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The fine- and generative spectra of varieties of monounary algebras

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Abstract. In this paper we present recursive formulas to compute the fine spectrum and generative spectrum of each of the varieties of monounary algebras. Hence, an asymptotic or log-asymptotic estimation for the number of n-generated and n-element algebras is given in every variety of monounary algebras. These results provide infinitely many examples of spectra with different orders of magnitude that are asymptotically bigger than any polynomial and smaller than any exponential function.

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1. Introduction

For a variety of algebras $\mathcal{V}$ let $g_{\mathcal{V}}(n)$ denote the number of n-generated algebras in $\mathcal{V}$, and let $f_{\mathcal{V}}(n)$ denote the number of n-element algebras in $\mathcal{V}$ up to isomorphism. The sequences $(g_{\mathcal{V}}(n))_{n\in\mathbb{N}}$ and $(f_{\mathcal{V}}(n))_{n\in\mathbb{N}}$ are called the *generative spectrum* and the *fine spectrum* of $\mathcal{V}$, respectively. For a detailed introduction into generative- and fine spectra, see [1]. The asymptotic behaviour of these sequences for certain varieties of algebras is often strongly related to the algebraic properties of the structures in the variety. For example, a finitely generated variety $\mathcal{V}$ of groups is nilpotent if and only if $g_{\mathcal{V}}(n)$ is at most polynomial, and a finite ring R generates a variety with at most exponential

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generative spectrum if and only if the square of the Jacobson radical of R is trivial [1]. The infinite counterpart of our problems is widely investigated in model theory. The famous Vaught conjecture says that the cardinality of the set of non-isomorphic models of any first-order theory in a countable language is either countable or continuum. In [4,5] the conjecture is verified for varieties of algebras.

A monounary algebra, $\mathcal{A} = (A; u)$ is an algebra with a single unary operation u. The theory of monounary algebras is well-developed, for a recent monograph see [9]. All varieties of monounary algebras were classified by Jacobs and Schwabauer [7]. Every variety of monounary algebras can be defined by a single identity. The variety $\mathcal{V}_{k,d}$ is defined by the equation $u^{k+d}(x) = u^k(x)$, and the variety $\mathcal{V}_k$ is defined by the equation $u^k(x) = u^k(y)$, where $u^0 = \mathrm{id}, u^1 = u$, and in general $u^{n+1} = u \circ u^n$.

In [6] a formula was obtained for the number of n-element monounary algebras. Let M_n and C_n denote the number of monounary algebras and connected monounary algebras, respectively. It was shown in [6] that $\log_\alpha C_n \sim \log_\alpha M_n \sim n$ for a constant $\alpha \approx 2.95576$. In our terminology, this result shows the log-asymptotic behaviour of the fine spectrum of the variety $\mathcal{V}_{0,0}$, the class of all monounary algebras. In [1], several results were proven about the growth rate of the generative spectrum of varieties. In many cases, the spectrum is at most polynomial (e.g., pure sets, vector spaces over finite fields) or at least exponential (e.g., Boolean algebras, semilattices). The variety $\mathcal{V}_2$ is mentioned in [1] as an interesting example for a locally finite variety whose generative spectrum is bigger than any polynomial and smaller than any exponential function. It was explicitly calculated there that the number of non-isomorphic n-generated algebras in $\mathcal{V}_2$ is bigger than $p(n)$ and smaller than $(n+1)^2 p(n)$, where $p(n)$ is the number of partitions of n. An asymptotic formula for the fine spectrum of $\mathcal{V}_2$ and the log-asymptotic behaviour of the fine spectrum of $\mathcal{V}_k$ were determined in [10] for all k.

The goal of the present paper is to obtain a recursive formula for the generative spectrum and fine spectrum of all the varieties $\mathcal{V}_{k,d}$ and $\mathcal{V}_k$, and to determine the log-asymptotic behaviour of these sequences. In some cases, we can even determine the asymptotic behaviour or provide an explicit formula for the fine- and generative spectra. The main results are presented in Theorems 5.1, 5.2, 6.1, 6.3.

2. Description of the varieties

2.1. Monounary algebras as directed graphs

The function u defines a directed graph on A. Let $G_A = (A; E)$, the vertex set is A and the edges are $E = \{(a, u(a)) \mid a \in A\}$. In G_A every vertex has out-degree 1, and every directed graph G with all vertices having out-degree 1 defines a monounary algebra on its vertex set, where $u(a)$ is the single vertex such that $(a, u(a))$ is an edge in G. Hence, a monounary algebra can be identified with a directed graph, where each vertex has out-degree 1. This identification gives rise to a number of notions. The algebra $(A; u)$ is connected if the graph G_A is connected. More generally, the connected components of $(A; u)$ are the connected components of G_A.

Let B be a connected component of $(A; u)$ with a cycle of length d. Then the graph G_B contains exactly one cycle. If we remove edges of this cycle in G_B, then we obtain d rooted trees whose edges are directed towards the root. These trees give a partition of the component B into d sets. We call this partition the cyclic partition of B. The roots of the trees in the cyclic partition are the vertices of the cycle, and an element a in the connected component is in the rooted tree with root r if and only if r is the first element of the cycle in the sequence $(u^k(a))_{k=0}^{\infty}$.

2.2. Varieties of monounary algebras

The notion of an equational class goes back to Birkhoff [2], who has shown that a class of algebras can be defined by a set of equations if and only if the class is closed under taking homomorphic images, subalgebras and (possibly infinite) direct products. Such classes are also called varieties. According to [7], every variety of monounary algebras can be defined by a single equation. Based on this result, they have given the following exhaustive list of varieties of monounary algebras.

- The varieties $\mathcal{V}_{k,d}$ are defined by the equation $u^k(x) = u^{k+d}(x)$, for $k \geq 0, d \geq 1$. An algebra $(A; u)$ is in $\mathcal{V}_{k,d}$ if and only if every connected component B of $(A; u)$ contains a cycle whose length divides d, and every rooted tree in the cyclic partition of B is of depth at most k. In particular, every n-generated algebra in $\mathcal{V}_{k,d}$ has at most $n(k+d)$ elements, showing that the generative spectra of these varieties are indeed sequences of integers.
- The class of all monounary algebras is $\mathcal{V}_{0,0}$ defined by the equation $x = x$. As there are infinitely many n-generated algebras in $\mathcal{V}_{0,0}$ for all n, the generative spectrum of this variety is not defined. In view of [6], the fine spectrum of $\mathcal{V}_{0,0}$ is $(M_n)_{n\in\mathbb{N}}$ and $\log M_n \sim n \log \alpha$, where $\alpha \approx 2.95576$.
- The varieties $\mathcal{V}_k$ are defined by the equation $u^k(x) = u^k(y)$, for $k \geq 1$. The classes $\mathcal{V}_k$ consist of connected monounary algebras. If $(A; u) \in \mathcal{V}_k$, then the cycle of $(A; u)$ is a loop, i.e., a single vertex r with $u(r) = r$. Thus G_A is a rooted tree with root r. This leads to the following combinatorial description: $(A; u) \in \mathcal{V}_k$ if and only if G_A is a rooted tree of depth at most k. In particular, the number of n-element algebras in $\mathcal{V}_k$ equals to the number of n-element rooted trees of depth at most k. The log-asymptotic behaviour of the fine spectrum of $\mathcal{V}_k$ was determined in [10]. The log-asymptotic behaviour of the generative spectrum can be computed in a similar fashion. As a preliminary observation, we note that the two spectra are not very far from each other, because an n-generated algebra has at most $nk + 1$ elements in this variety, and every n-element algebra is n-generated. The detailed computation and the results are presented in Sections 5 and 6.

- $\mathcal{V}_0$ consists of the isomorphism type of the one-element algebra, and it is defined by the equation $x = y$. The problem of computing the generative spectrum and fine spectrum of $\mathcal{V}_0$ is trivial.

