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Subcompletions of representable relation algebras

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In memoriam, Bjarni Jónsson.

Abstract. The variety of representable relation algebras is closed under canonical extensions but not closed under completions. What variety of relation algebras is generated by completions of representable relation algebras? Does it contain all relation algebras? It contains all representable finite relation algebras, and this paper shows that it contains many non-representable finite relation algebras as well. For example, every Monk algebra with six or more special elements (called "colors") is a subalgebra of the completion of an atomic symmetric integral representable relation algebra whose finitely-generated subalgebras are finite.

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1. Introduction and background

This section defines the concepts in the abstract, describes the main result, and includes a brief review of relevant material concerning relation algebras, representability, canonical extensions, and completions.

1.1.

A *relation algebra* $\mathfrak{A}$ is an algebra of the form

$$\mathfrak{A} = \langle A, +, \cdot, ^{-}, 0, ;, ^{\smile}, 1' \rangle ,$$

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where $\langle A, +, \cdot, ^{-}, 0\rangle$ is Boolean algebra, satisfying the *identity law*,

$$x;1' = 1';x = x,$$

the *exchange law*,

$$x;y \cdot z = 0 \iff y \cdot \breve{x};z = 0 \iff x \cdot z;\breve{y} = 0,$$

and the *associative law*,

$$x;(y;z) = (x;y);z.$$

The element 1' is the *identity*, $0' = \overline{1'}$ is the *diversity*, 0 is the *zero*, and $1 = \overline{0}$ is the (Boolean) *unit*. RA is the class of relation algebras, and NA, the class of *non-associative relation algebras*, is the class of algebras obtained by dropping the associative law from this list of membership requirements.

1.2.

The exchange and identity laws are enough to show that non-associative relation algebras are *additive*, *i.e.*, distributive laws hold:

$$x;(y+z) = x;y + x;z,$$
$$(y+z);x = y;x + z;x,$$
$$(x+y)^{\smile} = \breve{x} + \breve{y},$$

and *normal* ($;$ and $^{\smile}$ distribute over $\sum\emptyset$ as well, *i.e.*, $0;x = x;0 = \breve{0} = 0$). Furthermore, if $\mathfrak{A} \in \mathsf{NA}$ is *complete* (the join of every subset of A exists in $\mathfrak{A}$) then $\mathfrak{A}$ is also *completely additive* ($;$ and $^{\smile}$ distribute over all non-empty joins, *e.g.*, $(\sum X)^{\smile} = \sum\{\breve{x} : x \in X\}$).

1.3.

Let $\mathbf{H}, \mathbf{S}$, and $\mathbf{P}$ be the closure operators that map a class K of algebras to the classes of algebras isomorphic to subalgebras, homomorphic images, and direct products of algebras in K, respectively. NA has an equational axiomatization. To get one, delete the associative law from Tarski's original ten axioms for relation algebras; see [29, Definition 1.2] and [33, p. 289, Theorem 314]. By the easy direction of Birkhoff's Theorem, NA and RA are closed under the varietal operators:

$$\mathbf{HSP}(\mathsf{NA}) = \mathsf{NA}, \qquad \mathbf{HSP}(\mathsf{RA}) = \mathsf{RA}.$$

1.4.

The primordial example of a relation algebra is the algebra $\mathfrak{Re}(U)$ of all binary relations on a set U,

$$\mathfrak{Re}(U) = \left\langle \mathcal{P}(U \times U), \cup, \cap, ^{-}, \emptyset, |, ^{-1}, Id_U \right\rangle,$$

where $\langle \mathcal{P}(U \times U), \cup, \cap, ^{-}, \emptyset\rangle$ is the Boolean algebra of all subsets of the Cartesian square $U \times U$, and, for all relations $R, S \subseteq U \times U$,

$$R|S = \{\langle u, w\rangle : \exists v\,(\langle u, v\rangle \in R \wedge \langle v, w\rangle \in S)\},$$
$$R^{-1} = \{\langle u, v\rangle : \langle v, u\rangle \in R\},$$
$$Id_U = \{\langle u, u\rangle : u \in U\}.$$

A Boolean algebra $\mathfrak{B}$ is *atomic* if every non-zero element contains an *atom* (a non-zero element that contains no other non-zero element). $At(\mathfrak{B})$ is the set of atoms of $\mathfrak{B}$. As a Boolean algebra, $\mathfrak{Re}(U)$ is clearly complete because it is closed under arbitrary unions. $\mathfrak{Re}(U)$ is also atomic. The atoms of $\mathfrak{Re}(U)$ are the sets of the form $\{\langle u, v\rangle\}$ with $u, v \in U$.

1.5.

$\mathfrak{Re}(U)$ is an *algebra of binary relations*, *i.e.*, the elements are binary relations and the operations are the usual set-theoretic ones. $\mathfrak{Re}(U)$ and all its subalgebras are algebras of binary relations that are *simple* (have no non-trivial homomorphic images), but direct products of algebras of binary relations are not simple and are easily seen to also be algebras of binary relations. For this reason the class RRA of *representable relation algebras* is defined by

$$\mathsf{RRA} = \mathbf{SP}\{\mathfrak{Re}(U) : U \text{ is a set}\}.$$

From its definition, $\mathsf{RRA} = \mathbf{SP}\mathsf{RRA}$. The central problem was to determine whether $\mathbf{H}\mathsf{RRA} = \mathsf{RRA}$. This was proved by Tarski [39] (after having been incorrectly disproved by Lyndon [26]). By the consequential direction of Birkhoff's Theorem, RRA has an equational axiomatization (and Lyndon [27] constructed the first explicit one).

1.6.

Every Boolean algebra has a canonical extension $\mathfrak{B}^+$ and a completion $\mathfrak{B}^c$. These constructions are due to the work of (at least) Dedekind, MacNeille, Stone, and Sikorski (consult [38]). Both extensions are complete. $\mathfrak{B}^+$ is always atomic; $\mathfrak{B}^c$ can go either way. $\mathfrak{B}^+$ destroys all infinite joins, while $\mathfrak{B}^c$ preserves all infinite joins, *i.e.*, if $X \subseteq B, X$ is infinite, $\sum X \in B$, and $\sum X \neq \sum Y$ for all finite subsets $Y \subset X$, then the join of X in $\mathfrak{B}^+$ is strictly smaller than $\sum X$, and the join of X in $\mathfrak{B}^c$ is the same as $\sum X$. Both extensions are isomorphic under isomorphisms that leave the elements of $\mathfrak{B}$ unchanged, and they are both minimal or generated by $\mathfrak{B}$ in the appropriate sense. These properties lead to natural descriptions. The elements of $\mathfrak{B}^c$ can be constructed as subsets of $\mathfrak{B}$, and the elements of $\mathfrak{B}^+$ can be constructed as sets of subsets of $\mathfrak{B}$, that is, $\mathfrak{B}^c$ can be found in $\mathcal{P}(B)$, while $\mathfrak{B}^+$ is found at the next level, $\mathcal{P}(\mathcal{P}(B))$. An element of $\mathfrak{B}^+$ is the join the of atoms below it, and each atom is the meet of the elements of $\mathfrak{B}$ lying above it, so every element of $\mathfrak{B}^+$ is the join of some meets of some subsets (ultrafilters, actually) of $\mathfrak{B}$. If we identify an element of $\mathfrak{B}$ with the set of elements of $\mathfrak{B}$ lying below it, then the elements of $\mathfrak{B}^c$ missing from $\mathfrak{B}$ are provided by those subsets of $\mathfrak{B}$ whose joins are not yet present in $\mathfrak{B}$, so every element of $\mathfrak{B}^c$ is the join of a subset of $\mathfrak{B}$.

1.7.

The next goal was, given a relation (or cylindric) algebra $\mathfrak{A}$ whose Boolean part is $\mathfrak{B}$, to show that there are relation (or cylindric) algebras $\mathfrak{A}^+$ and $\mathfrak{A}^c$ whose Boolean parts are $\mathfrak{B}^+$ and $\mathfrak{B}^c$, respectively, such that $\mathfrak{A} \subseteq \mathfrak{A}^+$ and $\mathfrak{A} \subseteq \mathfrak{A}^c$. This was done by Jónsson–Tarski [21,22] and Monk [37], respectively, by extending the operators of $\mathfrak{A}$ on $\mathfrak{B}$ to operators on $\mathfrak{B}^+$ and $\mathfrak{B}^c$ along the lines described above. For example, a unary operator f on $\mathfrak{B}$ (like conversion or cylindrification) extends to unary operators f^c on $\mathfrak{B}^c$ and f^+ on $\mathfrak{B}^+$ defined by

$$f^c(x) = \sum_{x \geq b \in B} f(b) \qquad \text{for } x \in B^c,$$

$$f^+(x) = \sum_{x \geq a \in At(B^+)} \left(\prod_{a \leq b \in B} f(b) \right) \qquad \text{for } x \in B^+.$$

Binary operators like ; are extended similarly. The key fact is that the axioms of relation and cylindric algebras are preserved by the extensions, in the sense that an equation holding for the operators on $\mathfrak{B}$ also holds for the extended operators on $\mathfrak{B}^c$ and $\mathfrak{B}^+$. Consequently cylindric and relation algebras are closed under the formation of canonical extensions and completions.

