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Algebra Universalis

Meet-irreducible congruence lattices

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Abstract. The system of all congruences of an algebra (A, F) forms a lattice, denoted $\mathrm{Con}(A, F)$. Further, the system of all congruence lattices of all algebras with the base set A forms a lattice $\mathcal{E}_A$. We deal with meet-irreducibility in $\mathcal{E}_A$ for a given finite set A. All meet-irreducible elements of $\mathcal{E}_A$ are congruence lattices of monounary algebras. Some types of meet-irreducible congruence lattices were described in Jakubíková-Studenovská et al. (2017). In this paper, we prove necessary and sufficient conditions under which $\mathrm{Con}(A, f)$ is meet-irreducible in the case when (A, f) is an algebra with short tails (i.e., $f(x)$ is cyclic for each $x \in A$) and in the case when (A, f) is an algebra with small cycles (every cycle contains at most two elements).

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1. Introduction

Without doubt, the study of congruences (i.e. reflexive, symmetric and transitive relations) is important in universal algebra. It is known that all congruences of an algebra $\mathcal{A}$ ordered by inclusion form an algebraic lattice, denoted $\mathrm{Con}\,\mathcal{A}$ (see e.g. [8]). In 1963, it was proved by Grätzer and Schmidt that every algebraic lattice is isomorphic to the congruence lattice of some algebra [3]. The congruence lattices have been intensively studied by several authors, currently e.g. [1,2] or [4].

For a given set A, the system $\mathcal{E}_A$ of all $\mathrm{Con}\,\mathcal{A}$, where $\mathcal{A}$ is an algebra with the base set A, forms a lattice (with respect to class-theoretical inclusion) [8]. This lattice has been investigated in [6], e.g. it was shown that $\mathcal{E}_A$ is atomistic and if $|A| \geq 4$, it is tolerance simple. Also, all join-irreducible congruence lattices were characterized in [6].

In this paper, we study meet-irreducibility in the lattice $\mathcal{E}_A$.

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Since $F \subseteq G$ implies $\mathrm{Con}(A, G) \subseteq \mathrm{Con}(A, F)$, all $\wedge$-irreducible elements in $\mathcal{E}_A$ must be of the form $\mathrm{Con}(A, f)$ for a single mapping f, otherwise $\mathrm{Con}(A, F)$ would be the intersection of all $\mathrm{Con}(A, f)$ where $f \in F$. Therefore, it is sufficient to explore meet-irreducibility of congruence lattices of monounary algebras (i.e. algebras with a single unary operation). While studying properties of monounary algebras, we often use that they can be easily visualized as a digraphs which are always planar, hence easy to draw [7].

In [6], Studenovská, Pöschel and Radeleczki presented some partial answers to the question which lattices $\mathrm{Con}(A, f)$ for a given finite set A are meet-irreducible, namely, in the case when each cycle contains only one element and in the case when f is a permutation. Further, every coatom is meet-irreducible in the lattice $\mathcal{E}_A$. The coatoms of $\mathcal{E}_A$ can be obtained directly from coatoms of the lattice $\mathcal{L}_A$, the lattice of all quasiorder lattices of all algebras with the base set A (see [6]); properties of the lattice $\mathcal{L}_A$ were studied in [8].

Our aim is to contribute to the characterization of meet-irreducible elements of $\mathcal{E}_A$. In what follows, we will investigate two kinds of monounary algebras (A, f) :

- $(*)$ (A, f) is with short tails ($f(x)$ is cyclic for each $x \in A$),
- $(**)$ (A, f) is with small cycles (every cycle of (A, f) contains at most two elements).

If (A, f) satisfies $(*)$ or $(**)$, we prove necessary and sufficient conditions under which $\mathrm{Con}(A, f)$ is meet-irreducible (for $(*)$ see Theorem 3.6; for $(**)$ see Theorem 5.19).

2. Preliminary

In the following, let A be a fixed finite set. Further, let $\mathbb{N} := \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

For a mapping $f : A \to A$, $f(a)$ denotes the image of the element $a \in A$ in the mapping f, and if $n \in \mathbb{N}$ then f^n denotes the n-fold composition of f. By convention, f^0 denotes the identity mapping id_A and $f^{-1}(a) = \{x \in A, f(x) = a\}$. The mapping $f : A \to A$ is called *trivial* if it is either identity $x \mapsto x$ or the constant mapping $x \mapsto a$. Otherwise it is called *nontrivial*.

The pair (A, f) is said to be a *monounary algebra*.

An element $x \in A$ is called *cyclic* if there exists $n \in \mathbb{N}$ such that $f^n(x) = x$, otherwise it is called *noncyclic*. In this case, the set $\{x, f^1(x), f^2(x), \dots, f^{n-1}(x)\}$ is called a *cycle* of (A, f). Since A is finite, for each $a \in A$ there exists $k \in \mathbb{N}_0$ such that $f^k(a)$ is cyclic. The cycle containing $f^k(a)$ will be denoted $C(a)$.

The set $B \subseteq A$ such that $C(x) = C(y)$ for every $x, y \in B$ is called a *component* of (A, f). The monounary algebra (A, f) is called *connected* if it contains only one component.

Notation 2.1. Let (A, f) be a monounary algebra. We denote $Z_f(x) = \{f^k(x); k = 0, 1, \dots\}$.

By *length of a cycle*, we will understand the number of elements of this cycle. The cycle containing n elements will be also called n-cycle. Further, the operation f is called *acyclic*, if each cycle of (A, f) has length 1.

Definition 2.2. Monounary algebra (A, f) will be called an *algebra with small cycles* if each cycle of (A, f) has at most 2 elements.

In [6], the following notations were introduced.

Notation 2.3. Let (A, f) be a monounary algebra. We say that f is of the type (I) or (II) if the following holds:

(I) f is nontrivial and $f^2 = f$,
(II) f is nontrivial, f^2 is a constant, say 0 and $|f^{-1}(0)| \geq 3$.

Figure 1 shows monounary algebras whose operations are of the type (I) and (II), respectively. Labeled elements are mandatory, all others are optional.

Notation 2.4. Let (A, f) be a monounary algebra. Let f be nontrivial and acyclic. We say that f satisfies condition (α) or (β) if the following holds:

(α) There exist distinct elements $0, 1, 2, 0', 1', 2' \in A$ such that $f(0) = f(1) = 0, f(2) = 1,\ f(0') = f(1') = 0', f(2') = 1'$.
(β) (A, f) is connected and there exist distinct elements $0, 1, 2, 1', 2' \in A$ such that $f(0) = f(1) = f(1') = 0, f(2) = 1, f(2') = 1'$.

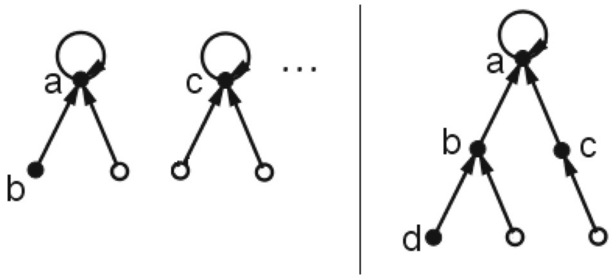

FIGURE 1. Operations of the type (I) and (II)

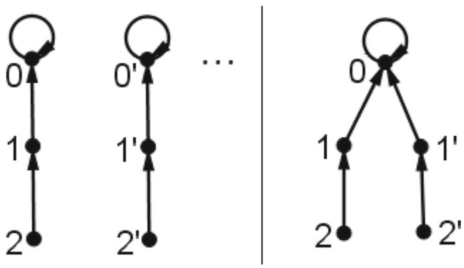

FIGURE 2. Operation satisfying the condition (α) or (β)

Figure 2 shows monounary algebras whose operations satisfy the conditions (α) or (β), respectively. Again, the labeled elements are mandatory, all others are optional.

Definition 2.5. If L is lattice, then nonunit element $a \in L$ is called *meet-irreducible* (shortly $\wedge$-irreducible) if $a = b_1 \wedge b_2$ implies $a \in \{b_1, b_2\}$. Similarly, nonzero element $a \in L$ is called *join-irreducible* ($\vee$-irreducible) if $a = b_1 \vee b_2$ implies $a \in \{b_1, b_2\}$ (see e.g. [9]).

Let us denote the least and the greatest congruence on the set A as $\Delta := \{(x, x), x \in A\}$ and $\nabla := A \times A$ respectively. For $x, y \in A$ let $\theta_f(x, y)$ be the smallest congruence on (A, f) such that $(x, y) \in \theta_f(x, y)$.

The following Lemma summarizes some properties of operations $f, g \in A^A$ with $\mathrm{Con}(A, f) \subseteq \mathrm{Con}(A, g)$, (see [6]).

Lemma 2.6. *Let $f, g \in A^A$ be nontrivial and* $\mathrm{Con}(A, f) \subseteq \mathrm{Con}(A, g)$. *Then we have*

(i) $\forall x, y \in A : (x, y) \in \varkappa \in \mathrm{Con}(A, f) \implies (g(x), g(y)) \in \varkappa$,
in particular we have $(g(x), g(y)) \in \theta_f(x, y)$ and $\theta_g(x, y) \subseteq \theta_f(x, y)$.

(ii) *Let B be a subalgebra of (A, f). Then either B is also a subalgebra of (A, g) or g is constant on B, where the constant does not belong to B.*

In [6], the following theorem, which describes the $\wedge$-irreducible congruence lattices of monounary algebras with acyclic operations, was proved:

Theorem 2.7. *A congruence lattice* $\mathrm{Con}(A, f)$ *with a nontrivial and acyclic $f \in A^A$ is $\wedge$-irreducible if and only if f is of type* (I) *or* (II) *or satisfies condition* (α) *or* (β).

Let $\mathrm{Eq}(A)$ denote the set of all *equivalence relations* on a given set A (i.e. reflexive, symmetric and transitive relations).

Notation 2.8. For $\varkappa \in \mathrm{Eq}(A)$ consider the corresponding partition $A/\varkappa$ into equivalence classes. If $C_1 = \{c_{11}, c_{12}, \dots\}$, $C_2 = \{c_{21}, c_{22}, \dots\}$, $\dots$, $C_k = \{c_{k1}, c_{k2}, \dots\}$ are the equivalence classes of $\varkappa$ with at least two elements, then we use the notation

$$\varkappa = [c_{11}, c_{12}, \dots]\,[c_{21}, c_{22}, \dots]\,\dots\,[c_{k1}, c_{k2}, \dots] \text{ or}$$
$$\varkappa = [C_1]\,[C_2]\,\dots\,[C_k].$$

3. Short tails

If f is a permutation on A then we also say that (A, f) is a permutation. In [6], it was described when the congruence lattice of (A, f) is $\wedge$-irreducible in the case that (A, f) is a permutation.

Theorem 3.1. *A congruence lattice* $\mathrm{Con}(A, f)$, $|A| \geq 3$ *with a nontrivial permutation f is $\wedge$-irreducible if and only if f is of prime power order p^m with at least two cycles of length p^m.*

Definition 3.2. Let $(\bar{A}, \bar{f})$ be a monounary algebra. Then $(\bar{A}, \bar{f})$ is said to be a *permutation-algebra with short tails* if there is a subalgebra (A, f) of $(\bar{A}, \bar{f})$ such that (A, f) is a permutation and $\bar{f}(x) \in A$ for each $x \in \bar{A}$. In this case (A, f) is called a *permutation-algebra corresponding to* $(\bar{A}, \bar{f})$.