  Note that pseudovarieties of monounary algebras are described in [8].

3. Generating functions

Definition 3.1. Throughout the paper log denotes the natural logarithm function, and L_m denotes the m-fold iterated logarithm function, namely $L_m(x) = \log\log\ldots\log x$. The exponential function e^x is denoted by $\exp(x)$. The number of positive divisors of n is denoted by $\tau(n)$.

We use the symbols $\sim, o(.), O(.)$ in the standard way. Given two series $a, b : \mathbb{N} \to \mathbb{R}$ with $b_n \neq 0$ for all but finitely many n, we put $a_n \sim b_n$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$, $a_n = o(b_n)$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, and $a_n = O(b_n)$ if $\frac{a_n}{b_n}$ is bounded. Furthermore, the expression "log-asymptotic behaviour of a_n" refers to an asymptotic estimation of the sequence $(\log a_n)_{n\in\mathbb{N}}$, i.e., a sequence $(b_n)_{n\in\mathbb{N}}$ with $\log a_n \sim b_n$. Note that finding such a sequence $(b_n)_{n\in\mathbb{N}}$ is usually less demanding than finding a sequence $(c_n)_{n\in\mathbb{N}}$ with $a_n \sim c_n$, or even one with $a_n = O(c_n)$: if $a_n = O(c_n)$, $c_n > 0$ for all but finitely many n and $\log a_n \to \infty$, then clearly $\log a_n \sim b_n$ for $b_n = \log c_n$.

Definition 3.2.
- For $k \geq 0$, $f_k(n)$ is the number of non-isomorphic n-element algebras in $\mathcal{V}_k$, which equals to the number of n-element rooted trees of depth at most k. The generating function of the sequence $(f_k(n))_{n=1}^{\infty}$ is denoted by $F_k(x) = \sum_{n=1}^{\infty} f_k(n)x^n$.
- For $k \geq 0$, $g_k^*(n)$ is the number of rooted trees of depth at most k with exactly n leaves, i.e., vertices with in-degree 0 in this context. Note that, by definition, the singleton tree has one leaf. The generating function of the sequence $(g_k^*(n))_{n=1}^{\infty}$ is denoted by $G_k^*(x) = \sum_{n=1}^{\infty} g_k^*(n)x^n$.
- For $k \geq 0$, $g_k(n)$ is the number of rooted trees of depth at most k with at most n leaves, which equals to the number of n-generated algebras in $\mathcal{V}_k$. The generating function of the sequence $(g_k(n))_{n=1}^{\infty}$ is denoted by $G_k(x) = \sum_{n=1}^{\infty} g_k(n)x^n$.
- For $k \geq 0, d \geq 0$, $f_{k,d,\mathrm{con}}(n)$ is the number of non-isomorphic connected n-element algebras in $\mathcal{V}_{k,d}$, which equals to the number of n-element digraphs with a directed cycle of length dividing d, such that by omitting the edges of the cycle the graph is partitioned into rooted trees of depth at most k, and the edges of each tree are directed towards the root. The generating function of the sequence $(f_{k,d,\mathrm{con}}(n))_{n=1}^{\infty}$ is denoted by $F_{k,d,\mathrm{con}}(x) = \sum_{n=1}^{\infty} f_{k,d,\mathrm{con}}(n)x^n$.
- For $k \geq 0, d \geq 0$, $f_{k,d}(n)$ is the number of non-isomorphic n-element algebras in $\mathcal{V}_{k,d}$. The generating function of the sequence $(f_{k,d}(n))_{n=1}^{\infty}$ is denoted by $F_{k,d}(x) = \sum_{n=1}^{\infty} f_{k,d}(n)x^n$.
- For $k \geq 0, d \geq 0$, $g_{k,d,\mathrm{con}}^*(n)$ is the number of non-isomorphic connected n-generated but not $(n-1)$-generated algebras in $\mathcal{V}_{k,d}$, which equals to the number of digraphs with exactly n leaves, containing a directed cycle

of length dividing d, such that by omitting the edges of the cycle the graph is partitioned into rooted trees of depth at most k, and the edges of each tree are directed towards the root. The generating function of the sequence $(g^*_{k,d,\mathrm{con}}(n))_{n=1}^\infty$ is denoted by $G^*_{k,d,\mathrm{con}}(x) = \sum_{n=1}^\infty g^*_{k,d,\mathrm{con}}(n)x^n$.

- For $k \geq 0, d \geq 0$, $g^*_{k,d}(n)$ is the number of non-isomorphic n-generated but not $(n-1)$-generated algebras in $\mathcal{V}_{k,d}$. The generating function of the sequence $(g^*_{k,d}(n))_{n=1}^\infty$ is denoted by $G^*_{k,d}(x) = \sum_{n=1}^\infty g^*_{k,d,\mathrm{con}}(n)x^n$.
- For $k \geq 0, d \geq 0$, $g_{k,d}(n)$ is the number of non-isomorphic n-generated algebras in $\mathcal{V}_{k,d}$. The generating function of the sequence $(g_{k,d}(n))_{n=1}^\infty$ is denoted by $G_{k,d}(x) = \sum_{n=1}^\infty g_{k,d,\mathrm{con}}(n)x^n$.

There are several recurrence formulas for the sequences defined in Definition 3.2, which we use to obtain the asymptotic estimations. All of these formulas can be written up in terms of the power series of the sequences.

Lemma 3.3. *The formal power series defined in Definition 3.2 satisfy the following formulas coefficient-wise for all integers $k, d \geq 0$.*

(1) $F_{k+1}(x) = x\exp(\sum_{m=1}^\infty \frac{1}{m} F_k(x^m))$.
(2) $G^*_{k+1}(x) = \exp(\sum_{m=1}^\infty \frac{1}{m} G^*_k(x^m)) + x - 1$.
(3) $F_{k,1,\mathrm{con}}(x) = F_k(x)$.
(4) $\frac{1}{d}(F_{k,1,\mathrm{con}}(x))^d \leq F_{k,d,\mathrm{con}}(x) \leq \sum_{t|d}(F_{k,1,\mathrm{con}}(x))^t$.
(5) $F_{k,d}(x) = \exp(\sum_{m=1}^\infty \frac{1}{m} F_{k,d,\mathrm{con}}(x^m)) - 1$.
(6) $G^*_{k,1,\mathrm{con}}(x) = G^*_k(x)$.
(7) $\frac{1}{d}(G^*_{k,1,\mathrm{con}}(x))^d \leq G^*_{k,d,\mathrm{con}}(x) \leq \sum_{t|d}(G^*_{k,1,\mathrm{con}}(x))^t$.
(8) $G^*_{k,d}(x) = \exp(\sum_{m=1}^\infty \frac{1}{m} G^*_{k,d,\mathrm{con}}(x^m)) - 1$.

Proof. Item (1) is shown in [10], see Theorem 2.2. The proof of item (2) is analogous.

Items (3) and (6) are straightforward from Definitions 3.2.