1.8.

Tarski's result that $\mathbf{H}\mathsf{RRA} = \mathsf{RRA}$ now becomes significant, for it might have been possible to show RRA is also closed under canonical extensions and completions by showing that equational axioms for RRA are preserved, just as they are for relation and cylindric algebras. Jónsson–Tarski [21,22] and Monk [37] proved that positive equations (ones not containing complementation) are preserved; this is enough for closure of RA under $(-)^+$ and $(-)^c$, but fails on RRA. Instead Monk proved RRA is closed under canonical extensions by a detour through cylindric algebras.

Monk's method used a descending chain of classes (each class contains the next) beginning with RA, converging on RRA (the intersection of the chain is RRA), and consisting of relation algebras obtained as definitional reducts of cylindric algebras of ever increasing finite dimension (see (5.2) below). A representable relation algebra $\mathfrak{A}$ belongs to every class in the chain, but the classes in the chain are closed under canonical extensions by the closure of finite dimensional cylindric algebras under canonical extensions and the close connections between cylindric algebras and their relation algebraic reducts, so $\mathfrak{A} \in \mathsf{RRA}$ because it belongs to every class in a chain converging on the representable algebras.

Monk's unpublished proof was reported in McKenzie's thesis [34]; the first published proof appeared in [28, Theorems 6(3), 8, 10], using a similar detour; see [33, §24, §52].

1.9.

One might have expected that RRA is also closed under completions, perhaps because it is a simpler situation, but it turns out that the causal link goes the other way around: if a variety of monotone bounded lattice expansions is closed under completions, then it is closed under canonical extensions [9]. RRA is a naturally occurring example of a canonical variety of Boolean algebra with operators that is not closed under completions, as was proved by Hodkinson [18].

An earlier example of an equation from [25] suggests why RRA might not be closed under completions. The *complex algebra* $\mathfrak{Cm}\,(G)$ of a group G is the Boolean algebra of all subsets of G supplemented with the binary operation of multiplying subsets (once known as complexes), the unary operation of forming the inverses of elements of a complex, and the singleton containing just the identity element of G. The complex algebra a group is a relation algebra that is simple and *integral* (the identity element 1' is an atom).

An element of a relation algebra is an *equivalence element* if it satisfies the equations asserting that it is transitive and symmetric, so called because $E \subseteq U \times U$ is an equivalence relation iff $E|E \subseteq E = E^{-1}$ iff E is an equivalence element of $\mathfrak{Re}(U)$. An element H of $\mathfrak{Cm}\,(G)$ is a subgroup of G iff $H;H \subseteq H = \breve{H}$, *i.e.*, H is an equivalence element of $\mathfrak{Cm}\,(G)$,

Let $\mathbb{Z}$ be the group of integers under addition. $\mathfrak{Cm}\,(\mathbb{Z})$ is large, but it has a countable subalgebra $\mathfrak{R}$ generated by its atoms. The subalgebra $\mathfrak{R}$ is *dense* in $\mathfrak{Cm}\,(\mathbb{Z})$, *i.e.*, every non-empty element of $\mathfrak{Cm}\,(\mathbb{Z})$ contains an element of $\mathfrak{R}$ (an atom, in fact), and $\mathfrak{Cm}\,(\mathbb{Z})$ is complete. This is enough to insure that $\mathfrak{Cm}\,(\mathbb{Z})$ is the completion of $\mathfrak{R}$. The elements of $\mathfrak{R}$ are only the finite subsets of $\mathbb{Z}$ and their complements. Consequently, $\mathfrak{R}$ satisfies an equation (from [25]) that says every equivalence element (subgroup) is either $\emptyset$ or 1' or G. That equation fails in $\mathfrak{Cm}\,(\mathbb{Z})$, because $\mathbb{Z}$ has non-trivial subgroups that are neither finite nor cofinite.

These phenomena occur in Hodkinson's proof and in this paper: complete atomic relation algebras have strange elements that prevent representability (sometimes by forming a finite non-representable relation algebra), yet their countable atom-generated subalgebras are representable.

1.10.

We have arrived at a chain of varieties

$$\mathsf{RRA} \subseteq \mathsf{RA} \subseteq \mathsf{NA},$$

where NA and RA are closed under $\mathbf{S}, \mathbf{H}, \mathbf{P}, (-)^+$, and $(-)^c$, while RRA is closed under $\mathbf{S}, \mathbf{H}, \mathbf{P}$, and $(-)^+$, but not closed under $(-)^c$. Let RRA^c be the class of completions of representable relation algebras,

$$\mathsf{RRA}^c = \{\mathfrak{A}^c : \mathfrak{A} \in \mathsf{RRA}\},$$

and let $V = \mathbf{HSP}\mathsf{RRA}^c$ be the variety of relation algebras generated by RRA^c.

Problem 1.1. (1) Is $V = \mathsf{RA}$? (2) Is V finitely axiomatizable? (3) Is V closed under canonical extensions? (4) Is V closed under completions? (5) Is membership in V decidable for finite algebras? (6) Does V contain any algebras that are not weakly representable (as defined in [20])?

All the algebras in RRA^c are complete, so $\mathsf{RRA} \not\subseteq \mathsf{RRA}^c$, but $\mathsf{RRA} \subseteq \mathbf{S}\mathsf{RRA}^c$ simply because every relation algebra is a subalgebra of its completion. $\mathbf{S}\mathsf{RRA}^c$ is the class named in the title of this paper. Finite relation algebras are completions only of themselves, so every finite algebra in RRA^c is in RRA. This paper shows that there are also many finite non-representable relation algebras in $\mathbf{S}\mathsf{RRA}^c$, thus making a contribution to Problem 1.1 (1) in the direction of showing $V = \mathsf{RA}$. On the other hand, some finite relation algebras have no proper simple extensions and are therefore neither representable nor in RRA^c; see [8] for examples. These are candidates for relation algebras that are not in V.

1.11.

As a partial positive answer to Problem 1.1 (1), it will be shown in this paper that every finite Monk algebra with six or more colors is in SRRA^c. Monk algebras are relation algebraic versions of cylindric algebras used by Monk [36] to prove that classes of finite-dimensional representable cylindric algebras are not finitely axiomatizable. Monk algebras are relation algebras that are *symmetric* (satisfy $\breve{x} = x$ for all x) and *integral* (1' is an atom and all other atoms are called *diversity atoms* because they are included in 0'). The simplest Monk algebras are the ones defined next.

Definition 1.2. For $4 \leq q \in \omega, \mathfrak{M}_q$ is the finite symmetric integral relation algebra with q atoms $\mathbf{e}_0 = 1', \mathbf{e}_1, \cdots, \mathbf{e}_{q-1}$ such that if a, b are distinct diversity atoms then $a;b = 0'$ and $a;a = \overline{a}$. The $q-1$ diversity atoms of $\mathfrak{M}_q$ are called the *colors* of $\mathfrak{M}_q$.

The relation algebras $\mathfrak{M}_q$ were first constructed in [30]; $\mathfrak{M}_q$ was called $\mathfrak{E}_q(\{2,3\})$ in [32, Definition 2.4, Problem 2.7]. In the notation of [33], $\mathfrak{M}_4 = 62_{65}$ and $\mathfrak{M}_5 = 3009_{3013}$. $\mathfrak{M}_4$ is isomorphic to a subalgebra of the complex algebra of the 13-element cyclic group $\mathbb{Z}_{13}$, and $\mathfrak{M}_4$ has exactly two other representations, both on 16-element sets. These happen to be the two good 3-colorings of K_{16}; see [23]. It is likely that $\mathfrak{M}_q$ is representable for all $q \geq 4$; see [29, Problem 2.7]. This has been shown for several hundred small values of q; see [1] and [24].

1.12.