Notice that to each $x \in \bar{A}$ there is a unique element in A, we denote it x', with $\bar{f}(x) = \bar{f}(x') = f(x')$.

Lemma 3.3. *Let $(\bar{A}, \bar{f})$ be a permutation-algebra with short tails. If the permutation-algebra (A, f) corresponding to $(\bar{A}, \bar{f})$ fails to be a transposition and* $\operatorname{Con}(A, f)$ *is $\wedge$-reducible in $\mathcal{E}_A$ then* $\operatorname{Con}(\bar{A}, \bar{f})$ *is $\wedge$-reducible in $\mathcal{E}_{\bar{A}}$.*

Proof. (A) Let the order of the permutation f be not a prime power, say $n = rs$ with $\gcd(r, s) = 1$. Put $g_r = f^r$, $g_s = f^s$. In view of the proof of Theorem 4.2 of [8] we obtain that $\operatorname{Con}(A, f)$ is $\wedge$-reducible,

$$\operatorname{Con}(A, f) \neq \operatorname{Con}(A, g_r), \operatorname{Con}(A, f) \neq \operatorname{Con}(A, g_s),$$
$$\operatorname{Con}(A, f) = \operatorname{Con}(A, g_r) \cap \operatorname{Con}(A, g_s).$$

Now define operations on $\bar{A}$ by setting $\bar{g}_r(x) = g_r(x')$, $\bar{g}_s(x) = g_s(x')$ for $x \in \bar{A}$. Then (A, g_r) is the permutation-algebra corresponding to $(\bar{A}, \bar{g}_r)$ and (A, g_s) is the permutation-algebra corresponding to $(\bar{A}, \bar{g}_s)$. It is clear to see that $\operatorname{Con}(\bar{A}, \bar{f}) \subsetneq \operatorname{Con}(\bar{A}, \bar{g}_r)$, $\operatorname{Con}(\bar{A}, \bar{f}) \subsetneq \operatorname{Con}(\bar{A}, \bar{g}_s)$. We will show that $\operatorname{Con}(\bar{A}, \bar{f}) = \operatorname{Con}(\bar{A}, \bar{g}_r) \cap \operatorname{Con}(\bar{A}, \bar{g}_s)$.

Let $x, y \in \bar{A}$. Since f is a permutation,

$$\begin{aligned}\theta_{\bar{f}}(x, y) &= [x, y] \vee \theta_{\bar{f}}(\bar{f}(x), \bar{f}(y)) = [x, y] \vee \theta_{\bar{f}}(\bar{f}(x'), \bar{f}(y')) \\ &= [x, y] \vee \theta_f(f(x'), f(y')) \vee \Delta_{\bar{A}} = [x, y] \vee \theta_f(x', y') \vee \Delta_{\bar{A}}.\end{aligned}$$

The elements x', y' belong to A, hence $\theta_f(x', y') = \theta_{g_r}(x', y') \vee \theta_{g_s}(x', y')$, which yields

$$\begin{aligned}[x, y] \vee \theta_f(x', y') \vee \Delta_{\bar{A}} &= [x, y] \vee \theta_{g_r}(x', y') \vee \theta_{g_s}(x', y') \vee \Delta_{\bar{A}} \\ &= [x, y] \vee \theta_{g_r}(g_r(x'), g_r(y')) \vee \theta_{g_s}(g_s(x'), g_s(y')) \vee \Delta_{\bar{A}} \\ &= [x, y] \vee \theta_{\bar{g}_r}(\bar{g}_r(x'), \bar{g}_r(y')) \vee \theta_{\bar{g}_s}(\bar{g}_s(x'), \bar{g}_s(y')) \\ &= \theta_{g_r}(x', y') \vee \theta_{g_s}(x', y'),\end{aligned}$$

hence $\theta_{\bar{f}}(x, y) = \theta_{\bar{g}_r}(x, y) \vee \theta_{\bar{g}_s}(x, y)$, and therefore $\operatorname{Con}(\bar{A}, \bar{f}) = \operatorname{Con}(\bar{A}, \bar{g}_r) \wedge \operatorname{Con}(\bar{A}, \bar{g}_s)$.

(B) It remains to consider the case when the order of f is a prime power p^m. Since $\operatorname{Con}(A, f)$ is $\wedge$-reducible, there is exactly one cycle (C_0) of length p^m, for simplicity let $(C_0) = (0, 1, \ldots, p^m - 1)$. We can exclude the case $p = 2$ and $m = 1$ since then f is a transposition; thus $p^m > 2$. In the proof of Theorem 4.2 of [8] there were indicated two unary operations g_1 and g_2 where

$$g_1(x) = \begin{cases} 0 & \text{if } x = p^{m-1} - 1, \\ f(x) & \text{otherwise}, \end{cases}$$

$$g_2(x) = \begin{cases} 1 & \text{if } x = p^{m-1}, \\ f(x) & \text{otherwise}. \end{cases}$$

Further, it was proved that

$$\begin{aligned}&\mathrm{Con}(A,f)\neq\mathrm{Con}(A,g_1),\quad \mathrm{Con}(A,f)\neq\mathrm{Con}(A,g_2),\\&\mathrm{Con}(A,f)=\mathrm{Con}(A,g_1)\cap\mathrm{Con}(A,g_2).\end{aligned}$$

Similarly as in the case (A), let us define operations on $\bar{A}$ by setting $\bar{g}_1(x)=g_1(x')$, $\bar{g}_2(x)=g_2(x')$ for $x\in\bar{A}$. Let $x,y\in\bar{A}$. Then

$$\begin{aligned}\theta_{\bar{f}}(x,y)&=[x,y]\vee\theta_{\bar{f}}(\bar{f}(x),\bar{f}(y))=[x,y]\vee\theta_{\bar{f}}(\bar{f}(x'),\bar{f}(y'))\\&=[x,y]\vee\theta_f(f(x'),f(y'))\vee\Delta_{\bar{A}}\\&\supseteq[x,y]\vee\theta_{g_1}(g_1(x'),g_1(y'))\vee\theta_{g_2}(g_2(x'),g_2(y'))\vee\Delta_{\bar{A}}.\end{aligned}$$

To finish the proof it remains to show the converse inclusion, i.e, that

$$\theta_f(f(x'),f(y'))\subseteq\theta_{g_1}(g_1(x'),g_1(y'))\vee\theta_{g_2}(g_2(x'),g_2(y')),$$

which is equivalent to

$$(f(x'),f(y'))\in\theta_{g_1}(g_1(x'),g_1(y'))\vee\theta_{g_2}(g_2(x'),g_2(y'))$$

The congruence of the right side will be denoted by α. Clearly, we can assume that $(f(x'),f(y'))\neq(g_1(x'),g_1(y'))$; without loss of generality let $f(x')\neq g_1(x')$. Then $x'=p^{m-1}-1$, $g_1(x')=0$, $g_2(x')=f(x')=p^{m-1}$. Next, we can assume that $(f(x'),f(y'))\neq(g_2(x'),g_2(y'))$. Since $f(x')=g_2(x')$, it yields $y'=p^{m-1}$, $g_2(y')=1$, $g_2(y')=f(y')=p^{m-1}+1$. This follows

$$\begin{aligned}(f(x'),f(y'))&=(p^{m-1},p^{m-1}+1),\\(0,p^{m-1}+1)&=(g_1(x'),g_1(y'))\in\alpha,\\(p^{m-1},1)&=(g_2(x'),g_2(y'))\in\alpha.\end{aligned}$$

Then

$$(g_1^{(p-1)\cdot p^{m-1}}(0),g_1^{(p-1)\cdot p^{m-1}}(p^{m-1}+1))\in\alpha.$$

Since $(p^{m-1}+1)+((p-1)\cdot p^{m-1}-2)=p^m-1$ holds, we obtain

$$\begin{aligned}g_1^{(p-1)\cdot p^{m-1}}(p^{m-1}+1)&=g_1^2(g_1^{(p-1)\cdot p^{m-1}-2}(p^{m-1}+1))\\&=g_1^2((p^{m-1}+1)+((p-1)\cdot p^{m-1}-2))=g_1^2(p^m-1)=1.\end{aligned}$$

Since $g_1^{(p-1)\cdot p^{m-1}}(0)=0$, it follows that $(0,1)\in\alpha$. Using transitivity

$$(p^{m-1})\ \alpha\ 1\ \alpha\ 0\ \alpha\ (p^{m-1}+1),$$

and hence $(f(x'),f(y'))\in\alpha$. □

Lemma 3.4. *Let $(\bar{A},\bar{f})$ be a permutation-algebra with short tails and let the permutation-algebra (A,f) corresponding to $(\bar{A},\bar{f})$ be a transposition, $|A|>2$. Then $\mathrm{Con}(\bar{A},\bar{f})$ is $\wedge$-irreducible in $\mathcal{E}_{\bar{A}}$ if and only if $\mathrm{Con}(A,f)$ is $\wedge$-irreducible in $\mathcal{E}_A$.*

Proof. By the assumption, there are elements $0,1\in A$ such that $f(0)=1$, $f(1)=0$ and $f(x)=x$ for $x\in A\backslash\{0,1\}$.

Let $|A| = 2$. Obviously, $\mathrm{Con}(A, f)$ is $\wedge$-irreducible in $\mathcal{E}_A$. Suppose that $\bar{f}(x) = 0$ for each $x \in \bar{A}\backslash A$. If g is a nontrivial operation on $\bar{A}$ with $\mathrm{Con}(\bar{A}, \bar{f}) \subsetneq \mathrm{Con}(\bar{A}, g)$, then there is $a \in \bar{A}\backslash A$ such that

$$g(x) = \begin{cases} 1 & \text{if } x = a, \\ a & \text{otherwise,} \end{cases}$$

which yields that $\mathrm{Con}(\bar{A}, \bar{f})$ is $\wedge$-irreducible in $\mathcal{E}_{\bar{A}}$. If $|\bar{f}^{-1}(0)| > 1$ and $|\bar{f}^{-1}(1)| > 1$, then no nontrivial operation g on $\bar{A}$ with $\mathrm{Con}(\bar{A}, \bar{f}) \subsetneq \mathrm{Con}(\bar{A}, g)$ exists, thus $\mathrm{Con}(\bar{A}, \bar{f})$ is $\wedge$-irreducible as well.