The proofs of items (5) and (8) are based on a similar argument, thus we only show item (5). For $1 \leq i \leq n$ let μ_i be the number of i-element connected components in the algebra $(A; u)$. Up to isomorphism, $(A; u)$ is determined by the isomorphism types of its connected components. There are $\binom{f_{k,d,\mathrm{con}}(j)+\mu_j-1}{\mu_j}$ ways to choose μ_j connected algebras in $\mathcal{V}_{k,d}$ of size j. Thus $f_{k,d}(n) = \sum_{\sum i\mu_i = n} \prod_{j=1}^n \binom{f_{k,d,\mathrm{con}}(j)+\mu_j-1}{\mu_j}$. According to the generalised binomial theorem, for every $|x| < 1$ we have that $(1-x^j)^{-f_{k,d,\mathrm{con}}(j)} = \sum_{\mu_j=0}^\infty \binom{-f_{k,d,\mathrm{con}}(j)}{\mu_j} \cdot \left(-x^j\right)^{\mu_j} = \sum_{\mu_j=0}^\infty \binom{f_{k,d,\mathrm{con}}(j)+\mu_j-1}{\mu_j} x^{j\mu_j}$. Thus for $n \geq 1$, $f_{k,d}(n)$ equals to the n-th coefficient in the power series $\prod_{j=1}^\infty (1-x^j)^{-f_{k,d,\mathrm{con}}(j)}$. The constant term of the power series $\prod_{j=1}^\infty (1-x^j)^{-f_{k,d,\mathrm{con}}(j)}$ is 1. Hence, $F_{k,d}(x) = \prod_{j=1}^\infty (1-x^j)^{-f_{k,d,\mathrm{con}}(j)} - 1 = \exp(\sum_{j=1}^\infty \log(1-x^j)^{-f_{k,d,\mathrm{con}}(j)}) - 1 = \exp(\sum_{j=1}^\infty f_{k,d,\mathrm{con}}(j)(-\log(1-x^j))) - 1$. By expanding $-\log(1-x)$ we obtain $F_{k,d}(x) = \exp(\sum_{j=1}^\infty f_{k,d,\mathrm{con}}(j) \sum_{m=1}^\infty \frac{1}{m} x^{jm}) - 1 = \exp(\sum_{m=1}^\infty \frac{1}{m} F_{k,d,\mathrm{con}}(x^m)) - 1$.

Finally, the proofs of items (4) and (7) are similar, thus we only show item (4). Let $(A; u)$ be a connected algebra in $\mathcal{V}_{k,d}$ such that the length of its cycle is t. Then $t|d$. Let $r_1,\dots,r_t$ be an enumeration of the elements of the cycle of $(A;u)$ such that $u(r_1)=r_2,\dots,u(r_t)=r_1$. This enumeration depends on the choice of r_1. By omitting the edges of the cycle of $(A;u)$, we obtain a partition of G_A into t rooted trees of depth at most k. The isomorphism type of the rooted tree with root r_i is denoted by x_i. Let us assign the t-tuple $(x_1,\dots,x_t)$ to $(A;u)$. Depending on the choice of r_1, it might be possible to assign more than one tuple to $(A;u)$. As there are t ways to choose r_1 with $t|d$, the number of tuples assigned to an algebra in $\mathcal{V}_{k,d}$ is at most d. Up to isomorphism, the algebra $(A;u)$ is uniquely determined by any of its assigned tuples. For $t|d$ let $S_{k,t}(n)$ be the set of tuples $(x_1,\dots,x_t)$ of isomorphism types of rooted trees with n elements altogether and of depth at most k. Let $s_{k,t}(n)=|S_{k,t}(n)|$. Every tuple in $S_{k,t}(n)$ is assigned to an n-element algebra in $\mathcal{V}_{k,d}$. Hence, the above argument shows that $\frac{1}{d}s_{k,d}(n)\le f_{k,d,\mathrm{con}}(n)\le\sum_{t|d}s_{k,t}(n)$. The number of tuples $(x_1,\dots,x_t)\in S_{k,t}(n)$ such that a rooted tree with isomorphism type x_i has μ_i vertices is $\prod_{i=1}^t f_{k,1,\mathrm{con}}(\mu_i)$. Thus $s_{k,t}(n)=\sum_{\mu_1+\dots+\mu_t=n}\prod_{i=1}^t f_{k,1,\mathrm{con}}(\mu_i)$, which is the n-th coefficient in the power series $(F_{k,1,\mathrm{con}}(x))^t$. □

The techniques used in Lemma 3.3 can be found in [3]. The following theorem is from [10]. Although in [10] these assertions were only shown for specific values of the parameters, the proof works in full generality without any modification.

Theorem 3.4. *Let $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ be sequences of positive integers, and let $A(x)=\sum_{n=1}^\infty a_nx^n$ and $B(x)=\sum_{n=1}^\infty b_nx^n$ be the generating functions of these sequences. Assume that $B(x)=\exp(\sum_{m=1}^\infty\frac{1}{m}A(x^m))$.*

(1) *If $\log a_n\sim C\sqrt{n}$ for some $C>0$, then $\log b_n\sim\frac{C^2}{4}\frac{n}{\log n}$.*
(2) *For $k\ge1$, if $\log a_n\sim C\frac{n}{L_k(n)}$ for some $C>0$, then $\log b_n\sim C\frac{n}{L_{k+1}(n)}$.*

We note that in [10] $B(x)$ is of the form $x\cdot\exp(\sum_{m=1}^\infty\frac{1}{m}A(x^m))$, that item (1) is proved for $a_n=f_2(n)$, $C=\pi\sqrt{\frac{2}{3}}$, and that item (2) is proved for $a_n=f_{k-2}(n)$, $k>4$, $C=\frac{\pi^2}{6}$.

4. Auxiliary computations

Lemma 4.1. *Let $K,C\in\mathbb{R}^+$, $s\in\mathbb{R}$. Let $a_n\sim Kn^s\exp(C\sqrt{n})$, and let $b_n=\sum_{i=1}^n a_i$. Then $b_n\sim\frac{2K}{C}n^{s+1/2}\exp(C\sqrt{n})$.*

Proof. As $a_n\to\infty$ and $a_n\sim Kn^s\exp(C\sqrt{n})$, we have that $b_n=\sum_{i=1}^n a_i\sim\sum_{i=1}^n Ki^s\exp(C\sqrt{i})$. Let $n_0=\lfloor n-2n^{2/3}+n^{1/3}\rfloor$. Then for all $i\le n_0$ we have $C\sqrt{i}\le C\sqrt{n}-Cn^{\frac{1}{6}}$ and $Ki^s\le Kn^s+K$. Thus $b_{n_0}=O(\sum_{i\le n_0}Ki^s\exp(C\sqrt{i}))$ $\le O((Kn^s+K)\exp(C\sqrt{n})\exp(-Cn^{1/6}))=o(a_n)$. Hence, by the monotonicity of n^s, and by using $n^s\sim n_0^s$, we obtain that $b_n\sim\sum_{n_0<i\le n}Ki^s\exp(C\sqrt{i})\sim$ $n^{s+1/2}\sum_{n_0<i\le n}\frac{K}{\sqrt{i}}\exp(C\sqrt{i})\sim n^{s+1/2}\sum_{i=1}^n\frac{K}{\sqrt{i}}\exp(C\sqrt{i})$.

The function $\frac{K}{\sqrt{x}}\exp(C\sqrt{x})$ is monotone for x large enough, thus replacing the sum $\sum_{i=1}^{n}\frac{K}{\sqrt{i}}\exp(C\sqrt{i})$ with the integral $\int\limits_{x=1}^{n}\frac{K}{\sqrt{x}}\exp(C\sqrt{x})$ introduces an error of order of magnitude $O(\frac{\exp(C\sqrt{n})}{\sqrt{n}})$. As $\frac{2K}{C}\exp(C\sqrt{x})$ is a primitive function of $\frac{K}{\sqrt{x}}\exp(C\sqrt{x})$, we have $\sum_{i=1}^{n}\frac{K}{\sqrt{i}}\exp(C\sqrt{i}) = O(\frac{\exp(C\sqrt{n})}{\sqrt{n}}) + \int\limits_{x=1}^{n}\frac{K}{\sqrt{x}}\exp(C\sqrt{x}) = O(\frac{\exp(C\sqrt{n})}{\sqrt{n}}) + \frac{2K}{C}\exp(C\sqrt{n})$. Consequently, the error $O(\frac{\exp(C\sqrt{n})}{\sqrt{n}})$ is negligible, and we obtain $\sum_{i=1}^{n}\frac{K}{\sqrt{i}}\exp(C\sqrt{i}) \sim \frac{2K}{C}\exp(C\sqrt{n})$, and $b_n \sim n^{s+1/2}\sum_{i=1}^{n}\frac{K}{\sqrt{i}}\exp(C\sqrt{i}) \sim \frac{2K}{C}n^{s+1/2}\exp(C\sqrt{n})$. □

Lemma 4.2. *Let $d \in \mathbb{N}$. Then $\max_{n_1+\cdots+n_d=n}\sum_{i=1}^{d}\sqrt{n_i} \sim \sqrt{dn}$ as $n \to \infty$.*

Proof. The function $\sqrt{x}$ is concave. This allows us to use Jensen's inequality. We consider all weigths equal to 1 and obtain

$$\sqrt{\frac{\sum_{i=1}^{d} n_i}{d}} \geq \frac{\sum_{i=1}^{d}\sqrt{n_i}}{d}$$

Thus $\sum_{i=1}^{d}\sqrt{n_i} \leq \sqrt{dn}$.