$\mathfrak{M}_q$ can also be described by cycles and atom structures, which are defined for algebras in NA. The *atom structure* [29, Definition 3.2] of an algebra $\mathfrak{A} \in \mathsf{NA}$ is

$$\mathfrak{At}(\mathfrak{A}) = \langle At(\mathfrak{A}), C, {}^{\smile}, I \rangle$$

where $At(\mathfrak{A})$ is the set of atoms of $\mathfrak{A}, C$ is the set of triples of atoms $\langle x, y, z \rangle$ such that $x;y \geq z, {}^{\smile}$ is the restriction of the converse operation of $\mathfrak{A}$ to the atoms of $\mathfrak{A}$ (possible since converses of atoms are atoms), and $I = \{x : 1' \geq x \in At(\mathfrak{A})\}$. In every NA, C is the union of sets of the form

$$[x, y, z] = \{\langle x, y, z \rangle, \langle \breve{x}, z, y \rangle, \langle y, \breve{z}, \breve{x} \rangle, \langle \breve{y}, \breve{x}, \breve{z} \rangle, \langle \breve{z}, x, \breve{y} \rangle, \langle z, \breve{y}, x \rangle\}, \quad (1.1)$$

where $x, y, z \in At(\mathfrak{A})$. Such sets are called *cycles*. If 1' is an atom of $\mathfrak{A}$, then the cycle $[x, y, z]$ is said to be an *identity cycle* if $1' \in \{x, y, z\}$, and a *diversity cycle* otherwise. If $\mathfrak{A}$ is symmetric, then a diversity cycle $[x, y, z]$ is said to be a *1-cycle*, *2-cycle*, or *3-cycle* if the cardinality $|\{x, y, z\}|$ is 1, 2, or 3, respectively. For example, the cycles of $\mathfrak{M}_q$ are all the 2-cycles and 3-cycles, but none of the 1-cycles.

1.13.

The *complex algebra* of the structure $\langle A, C, \breve{}, I\rangle$, where A is a set, C is a ternary relation on A, $\breve{}$ is a unary operation on A, and $I \subseteq A$, is the Boolean algebra of all subsets of A supplemented with I as a distinguished element, the unary complex converse operation defined by $\breve{X} = \{\breve{x} : x \in X\}$ for all $X \subseteq A$, and the complex relative multiplication defined by

$$X;Y = \{z : \exists x \in X,\, \exists y \in Y,\, \langle x, y, z\rangle \in C\} \tag{1.2}$$

for all $X, Y \subseteq A$. Every complete atomic NA is isomorphic to the complex algebra of its atom structure [29, Theorem 3.13(2)]. For example, if U is any set, then $\mathfrak{Re}(U)$ is equal (hence isomorphic) to the complex algebra of the structure $\langle A, C, \breve{}, I\rangle$, where

$$\begin{aligned} A &= \{\langle u, v\rangle : u, v \in U\}, \\ C &= \{\langle\langle u, v\rangle, \langle v, w\rangle, \langle u, w\rangle\rangle : u, v, w \in U\}, \\ \langle u, v\rangle\breve{} &= \langle v, u\rangle \text{ for all } u, v \in U, \\ I &= \{\langle u, u\rangle : u \in U\}. \end{aligned}$$

1.14.

The most general definition of Monk algebra would be a finite relation algebra that has some $\mathfrak{M}_q$ as a subalgebra. Here we restrict the definition to those algebras that can be obtained from $\mathfrak{M}_q$ by a process known as "splitting atoms", after the analogy between this and the process called "splitting elements" in cylindric algebras; see [10, p. 386, p. 390].

To "split an atom" in a relation algebra is (omitting some details) to create a copy of that atom, add the copy to the atom structure of the relation algebra, and extend the ternary relation of that atom structure to a larger ternary relation that has all the triples of the old ternary relation plus all those triples obtainable from them by replacing some or all occurrences of the chosen atom with its copy. If the chosen atom meets certain minimal conditions (see [3]) it is called "splittable" and the complex algebra of the extended structure is a relation algebra. Repetition of this process produces algebras "obtained by splitting".

Definition 1.3 (Andréka–Maddux–Németi [3]). Let $\mathfrak{A}$ and $\mathfrak{B}$ be atomic relation algebras. We say that $\mathfrak{A}$ is *obtained* from $\mathfrak{B}$ *by splitting* if $\mathfrak{B} \subseteq \mathfrak{A}$, every atom x of $\mathfrak{A}$ is contained in an atom $\xi(x)$ of $\mathfrak{B}$, called the *cover* of x, and for all $x, y \in At\,\mathfrak{A}$, if $x, y \leq 0'$ then

$$x;y = \begin{cases} \xi(x);\xi(y) \cdot 0' & \text{if } x \neq \breve{y} \\ \xi(x);\xi(y) & \text{if } x = \breve{y}. \end{cases} \tag{1.3}$$

Definition 1.4 (Andréka–Maddux–Németi [3, Example 6]). A *Monk algebra* is an atomic symmetric integral relation algebra obtained by splitting from some $\mathfrak{M}_q$, $4 \leq q \in \omega$.

1.15.

Monk [36] invented a method for constructing non-representable cylindric algebras. He used it to prove that the variety CA_n of n-dimensional cylindric algebras has no finite equational axiomatization. The basic idea will be described here for relation algebras. Monk [35] had already shown RRA is not a finitely based variety by a related method, so perhaps this was his starting point, prior to working out the cylindric algebraic details.

Let $\mathfrak{A}$ be a finite integral relation algebra, not necessarily symmetric. Suppose $\mathfrak{A}$ extends $\mathfrak{M}_q, 4 \leq q \in \omega$. In general, a finite integral relation algebra is representable iff if is isomorphic to a subalgebra of $\mathfrak{Re}(U)$ for some set U. Suppose $\mathfrak{A} \in \mathsf{RRA}$. Then $\mathfrak{A}$ can be embedded in $\mathfrak{Re}(U)$ by an injective homomorphism

$$\varphi\colon A \to \mathcal{P}(U \times U).$$

For any $u, v \in U$,

$$\langle u, v\rangle \in U \times U = \varphi(1) = \varphi\left(\sum_{a \in At(\mathfrak{M}_q)} a\right) = \bigcup_{a \in At(\mathfrak{M}_q)} \varphi(a),$$

so $\langle u, v\rangle \in \varphi(a)$ for some atom a that is uniquely determined by the pair $\langle u, v\rangle$. For any atom a of $\mathfrak{M}_q, \varphi(a)$ is a symmetric relation, hence

$$\langle u, v\rangle \in \varphi(x) \iff \langle v, u\rangle \in \varphi(x)$$

and we may let $\varphi(x)$ be the color of the subset $\{u, v\}$. The color is a diversity atom just in case u and v are distinct. The color of $\{u\}$ is 1'. A *monochromatic triangle* is a set $\{u, v, w\} \subseteq U$ with three elements such that $\{u, v\}, \{v, w\}$, and $\{u, w\}$ all have the same color, say $a \in At\,(\mathfrak{M}_q)$. Translated back to ordered pairs and φ, this implies $\langle u, v\rangle, \langle v, w\rangle, \langle u, w\rangle \in \varphi(a)$, hence $\langle u, w\rangle \in \varphi(a \cdot a;a) \neq \emptyset$, contradicting the fact that $a \cdot a;a = 0$ in $\mathfrak{M}_q$ because $\mathfrak{M}_q$ has no 1-cycles. Thus the edge-coloring, with q colors, of the complete graph on vertex set U has no monochromatic triangles, *i.e.*, the edge-coloring is *good* according to [4]. But then $|U| \leq q! \sum_{k=0}^{q} \frac{1}{k!}$ by [4, Proposition 19.3.1]. This bound originates with [7] and is a special case of the Finite Ramsey Theorem.