Let $|A| \geq 3$. Then $\mathrm{Con}(A, f)$ is $\wedge$-reducible in $\mathcal{E}_A$. Suppose that $\bar{A} \neq A$. We define three operations g_1, g_2, g_3 on $\bar{A}$ by putting

$$g_1(x) = \begin{cases} 0 & \text{if } \bar{f}(x) = 1, \\ 1 & \text{if } \bar{f}(x) = 0, \\ \bar{f}(x) & \text{otherwise,} \end{cases}$$

$$g_2(x) = \begin{cases} 0 & \text{if } \bar{f}(x) = 1, \\ 1 & \text{otherwise,} \end{cases}$$

$$g_3(x) = \begin{cases} 1 & \text{if } \bar{f}(x) = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, the operations are nontrivial and it is easy to show that $\mathrm{Con}(\bar{A}, \bar{f}) \subsetneq \mathrm{Con}(\bar{A}, g_i)$, $i = 1, 2, 3$. We need to prove $\theta_{\bar{f}}(x, y) = \theta_{g_1}(x, y) \vee \theta_{g_2}(x, y) \vee \theta_{g_3}(x, y)$ for each $x, y \in \bar{A}$. Let α be the congruence on the right side. For simplicity, we name an element $x \in \bar{A}\backslash\{0, 1\}$ by: a if $\bar{f}(x) = 0$; b if $\bar{f}(x) = 1$; u if $\bar{f}(x) = x$; v if $x \neq \bar{f}(x) \notin \{0, 1\}$. Then $(x, y) \in \{(0, u), (a, 0), (a, 1), (a, b)\}$ implies $\theta_{\bar{f}}(x, y) = \theta_{g_2}(x, y)$. Since $\theta_{\bar{f}}(0, v) = [0, 1, u, v]$ for $u = \bar{f}(v)$ and

$$(0, v) \in \alpha \implies (0, u) = (g_1(0), g_1(v)) \in \alpha \implies (0, 1) = (g_2(0), g_2(u)) \in \alpha,$$

we get $\theta_{\bar{f}}(0, v) = [0, 1, u, v] = \theta_{g_1}(0, v) \vee \theta_{g_2}(0, v)$. Analogously, $\theta_{\bar{f}}(a, u) = [0, 1, a, u]$,

$$(a, u) \in \alpha \implies (1, u) = (g_1(a), g_1(u)) \in \alpha \implies (1, 0) = (g_3(1), g_3(u)) \in \alpha.$$

Further, $\theta_{\bar{f}}(a, v) = [a, v][0, 1, u]$ for $u = \bar{f}(v)$,

$$\begin{aligned} (a, v) \in \alpha &\implies (1, u) = (g_1(a), g_1(v)) \in \alpha, \\ (a, v) \in \alpha &\implies (1, 0) = (g_3(a), g_3(v)) \in \alpha. \end{aligned}$$

This completes the proof: $\mathrm{Con}(\bar{A}, \bar{f})$ is $\wedge$-reducible in $\mathcal{E}_{\bar{A}}$. □

Lemma 3.5. *Let* $(\bar{A}, \bar{f})$ *be a permutation-algebra with short tails and let* (A, f) *be the permutation-algebra corresponding to* $(\bar{A}, \bar{f})$ *with* f *nontrivial. If* $\mathrm{Con}(A, f)$ *is* $\wedge$*-irreducible in* $\mathcal{E}_A$ *then* $\mathrm{Con}(\bar{A}, \bar{f})$ *is* $\wedge$*-irreducible in* $\mathcal{E}_{\bar{A}}$.

Proof. Suppose that $\mathrm{Con}(A, f)$ is $\wedge$-irreducible in $\mathcal{E}_A$ and, by the way of contradiction, assume that $\mathrm{Con}(\bar{A}, \bar{f})$ is $\wedge$-reducible in $\mathcal{E}_{\bar{A}}$. There exist nontrivial operations h_i, $i \in I$ such that

$$\mathrm{Con}(\bar{A},\bar{f}) = \bigcap_{i\in I}\mathrm{Con}(\bar{A},h_i), (\forall i\in I)\,\mathrm{Con}(\bar{A},\bar{f}) \subsetneq \mathrm{Con}(\bar{A},h_i).$$

First assume that there is $i\in I$ such that A fails to be a subalgebra of $(\bar{A},h_i)$. Then there are $a\in A, b\in \bar{A}\backslash A$ with $h_i(a)=b$. Also, $\bar{A}\backslash\{b\}$ is a subalgebra of $(\bar{A},\bar{f})$ and it is not a subalgebra of $(\bar{A},h_i)$. According to Lemma 2.6, $h_i(x)=b$ for each $x\in\bar{A}\backslash\{b\}$. The operation h_i is nontrivial and

$$(h_i(b),b)) = (h_i(b),h_i(b')) \in \theta_{h_i}(b,b') \subseteq \theta_{\bar{f}}(b,b') = [b,b'],$$

hence $h_i(b)=b'$. If there is $c\in\bar{A}\backslash(A\cup\{b\})$, then

$$\begin{aligned}(b,b') &= (h_i(c),h_i(b)) \in \theta_{h_i}(c,b) \subseteq \theta_{\bar{f}}(c,b)\\ &= [c,b]\vee\theta_{\bar{f}}(\bar{f}(c),\bar{f}(b)) = [c,b]\vee\theta_{\bar{f}}(c',b') = [c,b][c',b',\dots]\dots\end{aligned}$$

which is a contradiction. Therefore $\bar{A}\backslash A=\{b\}$. This implies that A is a subalgebra of $(\bar{A},h_j)$ for each $j\neq i$ and we may denote $g_j(x)=h_j(x)$ for $x\in A$. Let $x,y\in A$. Then

$$\begin{aligned}\theta_f(x,y)\vee\Delta_{\bar{A}} &= \theta_{\bar{f}}(x,y) = \bigvee_{j\in I}\theta_{h_j}(x,y) = \theta_{h_i}(x,y)\vee\bigvee_{j\in I\backslash\{i\}}\theta_{h_j}(x,y)\\ &= [x,y]\vee\bigvee_{j\in I\backslash\{i\}}\theta_{h_j}(x,y) = \bigvee_{j\in I\backslash\{i\}}\theta_{h_j}(x,y)\\ &= \bigvee_{j\in I\backslash\{i\}}\theta_{g_j}(x,y)\vee\Delta_{\bar{A}},\end{aligned}$$

which implies $\theta_f(x,y)=\bigvee_{j\in I\backslash\{i\}}\theta_{g_j}(x,y)$.

Now let J be the set of all $j\in I$ such that A is a subalgebra of $(\bar{A},h_j)$ and g_j is nontrivial. Then

$$\theta_f(x,y) = \bigvee_{j\in J}\theta_{g_j}(x,y).$$

According to the assumption that $\mathrm{Con}(A,f)$ is $\wedge$-irreducible in $\mathcal{E}_A$, there exists $j\in J$ such that $\mathrm{Con}(A,f)=\mathrm{Con}(A,g_j)$. In the paper [5] there were studied pairs of monounary algebras with coinciding congruence lattices. We will use the result that if one of the operations is a permutation, then so is the other (see [5], Theorem 6.10). This implies that if $\mathrm{Con}(A,g_j)=\mathrm{Con}(A,f)$, then g_j is a permutation and for $a,b\in\bar{A}$,

$$\begin{aligned}\theta_{\bar{f}}(a,b) &= [a,b]\vee\theta_{\bar{f}}(\bar{f}(a),\bar{f}(b)) = [a,b]\vee\theta_{\bar{f}}(\bar{f}(a'),\bar{f}(b'))\\ &= [a,b]\vee\theta_f(f(a'),f(b'))\vee\Delta_{\bar{A}} = [a,b]\vee\theta_f(a',b')\vee\Delta_{\bar{A}}\\ &= [a,b]\vee\theta_{g_j}(a',b')\vee\Delta_{\bar{A}} = [a,b]\vee\theta_{g_j}(g_j(a'),g_j(b'))\vee\Delta_{\bar{A}}\\ &= [a,b]\vee\theta_{g_j}(g_j(a'),g_j(b'))\vee\Delta_{\bar{A}} = [a,b]\vee\theta_{h_j}(h_j(a'),h_j(b'))\\ &= [a,b]\vee\theta_{h_j}(h_j(a),h_j(b)) = \theta_{h_j}(a,b).\end{aligned}$$

Hence $\mathrm{Con}(\bar{A},\bar{f})=\mathrm{Con}(\bar{A},h_j)$ and this is a contradiction. □

In the following theorem, we assume that $\bar{A}\backslash A\neq\emptyset$. Then in view of the Lemmas 3.3, 3.4 and 3.5 we obtain:

Theorem 3.6. *Let* $(\bar{A}, \bar{f})$ *be a permutation-algebra with short tails and let* (A, f) *be the permutation-algebra corresponding to* $(\bar{A}, \bar{f})$. *Then* $\mathrm{Con}(\bar{A}, \bar{f})$ *is* $\wedge$*-irreducible in* $\mathcal{E}_{\bar{A}}$ *if and only if either* $|A| = 2$ *or* $|A| > 2$ *and* $\mathrm{Con}(A, f)$ *is* $\wedge$*-irreducible in* $\mathcal{E}_A$.

4. Small cycles: ∧-reducible cases

In the following sections, we will consider monounary algebras with small cycles.

Lemma 4.1. *Suppose that* (A, f) *is a monounary algebra,* $A = K \cup L$ *such that* $L \neq \emptyset$ *and* $(L, f \restriction L)$ *is a permutation-algebra with short tails. If* K *is a component of* (A, f) *and there are distinct elements* $0, 1, 2 \in K$ *with* $f(1) = f(0) = 0$, $f(2) = 1$, *then* $\mathrm{Con}(A, f)$ *is* $\wedge$*-reducible.*

Proof. We define the following operations on A:

$$g_1(x) = \begin{cases} f(x) & \text{if } x \in K, \\ 0 & \text{otherwise,} \end{cases}$$

$$g_2(x) = \begin{cases} 0 & \text{if } x \in K, \\ f(x) & \text{otherwise.} \end{cases}$$

Clearly, g_1, g_2 are nontrivial and $\mathrm{Con}(A, f) \neq \mathrm{Con}(A, g_1), \mathrm{Con}(A, f) \neq \mathrm{Con}(A, g_2)$. To prove that $\mathrm{Con}(A, f) = \mathrm{Con}(A, g_1) \wedge \mathrm{Con}(A, g_2)$, we prove that $\theta_f(x, y) = \theta_{g_1}(x, y) \vee \theta_{g_2}(x, y)$ for every $x, y \in A$.