Let $n = kd + r$ with $k, r \in \mathbb{N}, r < d$. Put $n_i = k + 1$ for $i = 1, \ldots, r$ and $n_i = k$ for $i = r + 1, \ldots, d$. We have $\sum_{i=1}^{d} n_i = kd + r = n$, and moreover $\sum_{i=1}^{d}\sqrt{n_i} \geq d\sqrt{k} = \sqrt{d(n-r)} \geq \sqrt{d(n-d)} \sim \sqrt{dn}$ as $n \to \infty$. □

Lemma 4.3. *Let $d \in \mathbb{N}, k \geq 1$. Let $(h(n))_{n\in\mathbb{N}}$ be a sequence such that $h(n) \sim C\frac{n}{L_k(n)}$ for some $C > 0$. Then $\max_{n_1+\cdots+n_d=n}\sum_{i=1}^{d} h(n_i) \sim C\frac{n}{L_k(n)}$ as $n \to \infty$.*

Proof. Let $\varepsilon > 0$. By calculating the derivative and the second derivative of the function $\frac{x}{L_k(x)}$, it can be shown that there exists a positive constant x_k such that $\frac{x}{L_k(x)}$ is positive, strictly monotone increasing and strictly concave on (x_k, ∞). Moreover, assume that x_k is large enough so that $|\frac{h(n)}{Cn/L_k(n)} - 1| < \varepsilon$ for all $x_k \leq n$. Let $M = \max(1, \max_{i\in[1,x_k]} h(i))$. Let $n > d(x_k + 1)$ be arbitrary. Let $n_1 \geq n_2 \geq \cdots \geq n_d$ be such that $\sum_{i=1}^{d} n_i = n$. As $n > d(x_k + 1)$, there exists a $1 \leq t \leq d$ such that $n_i > x_k$ if and only if $i \leq t$. We give an upper bound for $\sum_{i=1}^{d} h(n_i)$.

By using the trivial estimation $h(n_i) \leq M$ for $i > t$, we have $\sum_{i=1}^{d} h(n_i) \leq dM + \sum_{i=1}^{t} h(n_i) \leq dM + \sum_{i=1}^{t}(1+\varepsilon)C\frac{n_i}{L_k(n_i)}$. Thus according to Jensen's inequality $\sum_{i=1}^{d} h(n_i) \leq dM + (1+\varepsilon)C\sum_{i=1}^{t}\frac{n_i}{L_k(n_i)} \leq dM + (1+\varepsilon)Ct\,(\frac{1}{t}\sum_{i=1}^{t}\frac{n_i}{L_k(n_i)}) \leq dM + (1+\varepsilon)Ct\frac{n/t}{L_k(n/t)} = dM + (1+\varepsilon)C\frac{n}{L_k(n/t)}$.

As the n_i were arbitrary, we have that $\max_{n_1+\cdots+n_d=n}(\sum_{i=1}^{d} h(n_i)) \leq dM + (1+\varepsilon)C\frac{n}{L_k(n/t)} \sim (1+\varepsilon)C\frac{n}{L_k(n)}$. A similar lower bound can be shown by setting all the n_i so that the difference of any two of them is at most 1. Hence, the lower estimation that we obtain in this way is asymptotically $(1-\varepsilon)C\frac{n}{L_k(n)}$. As $\varepsilon>0$ was arbitrary, we have that $\max_{n_1+\cdots+n_d=n}(\sum_{i=1}^{d} h(n_i)) \sim C\frac{n}{L_k(n)}$. $\square$

Lemma 4.4. *Let $\tau \in \mathbb{N}$, and let $1 = d_1, d_2, \ldots, d_\tau$ be natural numbers. For $n \in \mathbb{N}$ let $w_{d_1,\ldots,d_\tau}(n)$ be the number of tuples $(\alpha_1,\ldots,\alpha_\tau)$ of non-negative integers such that $\alpha_1 d_1 + \cdots + \alpha_\tau d_\tau = n$. Then $w_1(n) = 1$ for all $n \in \mathbb{N}$ and for $\tau \geq 2$ we have $w_{d_1,\ldots,d_\tau}(n) = \frac{1}{(\tau-1)!d_1d_2\cdots d_\tau}n^{\tau-1} + O(n^{\tau-2})$.*

Proof. We prove the statement by induction on τ. By definition, $w_1(n) = 1$ for all $n \in \mathbb{N}$. Let $\tau = 2$. Then we have $\lfloor \frac{n}{d_2} \rfloor + 1$ choices for α_2, and α_1 is uniquely determined by α_2. Thus $w_{1,d_2}(n) = \lfloor \frac{n}{d_2} \rfloor + 1 = \frac{n}{d_2} + O(1)$.

Assume that $\tau \geq 3$, and that the assertion is true for $(\tau-1)$. We show that the statement holds for τ. By rearranging the terms of $\alpha_1 d_1 + \cdots + \alpha_\tau d_\tau = n$ we obtain $\alpha_1 d_1 + \cdots + \alpha_{\tau-1} d_{\tau-1} = n - \alpha_\tau d_\tau$. Thus

$$\begin{aligned}
w_{d_1,\ldots,d_\tau}(n) &= \sum_{\alpha_\tau=0}^{\lfloor n/d_\tau \rfloor} w_{d_1,\ldots,d_{\tau-1}}(n-\alpha_\tau d_\tau)\\
&= \sum_{\alpha_\tau=0}^{\lfloor n/d_\tau \rfloor} \frac{1}{(\tau-2)!d_1d_2\cdots d_{\tau-1}}(n-\alpha_\tau d_\tau)^{\tau-2} + O(n^{\tau-2})\\
&= \frac{d_\tau^{\tau-2}}{(\tau-2)!d_1d_2\cdots d_{\tau-1}} \sum_{\alpha_\tau=0}^{\lfloor n/d_\tau \rfloor} \left(\frac{n}{d_\tau} - \alpha_\tau\right)^{\tau-2} + O(n^{\tau-2})\\
&= \frac{d_\tau^{\tau-2}}{(\tau-2)!d_1d_2\cdots d_{\tau-1}} \int_{x=0}^{\lfloor n/d_\tau \rfloor} \left(\frac{n}{d_\tau} - x\right)^{\tau-2} \mathrm{d}x + O(n^{\tau-2})\\
&= \frac{d_\tau^{\tau-2}}{(\tau-2)!d_1d_2\cdots d_{\tau-1}} \left(\frac{n}{d_\tau}\right)^{\tau-1} /(\tau-1) + O(n^{\tau-2})\\
&= \frac{1}{(\tau-1)!d_1d_2\cdots d_\tau} n^{\tau-1} + O(n^{\tau-2})
\end{aligned}$$

$\square$

The following lemmas are used to determine the log-asymptotic behaviour of the generative- and fine spectra of $\mathcal{V}_{1,d}$ for $d \geq 2$.

