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Extending partial projective planes

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In honor of my colleagues, Ralph Freese and Bill Lampe.

Abstract. This note discusses a computational method for constructing finite projective planes.

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1. Introduction

There are a number of interesting problems concerning finite non-desarguesian projective planes. One would hope that these problems would admit an algebraic, or geometric, or combinatorial solution. But it may just be that the existence, or non-existence, of certain types of planes is an accident of nature. With that in mind, since 1999 the author has been trying various computer programs to construct non-desarguesian projective planes. While all these attempts have failed, hope springs eternal, and this note describes a set of problems and some ideas for addressing them.

This note is based on a talk given to the Courant Institute geometry seminar in October 2017. The author appreciates the hospitality and encouragement from the participants of the seminar.

2. Basics

Recall that a *projective plane* is an incidence structure of points and lines satisfying these axioms.

- Two points determine a unique line.

Presented by W. DeMeo.
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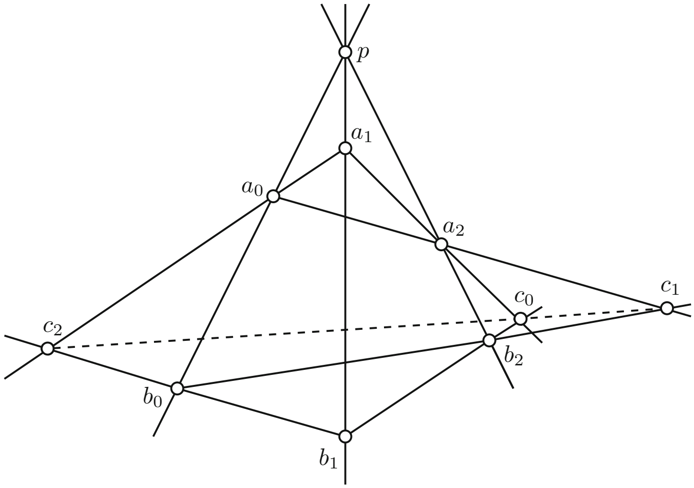
FIGURE 1. Desargues' Law: If the lines $a_0 \vee b_0$, $a_1 \vee b_1$, and $a_2 \vee b_2$ intersect in a point p, then the points c_0, c_1, and c_2 are colinear, where $c_i = (a_j \vee a_k) \wedge (b_j \vee b_k)$ for $\{i, j, k\} = \{0, 1, 2\}$. In a failure of Desargues' Law, the lines $c_0 \vee c_1$, $c_0 \vee c_2$, and $c_1 \vee c_2$ are distinct.



- Two lines intersect in a unique point.
- There exist four points with no three on a line.

Each finite plane has an *order* n such that there are

- $n + 1$ points on each line,
- $n + 1$ lines through each point,
- $n^2 + n + 1$ total points,
- $n^2 + n + 1$ total lines.

An elementary construction allows us to construct a projective plane starting from the affine plane over any division ring. Indeed, every projective plane can be coordinatized by a ternary ring [12]. For the basic combinatorics and coordinatization of projective planes, see [1,7,13].

Desargues' Law, a property that holds in some projective planes and not others, is illustrated in Figure 1 and explained in the caption. Planes that satisfy Desargues' Law are called *desarguesian*. A projective plane is desarguesian if and only if it can be coordinatized by a division ring. Thus from finite fields we obtain projective planes of order q for any prime power $q > 1$. The same construction yields non-desarguesian finite projective planes coordinatized by various finite quasi-fields; these are of prime power order $q \geq 9$.

There are four isomorphism types of planes of order 9, including the one coordinatized by a field of order 9, and a non-desarguesian plane coordinatized by a Hall quasi-field, and its dual. The fourth type, the Hughes plane, admits no such nice description.

A classic result of Bruck and Ryser [4] shows that some orders are impossible: *If* $n \equiv 1$ *or* $2 \bmod 4$ *and there is a plane of order* n*, then* n *is a sum of two squares.* Beyond the Bruck–Ryser Theorem, only one more restriction is known: Lam, Theil and Swiercz [14] proved that there is no plane of order 10.