First, we show that $\theta_{g_1}(x, y), \theta_{g_2}(x, y) \subseteq \theta_f(x, y)$ for every $x, y \in A$. Clearly, it is sufficient to consider the case when $x \in K, y \in L$. Then

$$\theta_{g_1}(x, y) = [x, y][0, f(x), f^2(x), f^3(x), \dots],$$
$$\theta_{g_2}(x, y) = [x, y][0, f(y), f^2(y), f^3(y), \dots].$$

Since

$$\theta_f(x, y) = [x, y][f(x), f^2(x), \dots, f(y), f^2(y), \dots],$$

it is clear to see that $\theta_{g_1}(x, y), \theta_{g_2}(x, y) \subseteq \theta_f(x, y)$. Therefore $\theta_{g_1}(x, y) \vee \theta_{g_2}(x, y) \subseteq \theta_f(x, y)$ for every $x, y \in A$. Now, to prove

$$\begin{aligned} \theta_f(x, y) &= [x, y] \vee \theta_f(f(x), f(y)) \subseteq \theta_{g_1}(x, y) \vee \theta_{g_2}(x, y) \\ &= [x, y] \vee \theta_{g_1}(g_1(x), g_1(y)) \vee \theta_{g_2}(g_2(x), g_2(y)), \end{aligned}$$

it is sufficient to show that

$$(f(x), f(y)) \in \theta_{g_1}(g_1(x), g_1(y)) \vee \theta_{g_2}(g_2(x), g_2(y)).$$

The congruence on the right side will be denoted by α. If $x, y \in K$ or $x, y \in L$, then $(f(x), f(y)) \in \alpha$ holds trivially. If $x \in K, y \in L$ then $g_1(x) = f(x), g_1(y) = 0, g_2(x) = 0, g_2(y) = f(y)$. This implies that $(f(x), 0) = (g_1(x), g_1(y)) \in \alpha$ and $(0, f(y)) = (g_2(x), g_2(y)) \in \alpha$, hence by transitivity $(f(x), f(y)) \in \alpha$. □

Lemma 4.2. *Let* (A, f) *be a monounary algebra with small cycles such that there are distinct elements* $0, 1, 2, 0', 1', 2'$ *with* $f(0) = f(1) = 0, f(2) = 1, f(0') = f(1') = 0', f(2') = 1'$. *Further, suppose that* (A, f) *contains a single two-element cycle* $\{a, b\}$ *and that cycle* $\{a, b\}$ *has only short tails. Then* $\mathrm{Con}(A, f)$ *is* $\wedge$*-reducible.*

Proof. We define the following operations on A:

$$g_1(x) = \begin{cases} a & \text{if } x \in f^{-1}(b), \\ b & \text{if } x \in f^{-1}(a), \\ f(x) & \text{otherwise,} \end{cases}$$

$$g_2(x) = \begin{cases} a & \text{if } x \in f^{-1}(b), \\ f(x) & \text{otherwise,} \end{cases}$$

$$g_3(x) = \begin{cases} b & \text{if } x \in f^{-1}(a), \\ f(x) & \text{otherwise.} \end{cases}$$

Clearly, g_1, g_2, g_3 are nontrivial and $\mathrm{Con}(A, f) \neq \mathrm{Con}(A, g_i)$ for $i \in \{1, 2, 3\}$. First, we show that for each $x, y \in A$ and for every $i \in \{1, 2, 3\}$ is

$$\theta_{g_i}(x, y) \subseteq \theta_f(x, y).$$

Denote the set of all elements of the component containing $\{a, b\}$ by K. If either $x, y \notin K$ or $x, y \in K$, $\theta_{g_i}(x, y) \subseteq \theta_f(x, y)$ holds trivially for every $i \in \{1, 2, 3, \}$.

Let $x \in \{a, b\}, y \notin K$. Then for each $i \in \{1, 2, 3\}$

$$\theta_{g_i}(x, y) = [x, y] \vee [\{g_i(x)\} \cup Z_f(f(y))] \subseteq \theta_f(x, y) = [Z_f(y) \cup \{a, b\}].$$

Similarly, if $x \in f^{-1}(a) \cup f^{-1}(b), y \notin K$, then for each $i \in \{1, 2, 3\}$

$$\theta_{g_i}(x, y) = [x, y][\{g_i(x)\} \cup Z_f(f(y))] \subseteq \theta_f(x, y) = [x, y][Z_f(f(y)) \cup \{a, b\}].$$

To prove that $\theta_f(x, y) = \theta_{g_1}(x, y) \vee \theta_{g_2}(x, y) \vee \theta_{g_3}(x, y)$, for each $x, y \in A$, it remains to show that

$$\theta_f(x, y) \subseteq \theta_{g_1}(x, y) \vee \theta_{g_2}(x, y) \vee \theta_{g_3}(x, y),$$

which is equivalent to

$$(f(x), f(y)) \in \theta_{g_1}(g_1(x), g_1(y)) \vee \theta_{g_2}(g_2(x), g_2(y)) \vee \theta_{g_3}(g_3(x), g_3(y)) = \alpha.$$

We can assume that $x \neq y, f(x) \neq f(y)$ and $(f(x), f(y)) \neq (g_2(x), g_2(y))$. Without loss of generality, let $f(x) \neq g_2(x)$. Then $x \in f^{-1}(b), f(x) = b$. Similarly, assume that $(b, f(y)) = (f(x), f(y)) \neq (g_3(x), g_3(y)) = (b, g_3(y))$. This yields $f(y) \neq g_3(y)$, hence $y \in f^{-1}(a)$ and $f(y) = a$. It follows that $\alpha = [a, b] \vee [a, a] \vee [b, b] = [a, b]$ and $(f(x), f(y)) = (b, a) \in \alpha$. □

Lemma 4.3. *Let* (A, f) *be a monounary algebra with* A *being the union of the disjoint sets* $\{0, 1, a, b\}, L$ *and* M *such that* $f(0) = 1, f(1) = f(a) = 0, f(b) = a$; *for every* $l \in L$ *either* $f(l) \in \{0, 1\}$ *or* $a \in Z_f(l)$; *and* $(M, f \restriction M)$ *is a permutation-algebra with short tails whose cycles have length* 1 *or* 2. *Then* $\mathrm{Con}(A, f)$ *is* $\wedge$*-reducible.*

Proof. We define the following operations on A

$$g_1(x) = \begin{cases} 1 & \text{if } f(x) = a, \\ f(x) & \text{otherwise}, \end{cases}$$

$$g_2(x) = \begin{cases} f(x) & \text{if } f(x) = a, \\ 1 & \text{otherwise}. \end{cases}$$

Let us denote set $\{l \in L, a \in Z_f(l)\}$ as B and set $L \backslash B$ as L'.

Obviously, g_1, g_2 are nontrivial and $\mathrm{Con}(A, f) \neq \mathrm{Con}(A, g_1), \mathrm{Con}(A, g_2)$.

To we show that $\theta_{g_1}(x, y) \subseteq \theta_f(x, y)$ for each $x, y \in A$, we consider the following four cases:

Case 1: If $x, y \in \{0, 1, a\} \cup L' \cup M$, this holds trivially.

Case 2: Let $x, y \in \{0, 1, a, b\} \cup B$. First, let $x, y \neq a$ and without loss of generality, let $x \in Z_f(y)$. If $d_f(y, x) \equiv 1 \pmod 2$ then

$$\theta_{g_1}(x, y) = [Z_{g_1}(y)] \subseteq [Z_f(y)] = \theta_f(x, y).$$

If $d_f(y, x) \equiv 0 \pmod 2$ then

$$\theta_{g_1}(x, y) = [x, y, f^2(x), f^2(y), ...] \vee [f(x), f(y), f^3(x), f^3(y), ...] \subseteq \theta_f(x, y).$$

Similarly, if $x \notin Z_{g_1}(y)$ and $y \notin Z_{g_1}(x)$, it is also easy to see that $\theta_{g_1}(x, y) \subseteq \theta_f(x, y)$. Now, let $x = a$. If either $d_f(y, a) \equiv 0 \bmod (2)$ or $y = 0$, then

$$\theta_{g_1}(x, y) = [a, y] \vee [0, f(y), f^3(y), ...] \vee [1, f^2(y), f^4(y), ...] \subseteq \theta_f(x, y).$$

Otherwise, $\theta_{g_1}(x, y) = [Z_{g_1}(y) \cup \{a\}] \subseteq \theta_f(x, y)$.

Case 3: Let $x \in B \cup \{b\}, y \in L'$. If $f(x) = a$ then trivially $\theta_{g_1}(x, y) \subseteq \theta_f(x, y)$. Otherwise

$$\theta_{g_1}(x, y) = [x, y] \vee \theta_{g_1}(g_1(x), g_1(y)) = [x, y] \vee \theta_{g_1}(f(x), f(y)).$$

Since $f(x), f(y) \in \{0, 1, b\} \cup B$, we get

$$\theta_{g_1}(x, y) = [x, y] \vee \theta_{g_1}(f(x), f(y)) \subseteq [x, y] \vee \theta_f(f(x), f(y)) = \theta_f(x, y).$$

Case 4: Let $x \in B \cup \{b\}, y \in M$. If $f(x) = a$ then trivially $\theta_{g_1}(x, y) \subseteq \theta_f(x, y)$. Otherwise

$$\begin{aligned} \theta_{g_1}(x, y) &= [x, y] \vee \theta_{g_1}(g_1(x), g_1(y)) = [x, y] \vee \theta_{g_1}(f(x), f(y)) \\ &\subseteq [x, y] \vee \theta_f(f(x), f(y)) = \theta_f(x, y). \end{aligned}$$

On the other hand, $\theta_{g_2}(x, y) \subseteq \theta_f(x, y)$ holds trivially for each $x, y \in A$.

It remains to show that $\theta_f(x, y) \subseteq \theta_{g_1}(x, y) \vee \theta_{g_2}(x, y)$. This is proved similarly like in the proof of Lemma 4.2. □

Lemma 4.4. *Let (A, f) be a monounary algebra with small cycles. Let it contain a component K such that there are distinct elements $0, 1, a, b, c, d$ with $f(0) = 1, f(1) = f(a) = f(c) = 0, f(b) = a, f(d) = c$ and for each $x \in K$ either $f(x) \in \{0, 1\}$ or $f^2(x) = 0$. Moreover, all other components of (A, f) contain only short tails. Then* $\mathrm{Con}(A, f)$ *is* $\wedge$*-reducible.*

Proof. Let us denote the set of elements x of A such that $f(x) \neq 1$ and $f^2(x) = 0$ as K'. We define the following operations on A:

$$g_1(x) = \begin{cases} 1 & \text{if } x \in K', \\ f(x) & \text{otherwise,} \end{cases}$$

$$g_2(x) = \begin{cases} f(x) & \text{if } x \in K', \\ 1 & \text{otherwise.} \end{cases}$$

Obviously, g_1, g_2 are nontrivial and $\mathrm{Con}(A, f) \neq \mathrm{Con}(A, g_1), \mathrm{Con}(A, g_2)$.

If $x, y \in A \backslash K'$ then clearly $\theta_{g_1}(x, y) = \theta_f(x, y)$. Let $x \in K'$. If $y \in K$ then clearly $\theta_{g_1}(x, y) \subseteq \theta_f(x, y)$. If $y \in A \backslash K$ then

$$\theta_{g_1}(x, y) = [x, y][1, f(y), f^3(y), \dots][0, f^2(y), f^4(y), \dots] \subseteq$$
$$\theta_f(x, y) = [x, y][a, 1, f(y), f^3(y), \dots][0, f^2(y), f^4(y), \dots].$$

Moreover, $\theta_{g_2}(x, y) \subseteq \theta_f(x, y)$ holds trivially for each $x, y \in A$.

In the view of the proof of Lemma 4.2, $\theta_f(x, y) \subseteq \theta_{g_1}(x, y) \vee \theta_{g_2}(x, y)$ holds for each $x, y \in A$. Then $\mathrm{Con}(A, f) = \mathrm{Con}(A, g_1) \wedge \mathrm{Con}(A, g_2)$, hence $\mathrm{Con}(A, f)$ is $\wedge$-reducible. □

5. Small cycles: ∧-irreducible cases

Now, we present the main result of this part - the characterization of the $\wedge$-irreducibility of the congruence lattices of monounary algebras with small cycles.