Lemma 4.5. *Let $a, b \in \mathbb{N}$. Then $\int_0^1 x^a(1-x)^b \,\mathrm{d}x = \frac{a!\cdot b!}{(a+b+1)!}$.*

Proof. The expression $\int_0^1 x^a(1-x)^b \,\mathrm{d}x$ is clearly symmetric in a and b. If $b = 0$, then $\int_0^1 x^a \,\mathrm{d}x = \frac{1}{a+1}$ holds, and by symmetry, the formula is also true when $a = 0$.

By the rule of partial integration, we obtain

$$(a+1)\int_0^1 x^a(1-x)^b \,\mathrm{d}x = \int_0^1 (x^{a+1})'(1-x)^b \,\mathrm{d}x$$
$$= -\int_0^1 x^{a+1}((1-x)^b)' \,\mathrm{d}x = b\int_0^1 x^{a+1}(1-x)^{b-1} \,\mathrm{d}x$$

Hence, the above formula is equivalent for pairs (a, b) and (a', b') if $a + b = a' + b'$. $\square$

Lemma 4.6. *Let $m \in \mathbb{N}^+$. For $x \geq 0$ define $S_{1,m}(x) = x^m$, and let*

$$S_{i+1,m}(x) = \int_0^x S_{i,m}(t)(x-t)^m \,\mathrm{d}t$$

for all integers $i \geq 2$. Then

$$S_{i,m}(x) = \frac{(m!)^i}{((m+1)\cdot i-1)!}\cdot x^{(m+1)\cdot i-1}$$

Proof. Induction on i with m fixed; the initial step $i = 1$ holds by definition. Assume that the formula is true for $i \geq 1$, and let us show it for $i+1$. By using the induction hypothesis, the integral form of $S_{i,m}(x)$ transforms to

$$S_{i+1,m}(x) = \int_0^x S_{i,m}(t)(x-t)^m \,\mathrm{d}t$$
$$= \int_0^t \frac{(m!)^i}{((m+1)\cdot i-1)!}\cdot t^{(m+1)\cdot i-1}(x-t)^m \,\mathrm{d}t$$

By applying the linear substitution $y = t/x$ and Lemma 4.5 we obtain

$$S_{i+1,m}(x) = \frac{(m!)^i}{((m+1)\cdot i-1)!}\cdot x^{(m+1)\cdot i-1+m+1}\int_0^1 y^{(m+1)\cdot i-1}(1-y)^m \,\mathrm{d}y$$
$$= \frac{(m!)^i}{((m+1)\cdot i-1)!}\cdot\frac{m!\cdot((m+1)\cdot i-1)!}{(m\cdot(i+1)+i-1)!}\cdot\frac{x^{(m+1)\cdot(i+1)-1}}{(m+1)\cdot(i+1)-1}$$
$$= \frac{(m!)^{i+1}}{((m+1)\cdot(i+1)-1)!}\cdot x^{(m+1)\cdot(i+1)-1}$$

$\square$

Lemma 4.7. *Let $K \in \mathbb{R}^+$, $m, n \in \mathbb{N}^+$. Then for $\log n \leq i \leq n^{\frac{m+2}{m+3}}$ we have*

$$\log\left(\max_i\left(\frac{K^i}{i!}\frac{(m!)^i}{((m+1)\cdot i-1)!}\cdot n^{(m+1)\cdot i-1}\right)\right)$$
$$= (m+2)\cdot\sqrt[m+2]{\frac{K\cdot m!}{(m+1)^{m+1}}}\cdot n^{\frac{m+1}{m+2}} + O(\log n)$$

Proof. By Stirling's formula, we have

$$\begin{aligned}
&\log\left(\max_i\left(\frac{K^i}{i!}\frac{(m!)^i}{((m+1)\cdot i-1)!}\cdot n^{(m+1)\cdot i-1}\right)\right)\\
&\quad=\max_i(i\log K-i\log i+i+i\log m!-\log((m+1)\cdot i-1)!\\
&\qquad+((m+1)\cdot i-1)\log n+O(\log i))\\
&\quad=\max_i(i\cdot(\log K-\log i+1+\log m!)-((m+1)\cdot i-1)\log((m+1)\cdot i-1)\\
&\qquad+((m+1)\cdot i-1)+((m+1)\cdot i-1)\log n)+O(\log n)\\
&\quad=\max_i(i\cdot(\log K-\log i+1+\log m!)-(m+1)\cdot i\log((m+1)\cdot i)\\
&\qquad+(m+1)\cdot i+(m+1)\cdot i\cdot\log n)+O(\log n)\\
&\quad=\max_i(i\cdot(\log K-\log i+1+\log m!-(m+1)\log(m+1)-(m+1)\log i\\
&\qquad+(m+1)+(m+1)\log n))+O(\log n)\\
&\quad=\max_i(i\cdot(\log K+m+2+\log m!-(m+1)\log(m+1)-(m+2)\log i\\
&\qquad+(m+1)\log n))+O(\log n)
\end{aligned}$$

We seek the maximum of this last expression in the interval $[\log n, n^{\frac{m+2}{m+3}}]$; the error this leads to has order of magnitude $O(\log n)$, as it is apparent from the derivative of the function we calculate now.

So let us define the following function

$$\begin{aligned}
u(x)=x\cdot(&\log K+m+2+\log m!-(m+1)\log(m+1)\\
&-(m+2)\log x+(m+1)\log n).
\end{aligned}$$

Then the derivative of this funtion is

$$u'(x)=\log K+\log m!-(m+1)\log(m+1)-(m+2)\log x+(m+1)\log n.$$

The equation $u'(x)=0$ has a unique solution, that is where $u(x)$ attains its maximum, namely $x_0=\sqrt[m+2]{\frac{K\cdot m!}{(m+1)^{m+1}}}\cdot n^{\frac{m+1}{m+2}}$. Note that if n is large enough, then indeed $x_0\in[\log n, n^{\frac{m+2}{m+3}}]$, and that $u(x)=x(m+2+u'(x))$. Thus $u(x_0)=(m+2)x_0$, which is equivalent to the statement of the lemma. □

Lemma 4.8. *Let $K\in\mathbb{R}^+$, $m,n\in\mathbb{N}^+$. Then for $0\le i\le n$ we have*

$$\begin{aligned}
&\log\left(\max_i\left(\frac{K^i}{i!}\cdot\sum_{r_1+\cdots+r_i=n}\prod_{j=1}^{i}r_j^m\right)\right)\\
&\qquad=(m+2)\cdot\sqrt[m+2]{\frac{K\cdot m!}{(m+1)^{m+1}}}\cdot n^{\frac{m+1}{m+2}}+O\left(n^{\frac{m+1}{m+3}}\right)
\end{aligned}$$

Proof. Let $T_{i,m}(n)=\sum_{r_1+\cdots+r_i=n}\prod_{j=1}^{i}r_j^m$ be the discrete variant of the integral $S_{i,m}(x)$ defined in Lemma 4.6, and let $U_{i,m}(n)=\frac{K^i}{i!}\cdot T_{i,m}(n)$.

The number of terms in the sum $T_{i,m}(n)$ is $\binom{n+i-1}{i-1}$, and each term is at most $(\frac{n}{i})^{im}$ by the inequality between the arithmetic and geometric means. Thus putting $V_{i,m}(n) := \frac{K^i}{i!}\binom{n+i-1}{i-1}(\frac{n}{i})^{im}$, we have $U_{i,m}(n) \le V_{i,m}(n)$.