However, the number of atoms in $\mathfrak{A}$ is a lower bound on the size of U. Indeed, for any fixed element $u \in U$ there is, for each diversity atom $a \in At\,(\mathfrak{A})$, at least one $v \in U$ such that $\langle u, v\rangle \in \varphi(a)$, because $\langle u, u\rangle \in \varphi(1') \subseteq \varphi(a;a) = \varphi(a)|\varphi(a)$, and distinct atoms a, b yield distinct elements, for if $\langle u, v\rangle \in \varphi(a)$ and $\langle u, w\rangle \in \varphi(b)$, then $\langle v, w\rangle \in \breve{a};b \leq 0'$ since $a \neq b$. There must be at least $|At\,(\mathfrak{A})\,|$ additional elements in U besides u, so we have

$$|At\,(\mathfrak{A})\,| \leq |U| \leq q! \sum_{k=0}^{q} \frac{1}{k!}.$$

The number of atoms in $\mathfrak{A}$ and q are essentially independent parameters, subject only to the requirement that the former is bigger than the latter. The number of atoms in $\mathfrak{A}$ can be increased by splitting without changing the fact that $\mathfrak{M}_q \subseteq \mathfrak{A}$. Eventually the inequalities above are violated, and the contradiction shows that $\mathfrak{A} \notin \mathsf{RRA}$. If we know nothing more about the structure of $\mathfrak{A}$ than $\mathfrak{M}_q \subseteq \mathfrak{A}$, then perhaps $\mathfrak{A} \notin \mathbf{S}\mathsf{RRA}^c$. But if we start with $\mathfrak{M}_q$ and get $\mathfrak{A}$ by splitting, then $\mathfrak{A}$ is a Monk algebra, according to Definition 1.4, so by Corollary 4.8, $\mathfrak{A} \in \mathbf{S}\mathsf{RRA}^c$ whenever $\mathfrak{M}_q$ has six or more colors. Note that $\mathfrak{A}$ must have enough atoms that $\mathfrak{A} \notin \mathsf{RRA}$ for this to be interesting, because if $\mathfrak{A} \in \mathsf{RRA}$ then $\mathfrak{A} \in \mathsf{RRA}^c$. Indeed, it is suspected that $\mathfrak{M}_q \in \mathsf{RRA}^c$ for all $q \geq 4$.

2. Special extensions

Assume $\mathfrak{A}$ is a Monk algebra obtained from $\mathfrak{M}_q$ by splitting. Consider a subalgebra $\mathfrak{E} \subseteq \mathfrak{M}_q \subseteq \mathfrak{A}$. Then the Monk algebra $\mathfrak{A}$ extends its subalgebra $\mathfrak{E}$ in a way that, for want of a better name, we simply call "special".

Definition 2.1. If $\mathfrak{A}$ and $\mathfrak{E}$ are finite symmetric integral relation algebras, then $\mathfrak{A}$ is said to be a *special extension* of $\mathfrak{E}$ if $\mathfrak{E} \subseteq \mathfrak{A}$ and for all diversity atoms $0' \geq a, b, c \in At(\mathfrak{E})$,

(1) if not $(a = b = c)$ and $a;b \geq c$ then $x;y \geq c$ whenever $a \geq x \in At(\mathfrak{A})$ and $b \geq y \in At(\mathfrak{A})$,
(2) if $a;a \geq a$ then $x;y \cdot a \neq 0$ whenever $a \geq x, y \in At(\mathfrak{A})$.

Every finite symmetric integral relation algebra is a special extension of itself. Every finite symmetric integral relation algebra is also a special extension of its minimum subalgebra, the one whose atoms are 1' and 0'.

Lemma 2.2. *Every Monk algebra obtained from $\mathfrak{M}_q$ by splitting is a special extension of every subalgebra of $\mathfrak{M}_q$.*

Proof. Assume $\mathfrak{M}_q \subseteq \mathfrak{A}, 4 \leq q, \mathfrak{A}$ is a Monk algebra obtained from $\mathfrak{M}_q$ by splitting, and $\xi(x)$ is the atom of $\mathfrak{M}_q$ containing the atom x of $\mathfrak{A}$. Consider a subalgebra $\mathfrak{E} \subseteq \mathfrak{M}_q \subseteq \mathfrak{A}$ and diversity atoms $0' \geq a, b, c \in At(\mathfrak{E})$.

To show part (1) of Definition 2.1, we assume not $(a = b = c), a;b \geq c, a \geq x \in At(\mathfrak{A})$, and $b \geq y \in At(\mathfrak{A})$. We want to prove $x;y \geq c$. Note that $x \leq \xi(x) \leq a$ and $y \leq \xi(y) \leq b$. There are two cases. If $\xi(x) \neq \xi(y)$ then $\xi(x);\xi(y) = 0'$ by Definition 1.2, and $x \neq y$, so by Definition 1.3,

$$x;y = \xi(x);\xi(y) \cdot 0' = 0' \geq c,$$

as desired. Assume $\xi(x) = \xi(y)$. Then $a = b$ since a and b are atoms of $\mathfrak{E}$ with a non-empty intersection, hence $a = b \neq c$ by the assumption not $(a = b = c)$. Since a and c are distinct atoms, $\overline{a} \geq c$. By Definitions 1.2 and 1.3 we get

$$x;x = \xi(x);\xi(x) = \overline{\xi(x)} \geq \overline{a} \geq c,$$

and if $x \neq y$ then

$$x;y = \xi(x);\xi(y) \cdot 0' = \xi(x);\xi(x) \cdot 0' \geq c.$$

For part (2) of Definition 2.1, we assume $a \geq x, y \in At(\mathfrak{A})$ and $a;a \geq a$. We wish to show that $x;y \cdot a \neq 0$. Again there are two cases. If $\xi(x) \neq \xi(y)$ then $\xi(x);\xi(y) = 0'$ by Definition 1.2, and $x \neq y$, so by Definition 1.3, $x;y =$

$\xi(x);\xi(y)\cdot 0' = 0' \geq a$, hence $x;y\cdot a \neq 0$. Assume $\xi(x)=\xi(y)=u$. Note that $x+y \leq u \leq a \leq 0'$, and a cannot be the atom u of $\mathfrak{M}_q$ because the assumption $a;a \geq a$ fails for all diversity atoms of $\mathfrak{M}_q$ by Definition 1.2. Since a is not an atom of $\mathfrak{M}_q$, a is the join of two or more atoms of $\mathfrak{M}_q$, hence there is some atom $v \in At(\mathfrak{M}_q)$ such that $u \neq v \leq a$, hence $\overline{u} \geq v$. By Definitions 1.2 and 1.3, either $x;y = u;u = \overline{u} \geq v$ or $x;y = u;u\cdot 0' = \overline{u}\cdot 0' \geq v$, so $0 \neq x;y\cdot v \leq x;y\cdot a$, as desired. □

Lemma 2.2 suggests that we consider an arbitrary subalgebra $\mathfrak{E}$ of $\mathfrak{M}_q$. Every subalgebra contains 1', but 1' is an atom in $\mathfrak{M}_q$, so it is an atom in $\mathfrak{E}$ as well. Thus $\mathfrak{E}$ is integral, but $\mathfrak{E}$ is also symmetric since it is a subalgebra of a symmetric algebra. The diversity atoms of $\mathfrak{E}$ are disjoint and join up to 0', so they partition the diversity atoms of $\mathfrak{M}_q$. In every relation algebra, the relative product $a;b$ of distinct diversity atoms a,b of $\mathfrak{E}$ is included in 0'. On the other hand, in this case we have $a;b \geq 0'$ because there are atoms x,y of $\mathfrak{M}_q$ such that $a \geq x, b \geq y$, and $a;b \geq x;y = 0'$ by Definition 1.2. Every diversity atom of $a \in \mathfrak{E}$ satisfies either $a;a = 1$ or $a;a = \overline{a}$, for if a is an atom of $\mathfrak{M}_q$ (as well as $\mathfrak{E}$) then $a;a = \overline{a}$ by Definition 1.2, while if a is not an atom of $\mathfrak{M}_q$, then it is the join of two or more atoms of $\mathfrak{M}_q$, say $a \geq \mathbf{e}_1 + \mathbf{e}_2$, so

$$\begin{aligned} a;a &\geq (\mathbf{e}_1+\mathbf{e}_2);(\mathbf{e}_1+\mathbf{e}_2) \\ &= \mathbf{e}_1;\mathbf{e}_1 + \mathbf{e}_1;\mathbf{e}_2 + \mathbf{e}_2;\mathbf{e}_1 + \mathbf{e}_2;\mathbf{e}_2 \\ &= \overline{\mathbf{e}_1} + 0' + 0' + \overline{\mathbf{e}_2} && \text{Definition 1.2} \\ &= 1. \end{aligned}$$

Every subalgebra of $\mathfrak{M}_q$ can therefore be characterized by just two parameters: α, the number of diversity atoms a satisfying $a;a = \overline{a}$, and β, the number of diversity atoms satisfying $a;a = 1$. The number of atoms in the subalgebra is $1+\alpha+\beta$. The only restrictions on these parameters are $\alpha + 2\beta < q$ and $0 < \alpha+\beta$.

3. An infinite atom structure from two finite algebras

In this section we use an arbitrary finite symmetric non-associative relation algebra $\mathfrak{A}$ in which 1' is an atom and its subalgebra $\mathfrak{E}$ to construct a complete atomic algebra $C_{\mathfrak{E}}(\mathfrak{A}) \in \mathsf{NA}$ that has subalgebras isomorphic to $\mathfrak{A}$ and $\mathfrak{E}$. To define a complete atomic NA, as in the following definition, it is enough to describe its atom structure.