Lemma 5.1. *Let (A, f) be a monounary algebra and let (A, g) be a monounary algebra such that $\mathrm{Con}(A, f) \subseteq \mathrm{Con}(A, g)$. Let there be distinct elements $0, 1, 2, 0', 1', 2' \in A$ with $f(1) = f(0) = 0$, $f(2) = 1$ and $f(0') \neq 0$, $f(1') = 0'$, $f(2') = 1'$ such that $0'$ is cyclic and $1'$ is noncyclic. Let there be an equivalence (with a simple nontrivial equivalence class) $[0, 2] \notin \mathrm{Con}(A, g)$. Then*

(i) *g and f agree on the set $\{0, 1, 2, 1', 2'\}$,*
(ii) *$g(x) \in Z_f(f(x))$ for each $x \in A$,*
(iii) *if $x \in A$ and $f(x)$ is noncyclic, then $g(x) = f(x)$.*

Proof. Put $D = C(0')$. The assumption yields that $\{0, 1, 2\}$ is a subalgebra of (A, g), otherwise g would be constant on $\{0, 1, 2\}$ and $[0, 2] \in \mathrm{Con}(A, g)$. Next we show that $\{0, 1\}$ is a subalgebra of (A, g). Suppose that this fails to hold. Then $g(0) = g(1) = 2$ and $g(2) \in \{0, 1, 2\}$. Since $[0, 2] \notin \mathrm{Con}(A, g)$, $g(2) = 1$. Then for a subalgebra $D \cup \{0, 1, 1'\}$ of (A, f) we get that g is constant on this set, hence $g(1') = 2$ and thus

$$(1, 2) = (g(2), g(1)') \in \theta_f(2, 1') = [2, 1'][\{1, 0\} \cup D],$$

which is a contradiction. Now, if $g(0) \neq 0$, then the previous result implies $g(0) = 1$. Because 1 does not belong to a subalgebra $D \cup \{0, 1'\}$ of (A, f), g is constant on this set, thus $g(2') = 1$. From this it follows

$$(g(2), 1) = (g(2), g(2')) \in \theta_f(2, 2') = [2, 2'][1, 1'][\{0\} \cup D],$$
$$(g(2), 1) = (g(2), g(0)) \in \theta_f(2, 0) = [0, 1, 2]$$

thus $g(2) = 1$, which contradicts $[0, 2] \notin \mathrm{Con}(A, g)$. Therefore $g(0) = 0$. Again using $[0, 2] \notin \mathrm{Con}(A, g)$, $g(2) \notin \{0, 2\}$. i.e., $g(2) = 1$. From

$$(1, g(2')) = (g(2), g(2')) \in \theta_f(2, 2') = [2, 2'][1, 1'][\{0\} \cup D],$$
$$(0, g(2')) = (g(0), g(2')) \in \theta_f(0, 2') = [\{0, 1', 2'\} \cup D]$$

it follows that $g(2') = 1'$. Then

$$(g(1), 1') = (g(1), g(2')) \in \theta_f(1, 2') = [1, 2'][\{0, 1'\} \cup D],$$
$$(g(1), 0) = (g(1), g(0)) \in \theta_f(1, 0) = [1, 0]$$

implies that $g(1) = 0$. Further, since $g(2') \in D \cup \{1', 2'\}$ which is a subalgebra of (A, f), the set $D \cup \{1', 2'\}$ is a subalgebra of (A, g). Next, from

$$(0, g(1')) = (g(1), g(1')) \in \theta_f(1, 1') = [1, 1'][\{0\} \cup D]$$

we get $g(1') \in \{0\} \cup D$, thus $g(1') = 0'$. Hence, we have proved that g and f agree on the set $\{0, 1, 2, 1', 2'\}$.

Since

$$(g(0'), 0) = (g(0'), g(0)) \in \theta_f(0', 0) = [\{0\} \cup D],$$
$$(g(0'), 0') = (g(0'), g(1')) \in \theta_f(0', 1') = [\{1'\} \cup D],$$

$g(0') \in D$. Let $x \in A$. If $f(x) = x$, then

$$(g(x), 0) = (g(x), g(0)) \in \theta_f(x, 0) = [x, 0],$$
$$(g(x), g(0')) \in \theta_f(x, 0') = [\{x\} \cup D],$$

thus $g(x) = x = f(x)$. If x belongs to a two-element cycle $\{x, x'\}$, then

$$(g(x), 0) = (g(x), g(0)) \in \theta_f(x, 0) = [x, x', 0],$$

and since $(0, g(0')) \notin \theta_f(x, 0')$, we get that $g(x) \in \{x, x'\} = Z_f(f(x))$. Now suppose that x is a noncyclic element. Then

$$(g(x), 0) = (g(x), g(1)) \in \theta_f(x, 1) = [x, 1][\{0\} \cup Z_f(f(x))],$$

thus $g(x) = 0$ or $g(x) \in Z_f(f(x))$. If $0 \notin Z_f(x)$, then $(0, 0') \notin \theta_f(x, 1')$ implies $g(x) \in Z_f(f(x))$. If $0 \in Z_f(x)$, then

$$(g(x), 0) = (g(x), g(1)) \in \theta_f(x, 1) = [x, 1][\{0\} \cup Z_f(f(x))],$$

implies $g(x) \in Z_f(f(x))$, which completes the proof of (ii).

Finally, suppose that $f(x)$ is noncyclic. If $x = 1$ or $f(x) = 1$, then according to (i), $g(x) = f(x)$. Otherwise from (ii) and

$$(g(x), 1') = (g(x), g(2')) \in \theta_f(x, 2') = [x, 2'][f(x), 1'][f^2(x), 0'] \dots,$$

it follows that $g(x) = f(x)$. □

Lemma 5.2. *Let the assumption of the above lemma be satisfied and let* $f(0') = 0'' \neq 0'$, $f(0'') = 0'$. *Then* $g(x) = f(x)$ *for each* $x \in A$.

Proof. From

$$(0', g(0'')) = (g(1'), g(0'')) \in \theta_f(1', 0'') = [1', 0''],$$

we get $g(0'') = 0' = f(0'')$. Next, $g(0') \in Z_f(f(0')) = \{0'', 0'\}$, thus

$$(1', g(0')) = (g(2'), g(0')) \in \theta_f(2', 0') = [2', 0'][1', 0'']$$

implies $g(0') = 0'' = f(0')$.

Let $x \in A \backslash \{0, 1, 2, 0', 1', 2', 0''\}$. By Lemma 5.1 (ii) and (iii), if $f(x) = x$, $f(x)$ is noncyclic or $f^2(x) = f(x)$, then $g(x) = f(x)$. Otherwise let $f(x) = y \neq x$. If $f(y) = x$, then

$$(g(x), 0'') = (g(x), g(0')) \in \theta_f(x, 0') = [x, 0'][y, 0''],$$

hence $g(x) = y = f(x)$. Now let there be $z \in A \backslash \{x, y\}$ with $f(y) = z = f^2(z)$. Since $\{y, z\}$ is a two-element cycle, we have already shown that $g(z) = y$, thus

$$(g(x), y) = (g(x), g(z)) \in \theta_f(x, z) = [x, z]$$

implies $g(x) = y = f(x)$ as well. □

Theorem 5.3. *Let (A, f) be a monounary algebra with small cycles and such that there are distinct elements $0, 1, 2, 0', 1', 2'$ with $f(1) = f(0) = 0$, $f(2) = 1$, $f(0') \neq 0$, $f(1') = 0'$, $f(2') = 1'$. Let $0'$ be cyclic and $1'$ be noncyclic. If*

(a) *(A, f) is acyclic, or*
(b) *$f(0') = 0'' \neq 0', f(0'') = 0'$, or*
(c) *(A, f) contains at least two 2-element cycles,*

then $\mathrm{Con}(A, f)$ *is $\wedge$-irreducible.*

Proof. (a) From Lemma 5.1 (see also [6] Theorem 6.4) it follows that if (A, f) is acyclic, then $\mathrm{Con}(A, f)$ is $\wedge$-irreducible.

(b) Suppose that there exists $0'' \in A$ with $f(0') = 0'' \neq 0'$. Since (A, f) possesses only small cycles, $f(0'') = 0'$. From Lemma 5.2 we conclude that

$$[0, 2] \in \bigcap \{\mathrm{Con}(A, g) : \mathrm{Con}(A, f) \subsetneq \mathrm{Con}(A, g)\}.$$

Since $\theta_f(0, 2) = [0, 1, 2]$ we have $[0, 2] \notin \mathrm{Con}(A, f)$ and the above intersection cannot be equal to $\mathrm{Con}(A, f)$. Therefore $\mathrm{Con}(A, f)$ is $\wedge$-irreducible.

(c) Suppose that $f(0') = 0'$ and that there exist two distinct two-element cycles $\{a, b\}$, $\{u, v\}$.

Let (A, g) be a monounary algebra with $\mathrm{Con}(A, f) \subseteq \mathrm{Con}(A, g)$. The assumption of Lemma 5.1 is satisfied, thus by (i) and (ii) of it, $g = f$ on the set K of the elements of all components with one-element cycles; and by (iii), if $f(x)$ is noncyclic then $g(x) = f(x)$. Also, $g(x) \in Z_f(f(x))$ for each $x \in A$.

Moreover,

$$(g(a), g(v)) \in \theta_f(a, v) = [a, v][b, u],$$
$$(g(a), g(u)) \in \theta_f(a, u) = [a, u][b, v],$$

which implies that either $g = f$ on the set $\{a, b, u, v\}$ or g is identity on $\{a, b, u, v\}$.

If $g = f$ on the set $\{a, b, u, v\}$, then clearly $g = f$ on A.

If g is identity on $\{a, b, u, v\}$, then $[a, 0] \in \operatorname{Con}(A, g)$. Therefore

$$[a, 0] \in \bigcap\{\operatorname{Con}(A, g) : \operatorname{Con}(A, f) \subsetneq \operatorname{Con}(A, g)\}.$$

However, $\theta_f(a, 0) = [a, b, 0]$, which implies that $[a, 0] \notin \operatorname{Con}(A, f)$ and the above intersection cannot be equal to $\operatorname{Con}(A, f)$. Therefore $\operatorname{Con}(A, f)$ is $\wedge$-irreducible. □

It remains to examine $\wedge$-irreducibility of $\operatorname{Con}(A, f)$ in the case when (A, f) contains at least one two-element cycle and each one-element cycle has only short tails.

Lemma 5.4. *Let (A, f) be a monounary algebra such that there are distinct elements $0, 1, a, d, b$ with $f(0) = 1$, $f(1) = f(a) = f(d) = 0$, $f(b) = a$ and let $\operatorname{Con}(A, f) \subseteq \operatorname{Con}(A, g)$. Then one of the following conditions is satisfied:*

(1) *g is an identity on the set $\{0, 1, a, b, d\}$*
(2) *g is a constant on $\{0, 1, a, b, d\}$*
(3) *g is equal to f on $\{0, 1, a, b, d\}$*
(4) *g is equal to f on $\{0, 1, a, d\}$ and $g(b) = 1$,*
(5) *$g(1) = g(a) = g(d) = 1$, $g(0) = g(b) = 0$,*
(6) *g is constant on the set $\{0, 1, a, d\}$, the constant is 1 and $g(b) = a$,*
(7) *g is constant on the set $\{0, 1, a, d\}$, the constant is a and $g(b) = 1$.*

Proof. Let the assumption be valid. According to

$$(g(0), g(1)) \in \theta_f(0, 1) = [0, 1],$$
$$(g(a), g(1)) \in \theta_f(a, 1) = [a, 1],$$

the following cases can occur:

(a) g is equal to f on the set $\{0, 1, a\}$,
(b) g is identity on $\{0, 1, a\}$,
(c) g is constant on $\{0, 1, a\}$
(d) $g(0) = 0, g(1) = g(a) = 1$,
(e) $g(0) = g(1) = 1, g(a) = a$.