It is easy to see that $\frac{V_{i+1,m}(n)}{V_{i,m}(n)} \le \frac{Kn^m(n+i)}{i^{m+1}(i+1)}$, which for $n^{\frac{m+2}{m+3}} < i \le n$ is at most $2K \cdot \frac{n^{m+1}}{i^{m+2}} < 2Kn^{-\frac{1}{m+3}} < 1$ for n large enough. Hence, the sequence $a_i = \log(V_{i,m}(n))$ is strictly monotone decreasing on $[n^{\frac{m+2}{m+3}}, n]$, and its value at $i_0 = \lceil n^{\frac{m+2}{m+3}} \rceil$ is at most

$$\begin{aligned}
\log\left(\frac{K^{i_0}}{i_0!}\cdot\binom{n+i_0-1}{i_0-1}\cdot\left(\frac{n}{i_0}\right)^{i_0 m}\right) &\le \log\left(\frac{(2K)^{i_0}\cdot n^{i_0}\cdot n^{i_0 m}}{i_0!\cdot i_0!\cdot i_0^{i_0 m}}\right)\\
&\le i_0(\log(2Ke^2) + (m+1)\log n - (m+2)\log i_0)\\
&\le i_0\left(\log(2Ke^2) + (m+1)\log n - (m+2)\cdot\frac{m+2}{m+3}\cdot\log n\right)\\
&\le i_0\left(-\frac{1}{m+3}\cdot\log n + O(1)\right) < 0
\end{aligned}$$

for n large enough.

Thus $U_{i,m}(n) \le V_{i,m}(n) \le V_{i_0,m}(n) < 1$ for n large enough for all $n^{\frac{m+2}{m+3}} < i \le n$.

For $i < \log n$ the trivial estimation $U_{i,m}(n) \le (2K)^{\log n}\cdot n^{\log n}\cdot n^{\log n}$ yields $\log(U_{i,m}(n)) \le 2\log^2 n + O(\log n) = O\left(n^{\frac{m+1}{m+3}}\right)$.

For $\log n \le i \le n^{\frac{m+2}{m+3}}$ we switch the sum $T_{i,m}(n) = \sum_{r_1+\cdots+r_i=n}\prod_{j=1}^i r_j^m = \sum_{r_1=0}^n\sum_{r_2=0}^{n-r_1}\cdots\sum_{r_{i-1}=0}^{n-r_1-\cdots-r_{i-2}}(r_1r_2\cdots r_{i-1})^m(n-r_1-\cdots-r_{i-1})^m$ to $S_{i,m}(n) = \int_{x_1=0}^{n}\int_{x_2=0}^{n-x_1}\cdots\int_{x_{i-1}=0}^{n-x_1-\cdots-x_{i-2}}(x_1x_2\cdots x_{i-1})^m(n-x_1-\cdots-x_{i-1}))^m\,\mathrm{d}x_{i-1}\cdots\mathrm{d}x_1$. This produces an error of order of magnitude $O\left(n^{\frac{m+1}{m+3}}\right)$, as $S_{i,m}(n-i) \le T_{i,m}(n) \le S_{i,m}(n+i)$, and by Lemma 4.6 we obtain

$$\begin{aligned}
\log\left(\frac{S_{i,m}(n+O(i))}{S_{i,m}(n)}\right) &\log\left(\frac{n+O(i)}{n}\right)^{(m+1)\cdot i-1}\\
&= \log\left(1+O\left(\frac{i}{n}\right)\right)^{(m+1)\cdot i-1}\\
&\le (m+1)\cdot n^{\frac{m+2}{m+3}}\cdot\log\left(1+O\left(n^{-\frac{1}{m+3}}\right)\right)\\
&= (m+1)\cdot n^{\frac{m+2}{m+3}}\cdot O\left(n^{-\frac{1}{m+3}}\right) = O\left(n^{\frac{m+1}{m+3}}\right)
\end{aligned}$$

The assertion then follows from Lemmas 4.6 and 4.7. □

5. Fine spectra

5.1. The fine spectrum of the varieties $\mathcal{V}_k$

In [10] recursive formulas and asymptotic estimations were given for the number of n-element rooted trees of depth k. Those results directly imply the following.

Theorem 5.1. *The sequences $f_k(n)$ satisfy the following asymptotic formulas.*

(1) $f_1(n) = 1$ *for all* $n \in \mathbb{N}$.
(2) $f_2(n) \sim \frac{1}{4\sqrt{3}n} \exp(\pi\sqrt{\frac{2}{3}}\sqrt{n})$.
(3) $\log f_k(n) \sim \frac{\pi^2}{6} \cdot \frac{n}{L_{k-2}(n)}$ *for* $k > 2$.

5.2. The fine spectrum of the varieties $\mathcal{V}_{k,d}$

Theorem 5.2. *The sequences $f_{k,d}(n)$ satisfy the following asymptotic formulas.*

(1) $\log f_{0,0}(n) \sim (\log \alpha)n$, *where* $\alpha \approx 2.95576$.
(2) $f_{0,1}(n) = 1$ *for all* $n \in \mathbb{N}$.
(3) $f_{0,d}(n) \sim \frac{1}{(\tau(d)-1)!d^{\tau(d)/2}} \cdot n^{\tau(d)-1}$ *for* $d \geq 2$.
(4) $f_{1,1}(n) = p(n) \sim \frac{1}{4\sqrt{3}n} \exp(\pi\sqrt{\frac{2}{3}}\sqrt{n})$.
(5) $\log f_{1,d}(n) \sim \frac{(d+1)\cdot \sqrt[d+1]{\zeta(d)}}{d} \cdot n^{\frac{d}{d+1}}$ *for* $d \geq 2$, *where* ζ *is the Riemann zeta function.*
(6) $\log f_{2,d}(n) \sim \frac{\pi^2 d}{6} \cdot \frac{n}{\log n}$ *for* $d \geq 1$.
(7) $\log f_{k,d}(n) \sim \frac{\pi^2}{6} \cdot \frac{n}{L_{k-1}(n)}$ *for* $k \geq 3, d \geq 1$.

Proof. For the proof of item (1) see [6].

Item (2) is straightforward from the definition of $f_{0,1}(n)$.

For item (3), let $1 = d_1, \ldots, d_{\tau(d)}$ be the positive divisors of d. An n-element algebra $(A; u)$ in the variety $\mathcal{V}_{0,d}$ consists of disjoint cycles with size in $\{d_1, \ldots, d_{\tau(d)}\}$. Let us denote the number of cycles in $(A; u)$ of size d_i by α_i. Then the isomorphism type of $(A; u)$ is uniquely determined by the tuple $(\alpha_1, \ldots, \alpha_{\tau(d)})$. According to Lemma 4.4 the number of such tuples is $\frac{1}{(\tau(d)-1)!d_1\cdots d_{\tau(d)}} n^{\tau(d)-1} + O(n^{\tau(d)-2}) = \frac{1}{(\tau(d)-1)!d^{\tau(d)/2}} n^{\tau(d)-1} + O(n^{\tau(d)-2})$.

For item (4), observe that an n-element algebra $(A; u)$ is in $\mathcal{V}_{1,1}$ if and only if it is the disjoint union of rooted trees of depth at most 1 with n vertices altogether, such that the edges are directed towards the root. A rooted tree with depth at most 1 is up to isomorphism uniquely determined by its size. Thus $(A; u)$ is up to isomorphism uniquely determined by the partition of n corresponding to the multi-set of the sizes of the rooted trees.