Definition 3.1. Assume $\mathfrak{E} \subseteq \mathfrak{A} \in \mathsf{NA}$ are finite symmetric non-associative relation algebras, and 1' is an atom of $\mathfrak{A}$. Then $C_{\mathfrak{E}}(\mathfrak{A})$ is a complete atomic NA with this atom structure: the atoms of $C_{\mathfrak{E}}(\mathfrak{A})$ are 1' and the ordered pair $x^{(i)}$ for every diversity atom x of $\mathfrak{A}$ and every index $i \in \omega$,

$$At(C_{\mathfrak{E}}(\mathfrak{A})) = \{1'\} \cup \{x^{(i)} : 0' \geq x \in At(\mathfrak{A}),\ i \in \omega\}, \tag{3.1}$$

the converse of every atom is itself, if $T \subseteq \omega^3$ is defined for $i,j,k \in \omega$ by

$$T(i,j,k) \iff (i \leq j = k) \vee (j \leq k = i) \vee (k \leq i = j),$$

and $\xi(x)$ is the atom of $\mathfrak{E}$ containing the atom x of $\mathfrak{A}$, then the cycles of $C_{\mathfrak{E}}(\mathfrak{A})$ are, for all $0' \geq x, y, z \in At\,(\mathfrak{A})$ and $i, j, k \in \omega$,

$$[1', 1', 1'], \quad [1', x^{(i)}, x^{(i)}], \tag{3.2}$$

$$[x^{(i)}, y^{(j)}, z^{(k)}] \quad \text{if} \;\; x{;}y \geq z \wedge (\xi(x) = \xi(y) = \xi(z) \;\Rightarrow\; T(i, j, k)). \tag{3.3}$$

For any $a \in A$ and $n \in \omega$, define the element $J(a, n)$ of $C_{\mathfrak{E}}(\mathfrak{A})$ by

$$J(a, n) = \sum\{x^{(i)} : 0' \cdot a \geq x \in At\,(\mathfrak{A})\,,\, n \leq i \in \omega\} + \sum\{1' : 1' \leq a\}. \tag{3.4}$$

If x is an atom of $\mathfrak{A}$, formula (3.4) takes the simpler form

$$J(x, n) = \begin{cases} \sum\{x^{(i)} : n \leq i \in \omega\} & \text{if} \quad x \leq 0' \\ 1' & \text{if} \quad x = 1' \end{cases}. \tag{3.5}$$

Lemma 3.2. *Assume* $0' \geq x, y \in At\,(\mathfrak{A})\,, i, j \in \omega,$ *and* $i \neq j$.

(1) *If* $\xi(x) = a$ *then*

$$x^{(i)}{;}x^{(i)} = J(0' \cdot \overline{a} \cdot x{;}x, 0) + \sum\{z^{(k)} : k \leq i,\; a \cdot x{;}x \geq z \in At\,(\mathfrak{A})\} + 1',$$

$$x^{(i)}{;}x^{(j)} = J(0' \cdot \overline{a} \cdot x{;}x, 0) + \sum\{z^{(\max(i,j))} : a \cdot x{;}x \geq z \in At\,(\mathfrak{A})\},$$

(2) *If* $\xi(x) \neq \xi(y)$ *then*

$$x^{(i)}{;}y^{(j)} = J(x{;}y, 0) = \sum\left\{z^{(k)} : x{;}y \geq z \in At\,(\mathfrak{A})\,,\, k \in \omega\right\},$$

(3) *If* $x \neq y$ *and* $\xi(x) = \xi(y) = a$ *then*

$$x^{(i)}{;}y^{(j)} = J(0' \cdot \overline{a} \cdot x{;}y, 0) + \sum\{z^{(\max(i,j))} : a \cdot x{;}y \geq z \in At\,(\mathfrak{A})\},$$

$$x^{(i)}{;}y^{(i)} = J(0' \cdot \overline{a} \cdot x{;}y, 0) + \sum\{z^{(k)} : k \leq i,\; a \cdot x{;}y \geq z \in At\,(\mathfrak{A})\}.$$

Start with a finite symmetric integral relation algebra $\mathfrak{A}$ in which every diversity atom is splittable in the sense of [3]. Let $\mathfrak{A}^{\omega} \supseteq \mathfrak{A}$ be the relation algebra obtained by splitting every diversity atom $a \in At\,(\mathfrak{A})$ into ω pieces $a^{(0)}, a^{(1)}, \cdots$ so that $a = \sum_{i \in \omega} a^{(i)}$. Splitting produces the maximum set of cycles in the extension $\mathfrak{A}^{\omega} \supseteq \mathfrak{A}$ that are consistent with containing $\mathfrak{A}$ as a subalgebra. Let $\mathfrak{E} \subseteq \mathfrak{A}$ be a subalgebra of $\mathfrak{A}$. From the atom structure of $\mathfrak{A}^{\omega}$ we obtain a new atom structure whose complex algebra is, in fact, isomorphic to $C_{\mathfrak{E}}(\mathfrak{A})$, by deleting all the diversity cycles $[a^{(i)}, b^{(j)}, c^{(k)}]$ of $\mathfrak{A}^{\omega}$ which have the property that all the atoms in the cycle lie below the same atom of $\mathfrak{E}$, and $T(i, j, k)$ fails to hold. This leaves only a thin remnant of the cycles of $\mathfrak{A}^{\omega}$ that we would classify as 1-cycles of $\mathfrak{E}$ (because their atoms all lie below a single atom of $\mathfrak{E}$). The set of 1-cycles produced by splitting is significantly reduced by imposing the thinning condition $T(i, j, k)$. Those cycles of $\mathfrak{A}$ that are covered by 1-cycles of $\mathfrak{E}$ are thinly reproduced in $C_{\mathfrak{E}}(\mathfrak{A})$, while the 2- and 3-cycles of $\mathfrak{A}$ that are covered by 2- or 3-cycles of $\mathfrak{E}$ are split into as many cycles as possible. Treating 1-, 2-, and 3-cycles differently in various combinations, either thinning or splitting each type of cycle, gives six more constructions that perhaps should be examined with regard to Problem 1.1(1).

Lemma 3.3. *$\mathfrak{A}$ is isomorphic, by $a \mapsto J(a,0)$, to a subalgebra $\mathfrak{A}'$ of $C_{\mathfrak{E}}(\mathfrak{A})$,*

$$\mathfrak{A} \cong \mathfrak{A}' \subseteq C_{\mathfrak{E}}(\mathfrak{A}).$$

Proof. Define the function $\varphi\colon \mathfrak{A} \to C_{\mathfrak{E}}(\mathfrak{A})$ by $\varphi(a) = J(a,0)$ for all $a \in A$. For a key part of the proof that φ embeds $\mathfrak{A}$ into $C_{\mathfrak{E}}(\mathfrak{A})$, assume $0' \geq x, y \in At(\mathfrak{A})$. We wish to prove that $\varphi(x);\varphi(y)$ and $\varphi(x;y)$ contain the same diversity atoms of $C_{\mathfrak{E}}(\mathfrak{A})$. (Proofs for the other parts, involving preservation by φ of the Boolean structure and identity element, are fairly easy.)

Consider an arbitrary diversity atom $z^{(k)} \in At(C_{\mathfrak{E}}(\mathfrak{A}))$, where $0' \geq z \in At(\mathfrak{A}), k \in \omega$. Assume $z^{(k)} \leq \varphi(x);\varphi(y)$. Then there are $u, v \in At(C_{\mathfrak{E}}(\mathfrak{A}))$ such that $z^{(k)} \leq u;v, \varphi(x) \geq u \in At(C_{\mathfrak{E}}(\mathfrak{A}))$, and $\varphi(y) \geq v \in At(C_{\mathfrak{E}}(\mathfrak{A}))$. By (3.5) there are some $i, j \in \omega$ such that $u = x^{(i)}$ and $v = y^{(j)}$. But then $[x^{(i)}, y^{(j)}, z^{(k)}]$ is a cycle of $C_{\mathfrak{E}}(\mathfrak{A})$, so $x;y \geq z$ in $\mathfrak{A}$, which implies $z^{(k)} \leq J(x;y,0)$, hence $z^{(k)} \leq \varphi(x;y)$. The argument is reversible. □

$C_{\mathfrak{E}}(\mathfrak{A})$ satisfies all the axioms for relation algebras except possibly the associative law, so $C_{\mathfrak{E}}(\mathfrak{A}) \in \mathsf{NA}$. Here is a computational lemma needed several times later.