Then according to

$$(g(d), g(1)) \in \theta_f(d, 1) = [d, 1],$$
$$(g(a), g(d)) \in \theta_f(a, d) = [a, d],$$
$$(g(b), g(0)) \in \theta_f(b, 0) = [b, 0][a, 1],$$
$$(g(b), g(d)) \in \theta_f(b, d) = [b, d][a, 0, 1],$$

if a) holds then $g(d) = 0, g(b) \in \{a, 1\}$, hence (3) or (4) is satisfied. If b) holds then $g(d) = d, g(b) = b$, hence (1) is satisfied. In the case c), either $g(d), g(b)$ both equal the constant or the constant is 1 and $g(d) = 1, g(b) = a$ or the constant is a and $g(d) = a, g(b) = 1$. So the case c) implies that either (2), (6) or (7) is satisfied. If d) holds then $g(d) = 1, g(b) = 0$, hence (5) is satisfied. Finally, if e) holds then $g(d) = d$ and $g(b) = b$ which yields $(b, 1) = (g(b), g(0)) \in \theta_f(b, 0) = [b, 0][a, 1]$, a contradiction. □

Lemma 5.5. *Let* (A, f) *be a monounary algebra such that there are distinct elements* $0, 1, a, b, d$ *with* $f(0) = f(d) = 1$, $f(1) = f(a) = 0$, $f(b) = a$ *and let* $\operatorname{Con}(A, f) \subseteq \operatorname{Con}(A, g)$*. Then one of the following conditions is satisfied:*

(1) g *is an identity on the set* $\{0, 1, a, b, d\}$
(2) g *is a constant on* $\{0, 1, a, b, d\}$
(3) g *is equal to* f *on* $\{0, 1, a, b, d\}$
(4) g *is equal to* f *on* $\{0, 1, a, d\}$ *and* $g(b) = 1$,
(5) $g(0) = g(b) = g(d) = 0$, $g(1) = g(a) = 1$,
(6) g *is constant on the set* $\{0, 1, a, d\}$*, the constant is* 1 *and* $g(b) = a$,
(7) g *is constant on the set* $\{0, 1, a, d\}$*, the constant is* a *and* $g(b) = 1$.

Proof. Let the assumptions be valid. According to proof of Lemma 5.4, cases a) – e) may occur. Moreover

$$\begin{aligned}
(g(0), g(d)) &\in \theta_f(0, d) = [0, d],\\
(g(0), g(b)) &\in \theta_f(0, b) = [0, b][1, a],\\
(g(a), g(d)) &\in \theta_f(a, d) = [a, d][0, 1],\\
(g(b), g(d)) &\in \theta_f(b, d) = [b, d][a, 1].
\end{aligned}$$

Then similarly to proof of the previous Lemma, we get that the conditions (1)–(7) are satisfied and that no other case may occur. □

In the following Lemmas 5.6–5.10 we will assume that:

- each one-element cycle has only short tails,
- there are distinct elements $0, 1, a, b, 0', 1' \in A$ such that $f(0) = 1$, $f(a) = f(1) = 0$, $f(b) = a$, $f(0') = 1'$, $f(1') = 0'$.

Lemma 5.6. *Let* (A, f) *be a monounary algebra. Suppose that* $\operatorname{Con}(A, f) \subseteq \operatorname{Con}(A, g)$ *and* $\rho = [0, 1][a, b] \notin \operatorname{Con}(A, g)$, $\pi = [a, 0'] \notin \operatorname{Con}(A, g)$*. Then one of the following conditions is satisfied:*

(1) g *is equal to* f *on the set* $\{0, 1, 0', 1', a, b\}$,
(2) $g(a) = 1$ *and* $g(x) = a, x \in \{0, 1, 0', 1', b\}$,
(3) $g(a) = a$ *and* $g(x) = 1, x \in \{0, 1, 0', 1', b\}$,
(4) g *is identity on the set* $\{0, 1, 0', 1', b\}$ *and* $g(a) = 1$.

Proof. Let the assumption be valid. If $\{0, 1, 0', 1'\}$ fails to be a subalgebra of (A, g), then g is constant on the set $\{0, 1, 0', 1'\}$ (the constant, say z, does not belong to $\{0, 1, 0', 1'\}$). From

$$(g(a), z) = (g(a), g(1)) \in \theta_f(a, 1) = [a, 1],$$

it follows that $g(a) = z$ or $g(a) = 1, z = a$. From

$$(g(b), z) = (g(b), g(0)) \in \theta_f(b, 0) = [b, 0][a, 1]$$

it follows that $g(b) = z$ or $g(b) = 0, z = b$ or $g(b) = 1, z = a$. Moreover, if $g(a) = g(b)$ then $\rho \in \operatorname{Con}(A, g)$, a contradiction. Hence only the following cases may occur:

(a) $z = b$, $g(a) = z$, $g(b) = 0$,
(b) $z = a$, $g(a) = z$, $g(b) = 1$,

(c) $z = a$, $g(a) = 1$, $g(b) = z$.

In the cases (a) or (b) we get $\pi \in \mathrm{Con}(A, g)$, a contradiction. Hence (a), (b) cannot occur. If (c) holds, then (2) is valid.

Otherwise, let $\{0, 1, 0', 1'\}$ be a subalgebra of (A, g). According to

$$\begin{aligned}(g(0), g(1)) &\in \theta_f(0, 1) = [0, 1],\\ (g(0'), g(1')) &\in \theta_f(0', 1') = [0', 1'],\\ (g(1), g(1')) &\in \theta_f(1, 1') = [1, 1'][0, 0'],\\ (g(0), g(1')) &\in \theta_f(0, 1') = [0, 1'],\end{aligned}$$

hence only the following cases may occur:

(d) g equals to f on the set $\{0, 1, 0', 1'\}$,
(e) g is a constant on the set $\{0, 1, 0', 1'\}$ such that the constant belongs to $\{0, 1, 0', 1'\}$,
(f) g is identity on $\{0, 1, 0', 1'\}$.

Moreover,

$$\begin{aligned}(g(a), g(1)) &\in \theta_f(a, 1) = [a, 1],\\ (g(b), g(0)) &\in \theta_f(b, 0) = [b, 0][a, 1].\end{aligned}$$

In the case d), we get $g(a) = 0$ and either $g(b) = a$ or $g(b) = 1$. If $g(b) = a$ then (1) is valid. If $g(b) = 1$, $\rho \in \mathrm{Con}(A, g)$, a contradiction.

If (e) holds, we denote the constant t. Let $t \neq 1$. Then $g(a) = 1$ and $\pi \in \mathrm{Con}(A, g)$, a contradiction. Hence $t = 1$. Then either $g(a) = 1$ which yields a contradiction like in the previous case, or $g(a) = a, g(b) \in \{1, a\}$. If $g(b) = a$ then $\rho \in \mathrm{Con}(A, g)$, a contradiction, hence $g(b) = 1$ and (3) is valid.

Finally, in the case (f), $g(a) = a$ or $g(a) = 1$. In the first case, $\pi \in \mathrm{Con}(A, g)$, a contradiction, hence $g(a) = 1$. Then $g(b) = 0$ or $g(b) = b$. Similarly, in the first case we get a contradiction with is $\rho \notin \mathrm{Con}(A, g)$, hence $g(b) = 1$ and (4) is valid. □

Lemma 5.7. *Let (A, f) be a monounary algebra. Suppose that $\mathrm{Con}(A, f) \subseteq \mathrm{Con}(A, g)$. If g equals f on the set $\{0, 1, 0', 1', a, b\}$ then the following holds:*

(i) *if x or $f(x)$ is cyclic then $g(x) = f(x)$,*
(ii) *for every $x \in A$, $g(x) \in \{f^{2k-1}(x) : k \in \mathbb{N}\}$.*

Proof. Let $x \in A$. If $f(x) = x$, then

$$\begin{aligned}(g(x), 1) = (g(x), g(0)) &\in \theta_f(x, 0) = [x, 0, 1],\\ (g(x), 1') = (g(x), g(0')) &\in \theta_f(x, 0') = [x, 0', 1'],\end{aligned}$$

which implies $g(x) = x = f(x)$. If x is noncyclic and $\{f(x)\}$ is a cycle, i.e., $f(x) = f^2(x) \neq x$ then

$$\begin{aligned}(g(x), 0) = (g(x), g(a)) &\in \theta_f(x, a) = [x, a][f(x), 0, 1],\\ (g(x), 1') = (g(x), g(0')) &\in \theta_f(x, 0') = [x, f(x), 0', 1'],\end{aligned}$$

hence $g(x) = f(x)$. If $\{x, f(x)\}$ is a two-element cycle distinct from $\{0, 1\}$, $\{0', 1'\}$, then

$$(g(x), 1) = (g(x), g(0)) \in \theta_f(x, 0) = [x, 0][f(x), 1],$$
$$(g(x), 0) = (g(x), g(1)) \in \theta_f(x, 1) = [x, 1][f(x), 0,],$$

which yields $g(x) = f(x)$. If x is noncyclic and $\{f(x), f^2(x)\}$ is a two-element cycle, then

$$(g(x), f(x)) = (g(x), f(f^2(x))) = (g(x), g(f^2(x))) \in \theta_f(x, f^2(x)) = [x, f^2(x)],$$

which yields $g(x) = f(x)$. Hence (i) is valid.

If $x \in \{0, 1, 0', 1', a, b\}$ or $f(x)$ is cyclic, then (ii) clearly holds. Now suppose that x is a noncyclic element and $\{f(x)\}$ fails to be a cycle. There is $n \in \mathbb{N}$ such that $f^n(x)$ is cyclic and $f^{n-1}(x)$ is noncyclic. If n is even then

$$\begin{aligned}(g(x), f^{n+1}(x)) &= (g(x), g(f^n(x))) \in \theta_f(x, f^n(x)) \\ &= [x, f^2(x), f^4(x), \dots, f^n(x)][f(x), f^3(x), \dots, f^{n+1}(x)]\end{aligned}$$

and if n is odd then

$$\begin{aligned}(g(x), f^n(x)) &= (g(x), g(f^{n+1}(x))) \in \theta_f(x, f^{n+1}(x)) \\ &= [x, f^2(x), f^4(x), \dots, f^{n+1}(x)][f(x), f^3(x), \dots, f^n(x)],\end{aligned}$$

which implies that (ii) is valid. □

Lemma 5.8. *Let (A, f) be a monounary algebra such that there are distinct elements $d, e \notin \{0, 1, a, b, 0', 1'\}$ with $f(d) = 1$, $f(e) = d$. Further, suppose that $\mathrm{Con}(A, f) \subseteq \mathrm{Con}(A, g)$ and $\rho = [0, 1][a, b] \notin \mathrm{Con}(A, g)$. Then $g(x) = f(x)$ for each $x \in A$.*

Proof. By assumptions and by Lemma 5.5, following cases may occur:

(a) g is equal to f on $\{0, 1, a, b, d\}$
(b) g is constant on the set $\{0, 1, a, d\}$, the constant is 1 and $g(b) = a$,
(c) g is constant on the set $\{0, 1, a, d\}$, the constant is a and $g(b) = 1$.