We show item (5). The number of n-element directed, connected unicyclic graphs with cycle length d is asymptotically $\frac{1}{d}\binom{n-1}{d-1}$. Thus the number of n-element directed, connected unicyclic graphs with cycle length dividing d is asymptotically $\sum_{t|d} \frac{1}{t}\binom{n-1}{t-1} = (1+O(\frac{1}{n}))\frac{1}{d!}n^{d-1}$. Let $a_n = \sum_{m|n} \frac{1}{m} f_{1,d,con}(\frac{n}{m})$. Then $f_{1,d}(n) \leq [x^n] \exp(\sum_{r=1}^{\infty} a_r x^r)$ by item (5) of Lemma 3.3, where $[x^n]H(x)$ denotes the coefficient of x^n in the power series $H(x)$. Furthermore, we have

$$a_n \le \left(1+O\left(\frac{1}{n}\right)\right)\sum_{m|n}\frac{1}{d!}\frac{1}{m}\left(\frac{n}{m}\right)^{d-1}$$
$$\le \left(1+O\left(\frac{1}{n}\right)\right)\frac{1}{d!}n^{d-1}\sum_{m=1}^{\infty}\left(\frac{1}{m}\right)^{d} = \left(1+O\left(\frac{1}{n}\right)\right)\frac{\zeta(d)}{d!}n^{d-1}$$

Putting $K_0 = \frac{\zeta(d)}{d!}, K = (1+\varepsilon)\cdot K_0$ and $m = d-1$, we obtain $a_n \le Kn^m$ for large enough n with arbitrary $\varepsilon > 0$. Thus

$$\log(f_{1,d}(n)) \le \log\left([x^n]\exp\left(\sum_{r=1}^{\infty}a_r x^r\right)\right)$$
$$\le (1+o(1))\log\left(\sum_{i=1}^{n}\frac{K^i}{i!}\sum_{r_1+\cdots+r_i=n}\prod_{j=1}^{i}r_j^m\right)$$
$$\le (1+o(1))\log\left(n\cdot\max_{i\in\{1,\ldots,n\}}\frac{K^i}{i!}\sum_{r_1+\cdots+r_i=n}\prod_{j=1}^{i}r_j^m\right).$$

Hence, Lemma 4.8 yields the asymptotic upper estimation

$$\log\left([x^n]\exp\left(\sum_{r=1}^{\infty}\frac{\zeta(d)}{d!}r^{d-1}x^r\right)\right) \sim (d+1)\cdot\sqrt[d+1]{\frac{\frac{\zeta(d)}{d!}\cdot(d-1)!}{d^d}}\cdot n^{\frac{d}{d+1}}$$
$$= \frac{(d+1)\cdot\sqrt[d+1]{\zeta(d)}}{d}\cdot n^{\frac{d}{d+1}}$$

for $\log f_{1,d}(n)$ as $\varepsilon \to 0$, as $K \to K_0$ then. The lower estimation can be obtained in a similar fashion. Let $\varepsilon > 0$ be fixed, and choose $k \in \mathbb{N}$ such that $\sum_{m=1}^{k}\left(\frac{1}{m}\right)^d \ge \zeta(d)-\varepsilon$. The only difference in the calculation compared to the upper estimation is that the inequality $(1-\varepsilon)\frac{\zeta(d)}{d!}n^{d-1} \le a_n$ does not hold for sufficiently large n, although for given ε, it is "often" true. The reason is that there are arbitrarily large numbers n with few divisors (e.g., primes), and for such an n we have $\sum_{m|n}\left(\frac{1}{m}\right)^d < \zeta(d)-\varepsilon$. So instead of $[x^n]\exp\left(\sum_{r=1}^{\infty}a_r x^r\right)$, it is better to compute $[x^n]\exp\left(\frac{1}{1-x}\sum_{r=1}^{\infty}a_r x^r\right)$, to even out the numbers with few divisors. This modification clearly has no effect on the log-asymptotics, and as every number is close to a number n that is divisible by the first k numbers, we obtain a power series in the exponential whose n-th coefficient is asymptotically bigger than $(1-\varepsilon)\frac{\zeta(d)}{d!}n^{d-1}$. Hence, the lower estimation $(d+1)\cdot\sqrt[d+1]{\frac{(1-\varepsilon)\frac{\zeta(d)}{d!}\cdot(d-1)!}{d^d}}\cdot n^{\frac{d}{d+1}} \le \log f_{1,d}(n)$ holds for all $\varepsilon > 0$ for sufficiently large n, which simplifies to the expression $\frac{(d+1)\cdot\sqrt[d+1]{(1-\varepsilon)\zeta(d)}}{d}\cdot n^{\frac{d}{d+1}} \le \log f_{1,d}(n)$ for large enough n.

We proceed with item (6). According to Lemma 3.3 item (4) and Theorem 5.1, $f_{2,1,\mathrm{con}}(n) = f_2(n) \sim \frac{1}{4n\sqrt{3}}\exp\left(\pi\sqrt{\frac{2n}{3}}\right)$. Lemma 3.3 item (3) yields $\frac{1}{d}\sum_{\mu_1+\cdots+\mu_d=n}\prod_{i=1}^{d}f_2(\mu_i) \le f_{2,d,\mathrm{con}}(n) \le \sum_{t|d}\sum_{\mu_1+\cdots+\mu_t=n}\prod_{i=1}^{t}f_2(\mu_i)$.

Asymptotically there are at most n^d terms in both the lower- and upper estimations, and according to Lemma 4.2 the logarithm of every term can be estimated by

$$\begin{aligned}
&\log\left(\max_{\mu_1+\cdots+\mu_t=n}\prod_{i=1}^{t} f_2(\mu_i)\right)\\
&\leq (1+o(1))\log\left(\max_{\mu_1+\cdots+\mu_t=n}\prod_{i=1}^{t}\frac{1}{4\mu_i\sqrt{3}}\exp\left(\pi\sqrt{\frac{2\mu_i}{3}}\right)\right)\\
&= O(t\log n)+(1+o(1))\pi\sqrt{\frac{2}{3}}\max_{\mu_1+\cdots+\mu_t=n}\sum_{i=1}^{t}\sqrt{\mu_i}\\
&\leq O(t\log n)+(1+o(1))\pi\sqrt{\frac{2}{3}}\sqrt{tn}\\
&\leq O(d\log n)+(1+o(1))\pi\sqrt{\frac{2}{3}}\sqrt{dn}\sim\pi\sqrt{\frac{2}{3}}\sqrt{dn}
\end{aligned}$$

Moreover, according to Lemma 4.2 the estimation is sharp when $t = d$ and the difference between any two of the n_i is at most 1. Such a term appears in both the lower- and upper estimations. As $\log n^d$ is negligible to $\pi\sqrt{\frac{2}{3}}\sqrt{dn}$, it makes no difference in the log-asymptotic estimations if we calculate with the biggest term or the sum of the terms. Hence, both the lower- and upper estimations we obtained for $\log f_{2,d,\mathrm{con}}(n)$ are asymptotically $\pi\sqrt{\frac{2}{3}}\sqrt{d}\sqrt{n}$, and consequently, so is $\log f_{2,d,\mathrm{con}}(n)$. By Lemma 3.3 item (5) and Theorem 3.4 item (1) we have that $\log f_{2,d}(n)\sim\frac{\pi^2 d}{6}\cdot\frac{n}{\log n}$.

Finally, we show item (7). From Lemma 3.3 item (3) and Theorem 5.1 we obtain $\log f_{k,1,\mathrm{con}}(n)=\log f_k(n)\sim\frac{\pi^2}{6}\cdot\frac{n}{L_{k-2}(n)}$. According to Lemma 3.3 item (4) we have $\frac{1}{d}\sum_{\mu_1+\cdots+\mu_d=n}\prod_{i=1}^{d} f_k(\mu_i)\leq f_{k,d,\mathrm{con}}(n)\leq\sum_{t|d}\sum_{\mu_1+\cdots+\mu_t=n}\prod_{i=1}^{t} f_k(\mu_i)$. Asymptotically there are at most n^d terms in both the lower- and upper estimations, which contributes a negligible factor. Using Lemma 4.3 the logarithm of every summand can be estimated from above by

$$\begin{aligned}
\log\left(\max_{\mu_1+\cdots+\mu_t=n}\prod_{i=1}^{t} f_k(\mu_i)\right) &= \max_{\mu_1+\cdots+\mu_t=n}\sum_{i=1}^{t}\log f_k(\mu_i)\\
&\leq (1+o(1))\max_{\mu_1+\cdots+\mu_t=n}\sum_{i=1}^{t}\frac{\pi^2}{6}\cdot\frac{\mu_i}{L_{k-2}(\mu_i)}\\
&\leq (1+o(1))\frac{\pi^2}{6}\cdot\frac{n}{L_{k-2}(n)}\sim\frac{\pi^2}{6}\cdot\frac{n}{L_{k-2}(n)}
\end{aligned}$$