Lemma 3.4. *Assume $\mathfrak{A}$ is a special extension of $\mathfrak{E}, a, b$ are distinct diversity atoms of $\mathfrak{E}$, and u, v are diversity atoms of $C_{\mathfrak{E}}(\mathfrak{A})$. If $u \leq J(a,0)$ and $v \leq J(b,0)$ then $u;v = J(a;b,0)$. In particular, if $a;b = 0'$, then $u;v = 0'$.*

Proof. From $u \leq J(a,0)$ and $v \leq J(b,0)$ we have $x^{(i)} = u, y^{(j)} = v, x \leq a = \xi(x)$, and $y \leq b = \xi(y)$, for some $x, y \in At(\mathfrak{A})$ and $i, j \in \omega$. The covers of x and y are different because $x \leq \xi(x) = a \neq b = \xi(y) \geq y$. Lemma 3.2(2) applies in this case and says that $x^{(i)};y^{(j)} = J(x;y,0)$. Note that $x;y \leq a;b$. Since $\mathfrak{A}$ is a special extension of $\mathfrak{E}$, we deduce from Definition 2.1(1) that every atom of $\mathfrak{E}$ below $a;b$ is also below $x;y$, hence $x;y = a;b$. We conclude that $u;v = x^{(i)};y^{(j)} = J(x;y,0) = J(a;b,0)$. If $a;b = 0'$, then $u;v = J(0',0)$, but $J(0',0)$ is the diversity element $0'$ of $C_{\mathfrak{E}}(\mathfrak{A})$, so $u;v = 0'$. □

4. Embedding Monk algebras

Theorem 4.1 below shows that when $\mathfrak{A}$ is a special extension of $\mathfrak{E}$ and $\mathfrak{R}$ is the subalgebra of $C_{\mathfrak{E}}(\mathfrak{A})$ generated by the atoms of $C_{\mathfrak{E}}(\mathfrak{A})$, the finitely generated subalgebras of $\mathfrak{R}$ are finite. Suppose $\mathfrak{A}$ is a Monk algebra obtained from $\mathfrak{M}_q$ by splitting and $\mathfrak{E}$ is a subalgebra of $\mathfrak{M}_q$:

$$\mathfrak{E} \subseteq \mathfrak{M}_q \subseteq \mathfrak{A}. \tag{4.1}$$

By Lemma 2.2, $\mathfrak{A}$ is a special extension of $\mathfrak{E}$, so Theorem 4.1 applies to $\mathfrak{A}$ and $\mathfrak{E}$. We show in Theorem 4.7(1)–(2) below that if, in addition, $\mathfrak{E}$ has a flexible trio (see Definition 4.5) then $\mathfrak{R}$ is representable because every finitely generated subalgebra of $\mathfrak{R}$ is included in a finite subalgebra of $\mathfrak{R}$ that has the 1-point extension property (see Definition 4.4). In example (4.1), if $7 \leq q$ ($\mathfrak{M}_q$ has at least six colors) then $\mathfrak{M}_q$ has a subalgebra $\mathfrak{E}$ with a flexible trio, so $\mathfrak{R} \in \mathsf{RRA}$ by Theorem 4.7(1)–(2). We show in Theorem 4.7(3) that if $\mathfrak{A}$ has no 1-cycles then the completion of $\mathfrak{R}$ is not representable. Theorem 4.7(3) applies to $\mathfrak{A}$ because Monk algebras have no 1-cycles. Corollary 4.8 accordingly says that every finite Monk algebra with six or more colors is a subalgebra of the non-representable completion of an atomic representable relation algebra whose finitely-generated subalgebras are finite.

The conclusion that $C_{\mathfrak{E}}(\mathfrak{A}) \notin \mathsf{RRA}$ can be obtained without Theorem 4.7 (3) in case the Monk algebra $\mathfrak{A}$ is non-representable, which happens if the number of atoms is large compared to the number of colors. In this case the completion of $\mathfrak{R}$ is non-representable simply because it has a non-representable subalgebra isomorphic to the non-representable Monk algebra $\mathfrak{A}$, so $C_{\mathfrak{E}}(\mathfrak{A}) \in \mathsf{RRA}^c$ and $\mathfrak{A} \in \mathbf{S}\mathsf{RRA}^c$.

In Theorem 4.1 we only assume the extension $\mathfrak{E} \subseteq \mathfrak{A}$ is special. In Theorem 4.7 we also consider what happens when, in addition, $\mathfrak{E}$ has a flexible trio and $\mathfrak{A}$ has no 1-cycles.

Theorem 4.1. *Assume $\mathfrak{A}$ and $\mathfrak{E}$ are finite symmetric integral relation algebras and $\mathfrak{A}$ is a special extension of $\mathfrak{E}$:*

$$\mathfrak{E} \subseteq \mathfrak{A}.$$

Let $\mathfrak{R} \subseteq C_{\mathfrak{E}}(\mathfrak{A})$ be the subalgebra of $C_{\mathfrak{E}}(\mathfrak{A})$ generated by $At\,(C_{\mathfrak{E}}(\mathfrak{A}))$.

(1) *$\mathfrak{R}$ is countable, atomic, symmetric, integral, and generated by its atoms.*
(2) *$C_{\mathfrak{E}}(\mathfrak{A})$ and $\mathfrak{R}$ have the same atom structure.*
(3) *$C_{\mathfrak{E}}(\mathfrak{A})$ is isomorphic to the complex algebra of the atom structure of $\mathfrak{R}$.*
(4) *$C_{\mathfrak{E}}(\mathfrak{A})$ is the completion of $\mathfrak{R}$.*
(5) *There are subalgebras $\mathfrak{E}' \cong \mathfrak{E}$ and $\mathfrak{A}' \cong \mathfrak{A}$ such that*

$$\mathfrak{E}' \subseteq \mathfrak{A}' \subseteq C_{\mathfrak{E}}(\mathfrak{A}).$$

(6) *Every finitely generated subalgebra of $\mathfrak{R}$ is finite.*

Proof. Parts (1)–(4) require only the assumption that $C_{\mathfrak{E}}(\mathfrak{A})$ is complete and atomic and $\mathfrak{R}$ is the subalgebra of $C_{\mathfrak{E}}(\mathfrak{A})$ generated by the atoms of $C_{\mathfrak{E}}(\mathfrak{A})$. Everything in parts (1)–(4) is either obvious or very easy to prove; see [29, Theorem 3.13] for part (3). Part (4) holds because $C_{\mathfrak{E}}(\mathfrak{A})$ is complete and $\mathfrak{R}$ is dense in $C_{\mathfrak{E}}(\mathfrak{A})$. Part (5) was proved in Lemma 3.3. The assumption that $\mathfrak{A}$ is a special extension of $\mathfrak{E}$ is needed only for the following Lemma 4.2, which is used to prove part (6).

Lemma 4.2. *For every $n \in \omega$, there is a subalgebra of $C_{\mathfrak{E}}(\mathfrak{A})$ whose set of atoms is*

$$\begin{aligned} Z_n = \{1'\} \cup \{x^{(i)} : 0' \geq x \in At\,(\mathfrak{A})\,,\, n > i \in \omega\} \\ \cup \{J(a,n) : 0' \geq a \in At\,(\mathfrak{E})\}. \end{aligned} \tag{4.2}$$

Proof. The elements of Z_n are disjoint and their join is 1, so the set of joins of subsets of Z_n is closed under the Boolean operations of $C_{\mathfrak{E}}(\mathfrak{A})$ and, under those operations, forms a Boolean algebra whose set of atoms is Z_n. The converse of everything in Z_n is again in Z_n because conversion is the identity function on $C_{\mathfrak{E}}(\mathfrak{A})$. What remains is to show the relative product $u;v$ of any two elements $u, v \in Z_n$ is the join of a subset of Z_n. For this it is enough to show that every element $w \in Z_n$ is contained in or disjoint from $u;v$. This is clearly true whenever $u = 1'$ or $v = 1'$ or w is itself an atom of $C_{\mathfrak{E}}(\mathfrak{A})$, so we may assume $w = J(a,n)$, for some $a \in At(\mathfrak{E})$, and $u + v \le 0'$. We will show that if $u;v$ has nonempty intersection with $J(a,n)$ then $u;v$ contains $J(a,n)$.