Assume that the case a) holds. Since

$$(g(e), 0) = (g(e), g(1)) \in \theta_f(e, 1) = [e, 1][d, 0],$$
$$(g(e), a) = (g(e), g(b)) \in \theta_f(e, b) = [e, b][d, a][0, 1],$$

we get $g(e) = d = f(e)$. Let $x \in A$. If x belongs to a component possessing a one-element cycle then by Lemma 5.7 (i), $g(x) = f(x)$. The remaining case is that $x \neq b, e$ belongs to a component possessing a two-element cycle but neither x nor $f(x)$ is cyclic. Without loss of generality, $a \notin Z_f(f(x))$. Since

$$(g(x), a) \in \theta_f(x, b) = [x, b][f(x), a][f^2(x), f^4(x), \dots, 0] \vee [f^3(x), f^5(x), \dots, 1],$$

Lemma 5.7 (ii) implies $g(x) = f(x)$.

If (b) holds then from (ii) of Lemma 5.7 and

$$(1, g(0')) = (g(0), g(0')) \in \theta_f(0, 0') = [0, 0'][1, 1'],$$

it follows that $g(0') = g(1') = 1$. Then from

$$(g(e), 1) = (g(e), g(1)) \in \theta_f(e, 1) = [e, 1][0, d],$$
$$(g(e), a) = (g(e), g(b)) \in \theta_f(e, b) = [e, b][d, a][1, 0],$$

we get $g(e) \in \{e, 1\} \cap \{d, a\}$, a contradiction. Similarly, if c) holds then we get $g(e) = a$ and $(g(e), g(b)) = (a, 1) \notin \theta_f(e, b) = [e, b][a, d][0, 1]$, a contradiction. □

Lemma 5.9. *Let* (A, f) *be a monounary algebra such that there are distinct elements* $c, d, e \notin \{0, 1, a, b, 0', 1'\}$ *with* $f(c) = b$, $f(d) = 0$, $f(e) = d$. *Further, suppose that* $\mathrm{Con}(A, f) \subseteq \mathrm{Con}(A, g)$ *and* $\rho = [0, 1][a, b] \notin \mathrm{Con}(A, g)$. *Then* $g(x) = f(x)$ *for each* $x \in A$.

Proof. By assumptions and by Lemma 5.4, following cases may occur:

(a) g is equal to f on $\{0, 1, a, b, d\}$
(b) g is constant on the set $\{0, 1, a, d\}$, the constant is 1 and $g(b) = a$,
(c) g is constant on the set $\{0, 1, a, d\}$, the constant is a and $g(b) = 1$.

If (a) holds then from

$$(g(e), a) = (g(e), g(b)) \in \theta_f(e, b) = [e, b][d, a],$$

we get $g(e) = d$. Further

$$(g(c), 0) = (g(c), g(1)) \in \theta_f(c, 1) = [a, c, 1][b, 0],$$
$$(g(c), d) = (g(c), g(e)) \in \theta_f(c, e) = [c, e][b, d][a, 0, 1],$$

yield that $g(c) = b = f(c)$. By Lemma 5.7, for every $x \in A$ such that either x or $f(x)$ is cyclic, it holds $g(x) = f(x)$. It remains to prove that $g(x) = f(x)$ for $x \neq b, e$ such that x belongs to a component possessing a two-element cycle but neither x nor $f(x)$ is cyclic. Without loss of generality, $a \notin Z_f(f(x))$. Since

$$(g(x), a) \in \theta_f(x, b) = [x, b][f(x), a][f^2(x), f^4(x), \dots, 0] \vee [f^3(x), f^5(x), \dots, 1],$$

Lemma 5.7 (ii) implies $g(x) = f(x)$.

If b) or c) hold then from

$$(g(0), g(0')) \in \theta_f(0, 0') = [0, 0'][1, 1'],$$
$$(g(0'), g(1')) \in \theta_f(0', 1') = [0', 1'],$$

it follows that $g(0') = g(1') = 1$. Moreover

$$(g(e), g(0)) \in \theta_f(e, 0) = [e, 0][d, 1],$$
$$(g(e), g(b)) \in \theta_f(e, b) = [e, b][d, a].$$

Then in the case b), we get $g(e) = d$ and from

$$(g(c), 1) = (g(c), g(d)) \in \theta_f(c, d) = [c, d][b, 0][a, 1],$$
$$(g(c), d) = (g(c), g(e)) \in \theta_f(c, e) = [c, e][b, d][a, 0, 1]$$

$g(c) \in \{a, 1\} \cap \{b, d\}$, a contradiction. Finally in the case c), $g(e) \in \{a\} \cap \{1\}$, a contradiction. □

Lemma 5.10. *Let (A, f) be a monounary algebra such that there are distinct elements $d, e \notin \{0, 1, a, b, 0', 1'\}$ with $f(d) = 0'$, $f(e) = d$. Further, suppose that $\mathrm{Con}(A, f) \subseteq \mathrm{Con}(A, g)$ and $\rho = [0, 1][a, b] \notin \mathrm{Con}(A, g)$, $\pi = [a, 0'] \notin \mathrm{Con}(A, g)$. Then $g(x) = f(x)$ for each $x \in A$.*

Proof. By assumption and by Lemma 5.6, the following cases may occur:

(a) g is equal to f on the set $\{0, 1, 0', 1', a, b\}$,
(b) $g(a) = 1$ and $g(x) = a, x \in \{0, 1, 0', 1', b\}$,
(c) $g(a) = a$ and $g(x) = 1, x \in \{0, 1, 0', 1', b\}$,
(d) g is identity on the set $\{0, 1, 0', 1', b\}$ and $g(a) = 1$.

From

$$(g(d), g(1')) \in \theta_f(d, 1') = [d, 1']$$

it follows that that in the cases (a)–(c), $g(d) = g(1')$. However, in cases (b), (c), we get a contradiction with $(1, a) = (g(d), g(1)) \in \theta_f(d, 1) = [d, 1, 1'][0, 0']$. Moreover,

$$(g(e), g(b)) \in \theta_f(e, b) = [e, b][d, a][0', 0][1', 1].$$

Then in the case a), $g(d) = 0', g(e) = d$. By Lemma 5.7, if either x or $f(x)$ is cyclic then $g(x) = f(x)$. It remains to prove that $g(x) = f(x)$ for $x \in A\backslash\{b, e\}$ such that $f(x)$ fails to be cyclic. Then either $a \notin Z_f(f(x))$ or $d \notin Z_f(f(x))$. Without loss of generality, let $a \notin Z_f(f(x))$. According to

$$\begin{aligned}(g(x), a) = (g(x), g(b)) &\in \theta_f(x, b)\\ &= [x, b][f(x), a][0, f^2(x), f^4(x), \dots][f^3(x), f^5(x), \dots],\\ (g(x), f^2(x)) = (g(x), g(f(x))) &\in \theta_f(x, f(x)) = [Z_f(x)],\end{aligned}$$

we obtain $g(x) = f(x)$. Hence $g(x) = f(x)$ for all $x \in A$.

Finally, in the case (d), we get $g(d) = 1'$ and $g(e) = e$ which yields a contradiction with $(g(e), g(a)) \in \theta_f(e, a) = [e, a][d, 0, 1'][0', 1]$. □

Theorem 5.11. *Let (A, f) be a monounary algebra with small cycles, let each one-element cycle have only short tails and assume that there are distinct elements $0, 1, 0', 1', a, b$ with $f(0) = 1$, $f(a) = f(1) = 0$, $f(b) = a$, $f(0') = 1'$, $f(1') = 0'$. If*

(a) *there exist elements $d, e \in A\backslash\{0, 1, a, b, 0', 1'\}$ such that $f(d) \in \{1, 0'\}$, $f(e) = d$, or*
(b) *there exist elements $c, d, e \in A\backslash\{0, 1, a, b, 0', 1'\}$ such that $f(c) = b$, $f(d) = 0$, $f(e) = d$,*

then $\mathrm{Con}(A, f)$ is $\wedge$-irreducible.

Proof. According to Lemmas 5.6, 5.8–5.10 we have

$$[a, b][0, 1] \in \bigcap\{\mathrm{Con}(A, g) : \mathrm{Con}(A, f) \subsetneq \mathrm{Con}(A, g)\}.$$

or

$$[a, 0'] \in \bigcap\{\mathrm{Con}(A, g) : \mathrm{Con}(A, f) \subsetneq \mathrm{Con}(A, g)\}.$$

Also, $[a,b][0,1] \notin \mathrm{Con}(A,f)$ because $\theta_f(a,b) = [a,b,0,1]$ and similarly $[a,0'] \notin \mathrm{Con}(A,f)$ because $\theta_f(a,0') = [a,0',1][0,1']$. Hence the above intersections fail to be equal to $\mathrm{Con}(A,f)$ thus $\mathrm{Con}(A,f)$ is $\wedge$-irreducible. □

In the following Lemmas 5.12–5.16 we will assume that:

- there is a single two-element cycle $\{0, 1\}$,
- each one-element cycle has only short tails,
- there are noncyclic elements $a, b \in A$ such that $f(a) = 0$, $f(b) = a$.

Lemma 5.12. *Let (A,f) be a monounary algebra such that there are distinct elements d, e with $f(d) = 0, f(e) = d$. Further, suppose that $\mathrm{Con}(A,f) \subseteq \mathrm{Con}(A,g)$ and $\rho = [a,e][0,1] \notin \mathrm{Con}(A,g)$. Then one of the following conditions is satisfied:*

(1) *g is equal to f on $\{0,1,a,b,d,e\}$*
(2) *g is constant on the set $\{0,1,a,d\}$, the constant is 1, $g(b) = a$, $g(e) = d$.*

Proof. Assume that our assumptions are satisfied. Then according to Lemma 5.4, one of the following cases occurs:

(a) g is an identity on the set $\{0,1,a,b,d\}$
(b) g is a constant on $\{0,1,a,b,d\}$
(c) g is equal to f on $\{0,1,a,b,d\}$
(d) g is equal to f on $\{0,1,a,d\}$ and $g(b) = 1$,
(e) $g(1) = g(a) = g(d) = 1$, $g(0) = g(b) = 0$,
(f) g is constant on the set $\{0,1,a,d\}$, the constant is 1 and $g(b) = a$,
(g) g is constant on the set $\{0,1,a,d\}$, the constant is a and $g(b) = 1$.

Then like in the proof of Lemma 4.2, it follows that the cases (a), (b), (d), (e), (g) yield contradiction. If (c) holds then we get $g(e) = d$ and (1) is valid. If (f) holds then $g(e) = d$ and (2) is valid. □

Lemma 5.13. *Let the assumption of Lemma 5.12 be satisfied and let there exist $c \in A$ such that $f(c) = b$. Then $g(x) = f(x)$ for each $x \in A$.*

Proof. It holds

$$(g(c), g(1)) \in \theta_f(c,1) = [c,1,a][b,0],$$
$$(g(c), g(e)) \in \theta_f(c,e) = [c,e][b,d][a,0,1],$$
$$(g(c), g(d)) \in \theta_f(c,d) = [c,d][b,0][a,1].$$

If (1) of Lemma 5.12 holds then $g(c) = b$, hence g is equal to f on $\{0,1,a,b,c,d,e\}$. Let $x \in A \setminus \{0,1,a,b,d,e\}$. If $f(x) \in \{0,1\}$, clearly $g(x) = f(x)$. Similarly if $C(x) \neq C(0)$, then clearly $g(x) = f(x)$. Then $b \notin Z_f(x)$ or $e \notin Z_f(x)$. Without loss of generality let $b \notin Z_f(x)$. Then

$$(g(x), a) \in \theta_f(x,b) = [x,b][f(x),a][0,f^2(x),f^4(x),\dots][1,f^3(x),f^5(x)\dots],$$
$$(g(x), 0) \in \theta_f(x,a) = [x,a][0,f(x),f^3(x),\dots][1,f^2(x),f^4(x)\dots],$$

which yields $g(x) = f(x)$. Therefore g is equal to f on A.