Moreover, according to the argument given in the proof of Lemmas 4.2 and 4.3 , the estimation is sharp when $t = d$ and the difference between any two of the n_i is at most 1. Such a term appears in both the lower- and upper estimations. Hence, both the lower- and upper estimations we obtained for $\log f_{k,d,\mathrm{con}}(n)$ are asymptotically $\frac{\pi^2}{6} \cdot \frac{n}{L_{k-2}(n)}$, and consequently, so is $\log f_{k,d,\mathrm{con}}(n)$. By Lemma 3.3 item (5) and Theorem 3.4 item (2) we have that $\log f_{k,d}(n) \sim \frac{\pi^2}{6} \cdot \frac{n}{L_{k-1}(n)}$. □

6. Generative spectra

6.1. The generative spectrum of the varieties $\mathcal{V}_k$

Theorem 6.1. *The sequences $g_k(n)$ satisfy the following asymptotic formulas.*

(1) $g_1(n) = n + 1$ *for all* $n \in \mathbb{N}$.
(2) $g_2(n) \sim \frac{\sqrt{3}}{2\pi^2} \exp\left(\pi\sqrt{\frac{2}{3}}\sqrt{n}\right)$.
(3) $\log g_k(n) \sim \frac{\pi^2}{6} \cdot \frac{n}{L_{k-2}(n)}$ *for* $k > 2$.

Proof. Item (1) holds by definition.

For item (2), we first give an asymptotic estimation for $g_2^*(n)$. If $(A; u)$ is an n-generated, but not $(n-1)$-generated algebra in $\mathcal{V}_2$, then G_A is a rooted tree of depth at most 2 with n leaves. Let two leaves x and y be equivalent if $u(x) = u(y)$. Leaves x such that $u(x)$ is the root form an equivalence class of $(n-i)$ elements, the others form a partition of an i-element set. The isomorphism type of $(A; u)$ is uniquely determined by the number i and the partition of the i-element set. Thus $g_2^*(n) = \sum_{i=1}^{n} p(i)$. According to the Hardy-Ramanujan formula, $p(n) \sim \frac{1}{4\sqrt{3}n} \exp\left(\pi\sqrt{\frac{2}{3}}\sqrt{n}\right)$. By Lemma 4.1 we obtain $g_2^*(n) \sim \frac{1}{2\sqrt{2}\pi\sqrt{n}} \exp\left(\pi\sqrt{\frac{2}{3}}\sqrt{n}\right)$. Hence, $g_2(n) = \sum_{i=1}^{n} g_2^*(i) \sim \frac{\sqrt{3}}{2\pi^2} \exp\left(\pi\sqrt{\frac{2}{3}}\sqrt{n}\right)$ by Lemma 4.1.

Finally, for item (3) it is enough to show that $\log g_k^*(n) \sim \frac{\pi^2}{6} \cdot \frac{n}{L_{k-2}(n)}$ for $k > 2$. We prove this estimation by induction on k. By Lemma 3.3 item (2) and Theorem 3.4 item (1), we obtain the result for $k = 3$. Assume that the statement is true for some $k \geq 3$. Then by Lemma 3.3 item (2) and Theorem 3.4 item (2), the assertion holds for $k + 1$, as well. □

Corollary 6.2. *The sequences $g_k^*(n)$ satisfy the asymptotic formulas*

(1) $g_2^*(n) \sim \frac{1}{2\sqrt{2}\pi\sqrt{n}} \exp\left(\pi\sqrt{\frac{2}{3}}\sqrt{n}\right)$ *and*
(2) $\log g_k^*(n) \sim \frac{\pi^2}{6} \cdot \frac{n}{L_{k-2}(n)}$ *for* $k > 2$.

6.2. The generative spectrum of the varieties $\mathcal{V}_{k,d}$

Theorem 6.3. *The sequences $g_{k,d}(n)$ satisfy the following asymptotic formulas.*

(1) $g_{0,d}(n) = \binom{\tau(d)+n}{n} - 1 \sim \frac{1}{\tau(d)!} n^{\tau(d)}$ *for* $d \geq 1$.

(2) $g_{1,1}(n) \sim \frac{\sqrt{3}}{2\pi^2} \exp\left(\pi\sqrt{\frac{2}{3}}\sqrt{n}\right)$.
(3) $\log g_{1,d}(n) \sim \frac{(d+1)\cdot \sqrt[d+1]{\zeta(d)}}{d} \cdot n^{\frac{d}{d+1}}$ *for* $d \geq 2$, *where* ζ *is the Riemann zeta function.*
(4) $\log g_{2,d}(n) \sim \frac{\pi^2 d}{6} \cdot \frac{n}{\log n}$ *for* $d \geq 1$.
(5) $\log g_{k,d}(n) \sim \frac{\pi^2}{6} \cdot \frac{n}{L_{k-1}(n)}$ *for* $k \geq 3, d \geq 1$.

Proof. For item (1) observe that an algebra $(A; u)$ in $\mathcal{V}_{0,d}$ is n-generated if and only if $(A; u)$ consists of at most n disjoint cycles. The length of a cycle can be any divisor of d. Thus up to isomorphism $(A; u)$ is uniquely determined by the multi-set of i numbers, with $i \leq n$, consisting of the sizes of the cycles in $(A; u)$, and these i numbers can be chosen from a $\tau(d)$-element set. Hence, $g_{0,d}(n) = \sum_{i=1}^{n} \binom{\tau(d)+i-1}{i} = \binom{\tau(d)+n}{n} - 1$.

For item (2) observe that there is a bijection between $\mathcal{V}_2$ and $\mathcal{V}_{1,1}$: if we omit the root of an algebra in $\mathcal{V}_2$ then we obtain an algebra in $\mathcal{V}_{1,1}$. Moreover, this bijection maps n-generated algebras in $\mathcal{V}_2$ to n-generated algebras in $\mathcal{V}_{1,1}$. Thus $g_{1,1}(n) = g_2(n)$, and we are done by Theorem 6.1 item (2).

The proof of item (3) is analogous to that of Theorem 5.2 item (5).

For items (4) and (5) it is enough to show that $\log g^*_{2,d}(n) \sim \frac{\pi^2 d}{6} \cdot \frac{n}{\log n}$ for $d \geq 1$ and $\log g^*_{k,d}(n) \sim \frac{\pi^2}{6} \cdot \frac{n}{L_{k-1}(n)}$ for $k \geq 3, d \geq 1$. By comparing Theorem 5.1 and Corollary 6.2 we obtain that $\log f_k(n) \sim \log g^*_k(n)$ for $k \geq 2$. In the statement of Lemma 3.3 items (3), (4) and (5) are analogous to items (6), (7) and (8). Hence, the proofs of the desired log-asymptotic estimations $\log g^*_{2,d}(n) \sim \frac{\pi^2 d}{6} \cdot \frac{n}{\log n}$ for $d \geq 1$ and $\log g^*_{k,d}(n) \sim \frac{\pi^2}{6} \cdot \frac{n}{L_{k-1}(n)}$ for $k \geq 3, d \geq 1$ are also analogous to the proofs of items (6) and (7) of Theorem 5.2. □

Many of our results determine the log-asymptotic behaviour of the fine- and generative spectra of varieties of monounary algebras. This can be considered as a first step towards more refined asymptotic formulas. As we mentioned in the introduction, our results provide infinitely many examples of spectra with log-asymptotic behaviour strictly between polynomials and exponential functions. It would be particularly interesting to find asymptotic formulas for $f_{1,d}(n)$ and $g_{1,d}(n)$, the two series that turned out to be the most challenging to deal with in the present paper. We are certain that such a result would require more advanced techniques, for example the saddle point method, which was effectively used in similar problems in the past, see [3].

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