Suppose $u;v \cdot J(a,n) \neq 0$, where $0' \geq u, v \in Z_n, 0' \geq a \in At(\mathfrak{E})$. Then there are $x, y, z \in At(\mathfrak{A})$ and $i, j, k \in \omega$ such that $x^{(i)} \leq u, y^{(j)} \leq v, z^{(k)} \leq J(a,n)$, and $x^{(i)};y^{(j)} \geq z^{(k)}$. For both cases below, note that $\xi(z) = a$ and $n \leq k$ by (3.4), and $x;y \geq z$ by (3.3).
Case 1. not $(\xi(x) = \xi(y) = \xi(z))$. From $x;y \geq z$ we get $x;y \cdot \xi(z) \neq 0$ since $0 \neq z \leq \xi(z)$, hence $x;y \geq \xi(z) = a$ by Definition 2.1(1) and $\xi(z)a \in At(\mathfrak{E})$. The implication in (3.3) has a false hypothesis and therefore holds trivially in this case for every atom of $\mathfrak{A}$ below a. It follows by (3.3) and (3.4) that $x^{(i)};y^{(j)} \geq J(a,0) \geq J(a,n)$.
Case 2. $\xi(x) = \xi(y) = \xi(z) = a$. In this case, by $x^{(i)};y^{(j)} \geq z^{(k)}$ and (3.3) we have $T(i,j,k)$. By $T(i,j,k)$ and $n \leq k$, either $j < n \leq i = k, i < n \leq j = k$, or $n \leq k \leq i = j$, hence either $u = J(a,n)$ and $v = y^{(j)}$, or $u = x^{(i)}$ and $v = J(a,n)$, or $u = v = J(a,n)$, respectively. Since $C_{\mathfrak{E}}(\mathfrak{A})$ is symmetric, the first two cases are really the same, and for them it is enough to prove $x^{(i)};J(a,n) \geq J(a,n)$ assuming $i < n$. In the third case it enough to show $x^{(n)};J(a,n) \geq J(a,n)$, for then $J(a,n);J(a,n) \geq J(a,n)$ follows by $x^{(n)} \leq J(a,n)$. We will do both inclusions together.

Suppose $w^{(l)} \leq J(a,n)$ where $n \leq l \in \omega$ and $w \leq a = \xi(w)$. From $x;y \geq z$ and $\xi(x) = \xi(y) = \xi(z) = a$ we get $a;a \cdot a \neq 0$, but $x \leq a$ and $w \leq a$, so $x;w \cdot a \neq 0$ by Definition 2.1(2). We may therefore choose an atom $t \in \mathfrak{A}$ such that $t \leq x;w \cdot a$. By assumption $i < n \leq l$, hence $T(i,l,l)$ and $T(n,l,l)$. From $\xi(t) = a, T(i,l,l)$, and $T(n,l,l)$, we conclude that $[x^{(i)}, t^{(l)}, w^{(l)}]$ and $[x^{(n)}, t^{(l)}, w^{(l)}]$ are cycles of $C_{\mathfrak{E}}(\mathfrak{A})$ by (3.3). Noting that $t^{(l)} \leq J(a,n)$ since $t \leq a$ and $n \leq l$, we have

$$w^{(l)} \leq x^{(i)};t^{(l)} \cdot x^{(n)};t^{(l)} \leq x^{(i)};J(a,n) \cdot x^{(n)};J(a,n).$$

Since this holds for all atoms $w^{(l)}$ below $J(a,n)$, we have proved

$$J(a,n) \leq x^{(i)};J(a,n) \cdot x^{(n)};J(a,n).$$

We have shown that every product of two elements of Z_n is the join of a subset of Z_n. It follows that $u;v = \sum\{w : u;v \geq w \in Z_n\}$ for all $u, v \in Z_n$. Hence, for all $U, V \subseteq Z_n$, we have

$$\begin{aligned} \sum U; \sum V &= \sum\{u;v : u \in U,\ v \in V\} \\ &= \sum\{\sum\{w : u;v \geq w \in Z_n\} : u \in U,\ v \in V\} \\ &= \sum\{w : u;v \geq w \in Z_n,\ u \in U,\ v \in V\} \\ &\in \{\sum X : X \subseteq Z_n\}. \end{aligned}$$

Therefore $\{\sum X : X \subseteq Z_n\}$ is closed under relative multiplication and is a subalgebra of $C_{\mathfrak{E}}(\mathfrak{A})$. □

The subalgebras mentioned in Lemma 4.2 may contain elements that do not appear in the subalgebra $\mathfrak{R}$ generated by the atoms of $C_{\mathfrak{E}}(\mathfrak{A})$. For example, let $\mathfrak{A}$ be a finite symmetric non-associative relation algebra in which

1' is an atom, there are at least two diversity atoms, and the atom structure has all 1-cycles, no 2-cycles, and no 3-cycles, *i.e.*, for distinct diversity atoms $a, b, a;a = 1' + a$ and $a;b = 0$. Let $\mathfrak{E} = \mathfrak{A}$. Since the only cycles in $\mathfrak{A}$ are 1-cycles, (3.3) implies that the relative product of any two atoms of $C_{\mathfrak{E}}(\mathfrak{A})$ is empty or an atom. Therefore $\mathfrak{R}$ is the subalgebra whose elements are joins of either finitely many, or else cofinitely many, atoms. For every diversity atom $a \in At(\mathfrak{A})$ and every $n \in \omega, J(a,n)$ is an element of Z_n but it is not in $\mathfrak{R}$ because it is the join of an infinite and coinfinite set of atoms of $C_{\mathfrak{E}}(\mathfrak{A})$ (for which we need to know there are at least two diversity atoms in $\mathfrak{A}$). In particular, $J(a,0)$ is an atom of $\mathfrak{E}' = \mathfrak{A}'$ that is not in $\mathfrak{R}$.

We return to the proof of Theorem 4.1. Suppose $\mathfrak{F}$ is a finitely generated subalgebra of $\mathfrak{R}$. Since $\mathfrak{R}$ is itself generated by $At(C_{\mathfrak{E}}(\mathfrak{A}))$, there is a finite set of atoms $X \subseteq At(C_{\mathfrak{E}}(\mathfrak{A}))$ such that $\mathfrak{F}$ is contained in the subalgebra of $C_{\mathfrak{E}}(\mathfrak{A})$ generated by X. Since X is finite and $At(C_{\mathfrak{E}}(\mathfrak{A})) \subseteq \bigcup_{n\in\omega} Z_n$, we may choose a sufficiently large $n \in \omega$ so that $X \subseteq Z_n$. By Lemma 4.2, $\mathfrak{F}$ is contained in the subalgebra $\{\sum X : X \subseteq Z_n\}$ of $C_{\mathfrak{E}}(\mathfrak{A})$ generated by Z_n. This subalgebra is finite since its set of atoms is the finite set Z_n, so $\mathfrak{F}$ is also finite. Hence (6) holds. □

As it happens, every finitely-generated subalgebra of $C_{\mathfrak{E}}(\mathfrak{A})$ (not just $\mathfrak{R}$) is also finite, even if the extension $\mathfrak{E} \subseteq \mathfrak{A}$ is not special. To prove this, one argues that for every finite subset F of $C_{\mathfrak{E}}(\mathfrak{A})$ there is some $n \in \omega$ and some finite partition $\mathcal{P}$ of $\{i : n \leq i \in \omega\}$ such that

$$\{1'\} \cup \{x^{(i)} : x \in At(\mathfrak{A}),\, n > i \in \omega\} \cup \{\textstyle\sum\{x^{(i)} : i \in P\} : x \in At(\mathfrak{A}),\, P \in \mathcal{P}\}$$

is the set of atoms of a subalgebra of $C_{\mathfrak{E}}(\mathfrak{A})$ that contains F. The remaining details of this proof are omitted since this fact is not needed and it is also not in itself enough to prove Lemma 4.2. On the other hand, changing $\mathfrak{E}$ to $\mathfrak{A}$ in (4.2) yields a special case that is easy to prove and needed later.

Lemma 4.3. *Assume $\mathfrak{A} \supseteq \mathfrak{E}$ are finite symmetric integral relation algebras. For every $n \in \omega$,*

$$\{1'\} \cup \{x^{(i)} : 0' \geq x \in At(\mathfrak{A}),\, n > i \in \omega\} \cup \{J(a,n) : 0' \geq a \in At(\mathfrak{A})\}. \quad (4.3)$$

is the set of atoms of a subalgebra of $C_{\mathfrak{E}}(\mathfrak{A})$.

Proof. The proof is similar to, but simpler than, the proof of Lemma 4.2. The closure of the set of joins of subsets of (4.3) under relative multiplication is an immediate consequence of Lemma 3.2(1)–(3). □

For the next theorem we need some definitions. A relation algebra has the 1-point extension property if, loosing speakly, every "finite partial representation" μ can be extended by one point wherever this is needed. We make this precise as follows.

Definition 4.4. For any $k \in \omega$ and any atomic relation algebra $\mathfrak{A}, B_k(\mathfrak{A})$ is the set of functions $\mu\colon k \times k \to At(\mathfrak{A})$ that satisfy the following conditions.