If (2) of Lemma 5.12 holds, then $g(c) \in \{c,a,1\} \cap \{b,d\}$, which is a contradiction. □

Theorem 5.14. *Let (A, f) be a monounary algebra with small cycles and assume that each one-element cycle has only short tails. Further, assume that there are distinct elements $0, 1, a, b, c, d, e$ with $f(0) = 1, f(1) = f(a) = f(d) = 0, f(b) = a$, $f(c) = b, f(e) = d$ and that (A, f) contains a single two-element cycle. Then* $\mathrm{Con}(A, f)$ *is $\wedge$-irreducible.*

Proof. According to Lemmas 5.12 and 5.13, we have

$$[a, e][0, 1] \in \bigcap\{\mathrm{Con}(A, g) : \mathrm{Con}(A, f) \subsetneq \mathrm{Con}(A, g)\}.$$

However, $[a, e][0, 1] \notin \mathrm{Con}(A, f)$ because $\theta_f(a, e) = [a, e][d, 0, 1]$. Therefore the above intersection fails to be equal to $\mathrm{Con}(A, f)$ which implies that $\mathrm{Con}(A, f)$ is $\wedge$-irreducible. □

Lemma 5.15. *Let (A, f) be a monounary algebra such that there are distinct elements $d, e \notin \{0, 1, a, b\}$ with $f(d) = 1$, $f(e) = d$ and let* $\mathrm{Con}(A, f) \subseteq \mathrm{Con}(A, g)$. *Then one of the following conditions is satisfied:*

(1) *g is equal to f on the set $\{0, 1, a, b, d, e\}$,*
(2) *g is identity on $\{0, 1, a, b, d, e\}$,*
(3) *g is constant on $\{0, 1, a, b, d, e\}$,*
(4) *g is equal to f on $\{0, 1, a, d\}$ and $g(e) = 0, g(b) = 1$,*
(5) *$g(1) = g(a) = g(e) = 1$, $g(0) = g(b) = g(d) = 0$.*

Proof. Let the assumptions be valid. According to

$$\begin{aligned}(g(0), g(1)) &\in \theta_f(0, 1) = [0, 1],\\ (g(a), g(1)) &\in \theta_f(a, 1) = [a, 1],\\ (g(0), g(d)) &\in \theta_f(0, d) = [0, d],\\ (g(a), g(d)) &\in \theta_f(a, d) = [a, d][0, 1],\end{aligned}$$

the following cases can occur:

(a) g is equal to f on the set $\{0, 1, a, d\}$,
(b) g is identity on $\{0, 1, a, d\}$
(c) g is constant on $\{0, 1, a, d\}$, the constant is $u \in \{0, 1, a, d\}$,
(d) g is constant on $\{0, 1, a, d\}$, the constant is $u \notin \{0, 1, a, d\}$,
(e) $g(0) = g(d) = 0$, $g(1) = g(a) = 1$.

Moreover,

$$\begin{aligned}(g(b), g(0)) &\in \theta_f(b, 0) = [b, 0][a, 1],\\ (g(e), g(1)) &\in \theta_f(e, 1) = [e, 1][d, 0],\\ (g(b), g(e)) &\in \theta_f(b, e) = [b, e][a, d][0, 1],\end{aligned}$$

which implies that in the case (a), either $g(b) = a, g(e) = d$ or $g(b) = 1, g(e) = 0$, i.e. either (1) or (4) is satisfied. If (b) holds then $g(b) = b, g(e) = e$, hence (2) is satisfied. In the case (c), $g(b) = g(e) = u$ and in (d), clearly $g(b) = g(e)$ and $u \in \{b, e\}$. Then from (c) and (d), we get (3). Finally, if (e) holds then $g(b) = 0, g(e) = 1$ and (5) is satisfied. □

Lemma 5.16. *Let* (A, f) *be a monounary algebra such that there are distinct elements* $d, e \notin \{0, 1, a, b\}$ *with* $f(d) = 1$*,* $f(e) = d$*. Further, suppose that* $\mathrm{Con}(A, f) \subseteq \mathrm{Con}(A, g)$ *and* $\rho = [e, b][0, 1] \notin \mathrm{Con}(A, g)$*. Then* $g(x) = f(x)$ *for every* $x \in A$*.*

Proof. According to Lemma 5.15, the only case when $\rho \notin \mathrm{Con}(A, g)$ is case (1), hence g is equal to f on the set $\{0, 1, a, b, d, e\}$.

Let $x \in A\backslash\{0, 1, a, b, d, e\}$. If $f(x) \in \{0, 1\}$, clearly $g(x) = f(x)$. Similarly if $C(x) \neq C(0)$, then clearly $g(x) = f(x)$. Otherwise either $b \notin Z_f(x)$ or $e \notin Z_f(x)$. Without loss of generality let $b \notin Z_f(x)$. Then

$$(g(x), a) \in \theta_f(x, b) = [x, b][f(x), a][0, f^2(x), f^4(x), \dots][1, f^3(x), f^5(x) \dots],$$
$$(g(x), 0) \in \theta_f(x, a) = [x, a][0, f(x), f^3(x), \dots][1, f^2(x), f^4(x) \dots],$$

which yields $g(x) = f(x)$. Therefore g is equal to f on A. □

Theorem 5.17. *Let* (A, f) *be a monounary algebra with small cycles and let each one-element cycle have only short tails. Further, assume that there are distinct elements* $0, 1, a, b, d, e$ *with* $f(0) = f(d) = 1$*,* $f(1) = f(a) = 0$*,* $f(b) = a$*,* $f(e) = d$ *and that* (A, f) *contains a single two-element cycle. Then* $\mathrm{Con}(A, f)$ *is* $\wedge$*-irreducible.*

Proof. According to Lemmas 5.15–5.16 we have

$$[e, b][0, 1] \in \bigcap\{\mathrm{Con}(A, g) : \mathrm{Con}(A, f) \subsetneq \mathrm{Con}(A, g)\}.$$

However, $[e, b][0, 1] \notin \mathrm{Con}(A, f)$ because $\theta_f(e, b) = [e, b][a, d][0, 1]$. Hence the above intersection fails to be equal to $\mathrm{Con}(A, f)$ and $\mathrm{Con}(A, f)$ is $\wedge$-irreducible. □

Notation 5.18. Let (A, f) be a monounary algebra with small cycles. We say that f satisfies condition (γ) or (δ) if the following holds:

(γ) there are distinct elements $0, 1, 2, 0', 1', 2'$ with $f(1) = f(0) = 0$, $f(2) = 1$, $f(0') \neq 0$, $f(1') = 0'$, $f(2') = 1'$ such that $0'$, $f(0')$ are cyclic, $1'$ is noncyclic and one of the following conditions is satisfied:
- (i) (A, f) is acyclic, or
- (ii) $f(0') = 0'' \neq 0'$, $f(0'') = 0'$, or
- (iii) $f(0') = 0'$ and (A, f) contains at least two 2-element cycles.

(δ) there are distinct elements $0, 1, a, b$ with $f(0) = 1$, $f(a) = f(1) = 0$, $f(b) = a$ and one of the following conditions is satisfied:
- (i) there exist $c, d, e \in A\backslash\{0, 1, a, b\}$ such that $f(c) = b$, $f(d) = 0$, $f(e) = d$, or
- (ii) there exist $d, e \in A\backslash\{0, 1, a, b\}$ such that $f(d) = 1$, $f(e) = d$, or
- (iii) there exist $0', 1', d, e \in A\backslash\{0, 1, a, b\}$ such that $f(0') = 1'$, $f(1') = f(d) = 0'$, $f(e) = d$.

Figures 3 and 4 illustrate the conditions (γ) and (δ) respectively. In each figure, the labeled elements are mandatory.

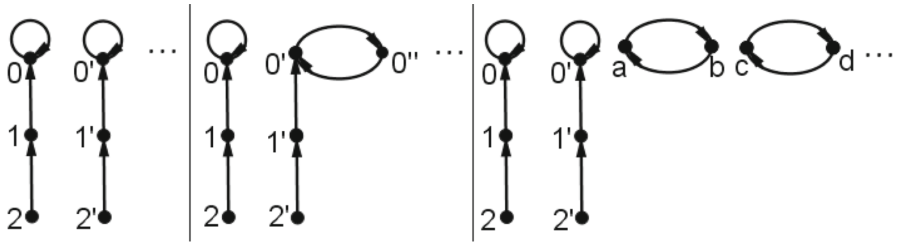

FIGURE 3. Operations satisfying the condition (γ)

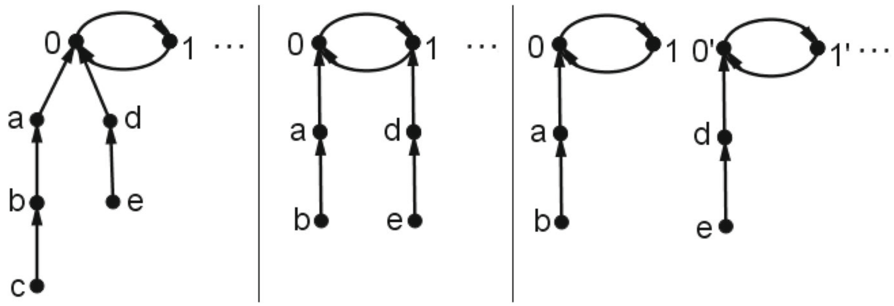

FIGURE 4. Operations satisfying the condition (δ)

Theorem 5.19. *Let (A, f) be a monounary algebra with small cycles and $|A| > 2$. Then* $\mathrm{Con}(A, f)$ *is $\wedge$-irreducible iff one of the following holds:*

(1) *(A, f) is connected and f is of type* (II) *or satisfies condition (β), or*
(2) *(A, f) is a permutation-algebra with short tails such that f is nontrivial and the corresponding permutation is either identity, or a two-element cycle, or (A, f) contains at least two nontrivial cycles, or*
(3) *f satisfies condition (γ) or (δ).*

Proof. If (1) holds, then according to Theorem 2.7, $\mathrm{Con}(A, f)$ is $\wedge$-irreducible. If (2) holds, then from Theorems 3.1 and 3.6 it follows that $\mathrm{Con}(A, f)$ is $\wedge$-irreducible. In the case (3), if f satisfies condition (γ) then Theorem 5.3 implies that $\mathrm{Con}(A, f)$ is $\wedge$-irreducible. Let f satisfy condition (δ). Theorems 5.11, 5.14 and 5.17 imply that if f satisfies (i)–(iii) of the condition (δ) then $\mathrm{Con}(A, f)$ is $\wedge$-irreducible

On the other hand, if (A, f) is a monounary algebra with small cycles and it fails to satisfy the conditions (1)–(3), then according to Theorems 2.7, 3.1, 3.6 and Lemmas 4.1–4.4, $\mathrm{Con}(A, f)$ is $\wedge$-reducible. □

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