That leaves the existence of a projective plane of the following orders unknown: 12, 15, 18, 20, 24, 26, 28, ...

A subplane of a finite projective plane need not have order dividing the order of the plane. Indeed, H. Neumann showed that every Hall plane has a subplane of order 2 (see [11]).

3. Partial projective planes

A *partial projective plane* is a collection of points and lines, and an incidence relation, so that

- two points lie on at most one line,
- two lines intersect in at most one point.

M. Hall showed that every finite partial plane can be extended to a projective plane (usually infinite) [12]. In retrospect, it is not hard to see how to build this free extension.

Now projective planes correspond to simple, complemented, modular lattices of height 3 (Birkhoff [2] and Menger [15]). To form a projective plane from such a lattice, take the points to be the elements of height 1, and the lines to be the elements of height 2. Upper semimodularity means that 2 points join to a unique line, while lower semimodularity means that 2 lines meet in a unique point. In a finite dimensional, complemented, modular lattice, every element is a join of atoms. Such a lattice is simple if and only if the join of any two atoms contains a third, which in geometric terms says that every line contains at least three points. To obtain four points in general position, choose two distinct lines, and take two points from each line, none of which is their point of intersection.

This suggests that we employ partial planes that are meet semilattices. A *semiplane* is a collection of lines and points, with an incidence relation, such that any two lines intersect in a unique point. A canonical example of a semiplane is formed by taking any subset of the lines of a plane, together with the points that are intersections of those lines.

4. Four questions

That brings us to four basic questions about finite projective planes.

- Is there a finite projective plane of non-prime-power order (necessarily non-desarguesian)?
- Is there a non-desarguesian plane of prime order?
- Does every finite non-desarguesian plane contain a subplane of order 2?
- Does every finite partial plane have an extension to a finite plane?

If we fix a desired order n, the general plan for extending a finite partial plane to a plane of order n is straightforward enough.

- Start with a semiplane that contains your desired configuration (e.g., a failure of Desargues' Law or a plane of order 2).
- As long as possible add lines, with their intersections with existing lines, one at a time, keeping a semiplane structure and at most $n + 1$ points-per-line.
- Intersections can be new points or old points.
- If you get $n^2 + n + 1$ lines, the semiplane is a plane [6].
- Otherwise, when adding a line is no longer possible, back up and try again.

Did we mention that you should be *patient*, as the program could take tens of thousands of years?

Nonetheless, there is a simple turnaround criterion from [16]. Given n and a semiplane $\Pi = \langle P_0, L_0, \leq_0 \rangle$, define

$$\rho_n(\Pi) = \sum_{\ell \in L_0} r_\Pi(\ell) + n^2 + n + 1 - |P_0| - |L_0|(n + 1)$$

where $r_\Pi(\ell)$ denotes the number of points on the line ℓ in the semiplane Π. If $\Pi = \langle P_0, L_0, \leq_0 \rangle$ can be extended to a projective plane $\Sigma = \langle P, L, \leq \rangle$ of order n, then

$$\rho_n(\Pi) = |\{p \in P : p \not\leq \ell \text{ for all } \ell \in L_0\}|.$$

Hence Π can be extended to a projective plane of order n only if $\rho_n(\Pi) \geq 0$.

There are nice extension theorems for some types of partial structures, summarized in Chapter 9 of Dénes and Keedwell [8], and updated in [9]. We note especially the results of Bruck [3] and Dow [10]. For a logical approach, see Conant and Kruckman [5].

5. Non-desarguesian planes

In order to apply this program to construct a non-desarguesian plane of questionable order, we must first extend a non-desarguesian configuration to a semi-plane. A non-desarguesian configuration has 10 points and 12 lines, see Figure 1. To form a semiplane, those lines can intersect in various ways: the intersections could be new points or old ones. The result is a semiplane with 12 lines and between 20 and 37 points. Seffrood proved that there are 875 such non-desarguesian semiplanes, which fall into 105 isomorphism classes. For some pairs (A,B) of the 105 types, if a plane contains a semiplane of type A then it contains one of type B. There are 15 non-desarguesian semiplanes that are minimal in the sense that every non-desarguesian plane must contain one of these 15 semiplanes. Thus a program to construct finite non-desarguesian planes can use one of these 15 minimal semiplanes as a starting configuration. (The results in this paragraph are from Seffrood and Nation [16].)

