\Diamond a \to a \parallel a$ by Proposition 3.2. Suppose that $\Diamond a \neq e$. Then $\Diamond(\Diamond a \to a) = \Diamond a \neq e$ by Lemma 5.2(3). But since $\mathbf{L} \in \mathbf{U}$, $\Diamond(\Diamond a \to a) = e$, a contradiction. Thus $\Diamond a = e$, which implies that $a \in L_2$. Consequently, $L = L_1 \cup L_2$.

(2) Let $a, b \in L_2$. By $(N2)$, $\Diamond(a \wedge b) \leq \Diamond a$ and $\Diamond(a \wedge b) \leq \Diamond b$. Hence $\Diamond(a \wedge b) \leq \Diamond a \wedge \Diamond b = \Diamond a \Diamond b \leq \Diamond(ab) = \Diamond(a \wedge b)$ by $(N3)$ and so $\Diamond(ab) = \Diamond(a \wedge b) = \Diamond a \wedge \Diamond b = e$, which implies that $ab, a \wedge b \in L_2$. By $(SN1)$, $\Diamond(a \vee b) = \Diamond a \vee \Diamond b = e$. Hence $a \vee b \in L_2$. If $a \leq b$, then $a \to b = e$ and so $a \to b \in L_2$. If $b < a$ or $a \parallel b$, then by Proposition 3.2, $a \to b = b$ or $a \to b \parallel a$. If $a \to b = b$, then $a \to b \in L_2$. If $a \to b \parallel a$, then since $\mathbf{L}$ is a $\Diamond$-chain, $\Diamond(a \to b) = \Diamond a = e$ by Lemma 5.2(3). Hence $a \to b \in L_2$. Consequently, $\mathbf{L}_2$ is a subalgebra of $\mathbf{L}$. Similarly, $\mathbf{L}_1$ is a subalgebra of $\mathbf{L}$. We claim that for all $a \in L_2$ and $b \in L_1 \backslash \{e\}$, $b < a$. Since $\Diamond b = b$, $b \nparallel a$ by Proposition 4.8. Suppose that there exist $b \in L_1 \backslash \{e\}$ and $a \in L_2$ such that $b \geq a$. Then by $(N2)$, $e = \Diamond a \leq \Diamond b = b < e$, a contradiction. Thus for all $a \in L_2$ and $b \in L_1 \backslash \{e\}$, $b < a$. It follows that $\mathbf{L} = \mathbf{L}_1 \oplus \mathbf{L}_2$.

(3) Let $\mathbf{L} \in \mathbf{U}$ such that $\mathbf{L}$ is finitely subdirectly irreducible and $\mathbf{L}_2$ is nontrivial. Suppose that $\mathbf{L}_2$ is not finitely subdirectly irreducible. Then there exist $a, b \in L_2$ such that $a \parallel b$ and $a \vee b = e$. Since $\mathbf{L} = \mathbf{L}_1 \oplus \mathbf{L}_2$, e is not join-irreducible in $\mathbf{L}$, which implies that $\mathbf{L}$ is not finitely subdirectly irreducible. It is a contradiction. Consequently, $\mathbf{L}_2$ is finitely subdirectly irreducible. Similarly, if $\mathbf{L} \in \mathbf{U}$ such that $\mathbf{L}$ is subdirectly irreducible and $\mathbf{L}_2$ is nontrivial, then $\mathbf{L}_2$ is subdirectly irreducible. $\square$

By Lemma 6.6, the subdirectly irreducible algebras in $\mathbf{U}$ are exactly the strong nuclear Brouwerian algebras $\mathbf{L}$ such that the set $\{a \in L : a < e\}$ has a greatest element and for each $a \in L$, the element $\Diamond a$ is a or e.

Theorem 6.7. ***CSGSM** is categorically equivalent to **U**.*

Proof. Let $\mathbf{L} \in \mathbf{SGSM}$. Then $\mathbf{L} \cong \mathbf{S}(\mathbf{L}^-_\Diamond)$, by Theorem 5.7. So by the above remarks, $\mathbf{L}$ is a subdirectly irreducible member of **CSGSM** if and only if $\mathbf{L}^-_\Diamond$ is a subdirectly irreducible member of $\mathbf{U}$, and the result follows from [8, Theorem 9.3]. □

Proposition 6.8. *Let $\boldsymbol{L}_1$ be a totally ordered strong nuclear Brouwerian algebra such that for all $a \in L_1, \Diamond a = a$. Let $\boldsymbol{L}_2 \in \boldsymbol{SNBrA}$ such that for all $b \in L_2, \Diamond b = e$. Let $\boldsymbol{L} = \boldsymbol{L}_1 \oplus \boldsymbol{L}_2$. Let $\boldsymbol{L}^\Diamond$ be the expansion of $\boldsymbol{L}$ in which $\Diamond a = a$ for all $a \in L_1$ and $\Diamond b = e$ for all $b \in L_2$. Then*

1. *$\boldsymbol{L}^\Diamond \in \boldsymbol{U}$;*
2. *if $\boldsymbol{L}_2$ is nontrivial, then $\boldsymbol{L}^\Diamond$ is [finitely] subdirectly irreducible if and only if $\boldsymbol{L}_2$ is [finitely] subdirectly irreducible.*

Proof. (1) It is clear that $\mathbf{L}^\Diamond \in \mathbf{SNBrA}$. Let $a \in L$. If $a \in L_1$, then $\Diamond(\Diamond a \to a) = \Diamond(a \to a) = \Diamond e = e$. If $a \in L_2$, then $\Diamond(\Diamond a \to a) = \Diamond(e \to a) = \Diamond a = e$. Hence $\mathbf{L}^\Diamond \in \mathbf{U}$.