So far, our programs have yielded

- semiplanes of order 11 with 40 lines (a plane has 133 lines),
- semiplanes of order 12 with 44 lines (a plane has 157 lines),
- semiplanes of order 13 with 48 lines (a plane has 183 lines),
- a semiplane of order 15 with 56 lines (a plane has 241 lines).

In each case, these semiplanes are maximal, in the sense that they cannot be extended to a projective plane of the given order. Most of the semiplanes have the full number of points.

This is not as bad as it first seems. When we tested the program by constructing a Hall plane of order 9, it turned out that once the semiplane had 35 lines, from that point on there was a unique choice for how to extend it with a new line. Thus the program extended the semiplane with 35 lines to a plane with 91 lines in a matter of seconds. Hence we suggest the following problem:

Find $f(n)$ such that every semiplane with at least $f(n)$ lines and at most $n+1$ points-per-line can be extended to a plane of order n.

There are results of this nature for latin squares. A projective plane of order n can be constructed from a set of $n-1$ mutually orthogonal $n \times n$ latin squares (MOLS), and *vice versa* [8,9]. Shrikhande [17] proved that, for $n > 4$, any set of $n-3$ mutually orthogonal $n \times n$ latin squares can be extended to a complete set of $n-1$ MOLS, and Bruck [3] proved the same result for any collection of $n - 1 - (2n)^{\frac{1}{4}}$ MOLS. (See Chapter 9 of [8].)

6. Other starting configurations

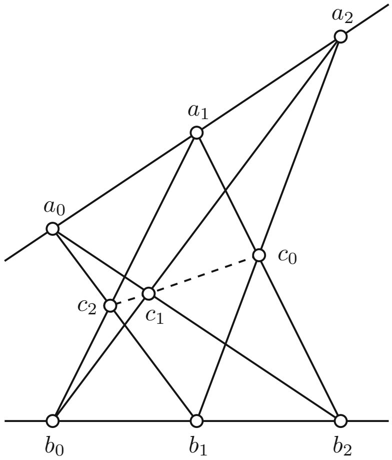
Pappus' Law is illustrated in Figure 2 and explained in the caption. A desarguesian plane can be coordinatized by a division ring; that division ring is commutative if and only if the plane satisfies Pappus' Law. Since every finite division ring is a field, this suggests that we start with a semiplane generated by a failure of Pappus' Law. A non-pappian configuration has 9 points and 11 lines, so there should be fewer options for completing it to a semiplane.

Hanna Neumann's Fano planes sitting inside Hall planes suggests another idea: *What happens if you start with a plane of order 2 and try to extend it to a plane of order n?* A starting semiplane with a Fano plane and one extra line would have 8 lines and 14 points as the only option.

Freese tried a variation on this theme, with a program to look for intermediate subplanes in a Hall plane, between a Fano subplane and the whole Hall plane [11]. Perhaps it is time to revisit this approach.

A theorem from folklore is that if a plane of order n has a subplane of order $r < n$, then either $n = r^2$ or $n \geq r^2 + r$. Thus possibly a plane of order 3 could be extended to a non-desarguesian plane of order 12 or more.

FIGURE 2. Pappus' Law: If the points a_0, a_1, and a_2 are colinear, and the points b_0, b_1, and b_2 are colinear, then the points c_0, c_1, and c_2 are colinear, where $c_i = (a_j \vee b_k) \wedge (b_j \vee a_k)$ for $\{i, j, k\} = \{0, 1, 2\}$. In a failure of Pappus' Law, $c_0 \vee c_1$, $c_0 \vee c_2$, and $c_1 \vee c_2$ are three distinct lines.



7. Conclusion

Of course, if a non-desarguesian projective plane of a given order does not exist, then no amount of subtle programming will matter. Nonetheless, it seems prudent to complement attempts to prove that they don't exist with searches to find them, in hopes that one or the other will succeed!

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