(2) By Lemma 6.6 (3) and the definition of $\mathbf{L}^\Diamond$, $\mathbf{L}^\Diamond$ is [finitely] subdirectly irreducible if and only if $\mathbf{L}_2$ is [finitely] subdirectly irreducible. □

For our purpose, we need the following result, which is essentially due to Maksimova (see [9, Chapter 6], [16] and [8, p.3213] for clarification).

Theorem 6.9 (Maksimova). *Let $\boldsymbol{K}$ be a variety consisting of expansions of Brouwerian algebras, where every member of $\boldsymbol{K}$ has the same congruences as its Brouwerian algebra reduct. Then the following conditions are equivalent:* (i) *$\boldsymbol{K}$ has the amalgamation property.* (ii) *The class of finitely subdirectly irreducible algebras in $\boldsymbol{K}$ has the amalgamation property.*

Lemma 6.10. *The variety $\boldsymbol{U}$ has the amalgamation property.*

Proof. Let $\overline{\mathbf{U}}$ be the class of all finitely subdirectly irreducible algebras in $\mathbf{U}$. Let $\mathbf{L}_1, \mathbf{L}_2 \in \overline{\mathbf{U}}$, and let $\mathbf{L}_0$ be a subalgebra both of $\mathbf{L}_1$ and of $\mathbf{L}_2$. Then $\mathbf{L}_0 \in \overline{\mathbf{U}}$. By Theorem 6.9, we only need show that there exist $\mathbf{B} \in \overline{\mathbf{U}}$ and embeddings $h_1 : \mathbf{L}_1 \to \mathbf{B}$ and $h_2 : \mathbf{L}_2 \to \mathbf{B}$, with $h_1 |_{L_0} = h_2 |_{L_0}$. We may assume without loss of generality that $L_0 = L_1 \cap L_2$.

By Lemma 6.6, we have $\mathbf{L}_i = \mathbf{L}_i' \oplus \mathbf{L}_i''$, where $i \in \{0, 1, 2\}$, the sets

$$L_i'' := \{a \in L_i : \Diamond a = e\} \quad \text{and} \quad L_i' := (L_i \backslash L_i'') \cup \{e\}$$

are subuniverses of $\mathbf{L}_i$ and, because $\mathbf{L}_0$ is a subalgebra of $\mathbf{L}_i$, we have

$$L_0' = L_1' \cap L_2' \quad \text{and} \quad L_0'' = L_1'' \cap L_2''.$$

Let $\mathbf{L}_i'$ and $\mathbf{L}_i''$ be the **BrA**-subreducts of $\mathbf{L}_i$ whose universes are L_i' and L_i'', respectively. Since by Lemma 6.6, $\mathbf{L}_i'' \in \mathbf{BrA}$ and are finitely subdirectly irreducible algebras, by Theorems 6.4 and 6.9, there exists a finitely subdirectly irreducible algebra $\mathbf{B}'' \in \mathbf{BrA}$ such that, for each $i \in \{1, 2\}$, there are embeddings $g_i'' : \mathbf{L}_i'' \to \mathbf{B}''$ with $g_1'' |_{L_0''} = g_2'' |_{L_0''}$. By Lemma 6.6, $\mathbf{L}_i'$ are chains. Hence by Theorem 6.9 and [8, Theorem 12.7], there exists a totally ordered algebra $\mathbf{B}' \in \mathbf{BrA}$ such that, for each $i \in \{1, 2\}$, there are embeddings $g_i' : \mathbf{L}_i' \to \mathbf{B}'$ with $g_1' |_{L_0'} = g_2' |_{L_0'}$. Let $\mathbf{B} = \mathbf{B}' \oplus \mathbf{B}''$, so $\mathbf{B} \in \mathbf{BrA}$. Let $\mathbf{B}^\Diamond$ be the expansion of $\mathbf{B}$ in which $\Diamond b = b$ for all $b \in B'$ and $\Diamond b = e$ for all $b \in B''$, so by Proposition 6.8, $\mathbf{B}^{\Diamond} \in \overline{\mathbf{U}}$. For each $i \in \{1, 2\}$, the relation $h_i = g_i' \cup g_i''$ is a function embedding $\mathbf{L}_i$ into $\mathbf{B}^{\Diamond}$, and $h_1 |_{L_0} = h_2 |_{L_0}$. It follows that the variety $\mathbf{U}$ has the amalgamation property. □

By Theorem 6.7, we have the following result, which generalizes [8, Theorem 13.3].

Theorem 6.11. *The variety* ***CSGSM*** *has the strong amalgamation property, and therefore the strong ES property.*

Proof. Amalgamation follows from Lemma 6.10 and the category equivalence in Theorem 6.7. Then strong amalgamation and the strong *ES* property follow from Theorems 6.2 and 6.1. □

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