(1) $\mu_{i,i} \leq 1'$ for all $i < k$,
(2) $\breve{\mu}_{i,j} = \mu_{j,i}$ for all $i, j < k$,

(3) $\mu_{i,l} ; \mu_{l,j} \geq \mu_{i,j}$ for all $i, j, l < k$.

The elements of $B_k(\mathfrak{A})$ are called *basic matrices*. A matrix μ satisfies the *identity condition* if, for all $l, m < k, \mu_{l,m} \leq 1$' iff $l = m$. The algebra $\mathfrak{A}$ has the *1-point extension property* if, assuming $\mu \in B_k(\mathfrak{A}), \mu$ satisfies the identity condition, x, y are diversity atoms of $\mathfrak{A}, i, j < k$, and $\mu_{i,j} \leq x ; y$, there is a basic matrix $\mu' \in B_{k+1}(\mathfrak{A})$ satisfying the identity condition such that $\mu'_{i,k} = x, \mu'_{k,j} = y$, and $\mu_{l,m} = \mu'_{l,m}$ for all $l, m < k$.

Definition 4.5. An atom a of a symmetric integral relation algebra is said to be *flexible atom* if $a ; a = 1$ and $x ; a = 0$' for all diversity atoms x distinct from a. Three diversity atoms a, b, c of a symmetric integral relation algebra $\mathfrak{A}$ are said to be a *flexible trio* if

$$a ; a = b ; b = c ; c = 1, \tag{4.4}$$

$$a ; b = a ; c = b ; c = 0\text{'}, \tag{4.5}$$

and, for every atom $x \notin \{1\text{'}, a, b, c\}$,

$$x ; a = x ; b = 0\text{'} \ \vee \ x ; a = x ; c = 0\text{'} \ \vee \ x ; b = x ; c = 0\text{'}. \tag{4.6}$$

Having a flexible atom is a sufficient condition for representability on an infinite set; see Comer [6, 5.3] or [31, Theorem 6]. Since every proper subalgebra of $\mathfrak{M}_q$ has at least one atom a satisfying $a ; a = 1$ and this atom is flexible by Definition 1.2, every proper subalgebra of $\mathfrak{M}_q$ is representable. Theorem A.1 (relegated to "Appendix A") shows that having a flexible trio is also sufficient for representability on an infinite set. The *Flexible Atom Conjecture* states that every finite symmetric integral relation algebra with a flexible atom is representable on a finite set. This has been proven in some special cases [2], and suggests a similar problem.

Problem 4.6 (Flexible Trio Conjecture). Show that every finite symmetric integral relation algebra with a flexible trio is representable on a finite set.

If each of a, b, and c is a flexible atom then a, b, c is a flexible trio. For example, if $q \geq 7$ and the diversity atoms of $\mathfrak{M}_q$ are grouped into three twos plus the rest, the resulting subalgebra has three flexible atoms that together make up a flexible trio. It can also happen that a, b, c is a flexible trio while none of a, b, c is flexible. For example, the symmetric integral relation algebra with seven atoms $1\text{'}, a, b, c, d, e, f$ and all diversity cycles except $[a, d, d], [b, e, e]$, and $[c, f, f]$ has no flexible atoms, but a, b, c is a flexible trio. (This is probably the unique smallest example among symmetric integral relation algebras.)

In Monk algebras with at least six colors there are at least three pairs of diversity atoms, so Theorem 4.7 below applies to them. However, many other algebras also satisfy its hypotheses. For an example, let $\mathfrak{A}$ be the symmetric integral relation algebra whose atoms are $1\text{'}, a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$, and whose diversity cycles consist of none of the 1-cycles, all of the 2-cycles, and all the 3-cycles except $[a_1, a_2, a_3], [b_1, b_2, b_3]$, and $[c_1, c_2, c_3]$. Let $\mathfrak{C}$ be the subalgebra of $\mathfrak{A}$ whose atoms are $1\text{'}, a = a_1 + a_2 + a_3, b = b_1 + b_2 + b_3$, and $c = c_1 + c_2 + c_3$. Then $\mathfrak{A}$ is a special extension of $\mathfrak{E}, a, b, c$ is a flexible trio of $\mathfrak{E}$, and $\mathfrak{A}$ has no 1-cycles, so Theorem 4.7 applies, but $\mathfrak{A}$ is not a Monk algebra (in the narrow sense of Definition 1.4). More than 3000 additional examples can be obtained by deleting any or all of the following 2-cycles: $[a_1, a_2, a_2], [a_2, a_3, a_3], [a_1, a_1, a_3], [b_1, b_2, b_2], [b_2, b_3, b_3], [b_1, b_1, b_3], [c_1, c_2, c_2], [c_2, c_3, c_3], [c_1, c_1, c_3]$, and restoring any or all of the deleted 3-cycles, but at least one of the 2- or 3-cycles must be deleted to insure $\mathfrak{A}$ is not a Monk algebra of Definition 1.4. This scheme retains enough 2-cycles that the extension is always special.

Theorem 4.7. *Assume $\mathfrak{E} \subseteq \mathfrak{A}$ are finite symmetric integral relation algebras, $\mathfrak{A}$ is a special extension of $\mathfrak{E}$, and $\mathfrak{E}$ has a flexible trio. If $\mathfrak{R}$ is the subalgebra of $C_{\mathfrak{E}}(\mathfrak{A})$ generated by $At\,(C_{\mathfrak{E}}(\mathfrak{A}))$, then*

(1) *every finitely generated subalgebra of $\mathfrak{R}$ is contained in a subalgebra of $C_{\mathfrak{E}}(\mathfrak{A})$ that has the 1-point extension property,*
(2) *$\mathfrak{R}$ is representable,*
(3) *if $\mathfrak{A}$ has no 1-cycles, i.e., $u;u \cdot u = 0$ whenever $1' \neq u \in At\,(\mathfrak{A})$, then the completion of $\mathfrak{R}$ is not representable.*

Proof. Suppose $\mathfrak{F}$ is a finitely generated subalgebra of $\mathfrak{R}$. By the argument at the end of the proof of Theorem 4.1, there is some $n \in \omega$ such that $\mathfrak{F}$ is contained in the subalgebra $\mathfrak{Z}_n$ of $C_{\mathfrak{E}}(\mathfrak{A})$ with atoms $At\,(\mathfrak{Z}_n) = Z_n$; see (4.2). Let a, b, c be a flexible trio of $\mathfrak{E}$. We will show that $J(a,n), J(b,n), J(c,n)$ is a flexible trio of $\mathfrak{Z}_n$. Consider the product of the first two elements of the trio. Note that $J(a,n);J(b,n) \leq 0'$ since $J(a,n)$ and $J(b,n)$ are disjoint atoms of $\mathfrak{R}_n$. We have

$$J(a,n);J(b,n) = \sum\{u;v : J(a,n) \geq u \in At\,(C_{\mathfrak{E}}(\mathfrak{A}))\,, \\ J(b,n) \geq v \in At\,(C_{\mathfrak{E}}(\mathfrak{A}))\}$$

but every disjunct $u;v$ in this last join is $0'$ by Lemma 3.4 and the assumption $a;b = 0'$, so $J(a,n);J(b,n) = 0'$. Similarly, $J(a,n);J(c,n) = 0' = J(b,n);J(c,n)$. Thus (4.5) holds.

For (4.6), consider a diversity atom of $\mathfrak{R}_n$ that is not one of $J(a,n), J(b,n)$, or $J(c,n)$. It is either an atom of $C_{\mathfrak{E}}(\mathfrak{A})$ or an atom of $\mathfrak{R}_n$ with the form $J(d,n)$, where d is a diversity atom of $\mathfrak{E}$ distinct from a, b, c.

We first consider $J(d,n)$. Now d multiplies to $0'$ with two of a, b, c by (4.6), say $a;d = b;d = 0'$. Choose atoms x, y of $\mathfrak{A}$ with $x \leq a$ and $y \leq d$. Then $J(a,n);J(d,n) \leq 0'$ since $J(a,n)$ and $J(d,n)$ are disjoint, and

$$J(a,n);J(d,n) \geq x^{(n)};y^{(n)},$$

but $x^{(n)};y^{(n)} = 0'$ by Lemma 3.4 because $a;d = 0'$, so $J(a,n);J(d,n) = 0'$. Similarly $J(b,n);J(d,n) \geq 0'$, so the atom $J(d,n)$ multiplies to $0'$ with two of $J(a,n), J(b,n), J(c,n)$, as desired.

Next consider a diversity atom u of $C_{\mathfrak{E}}(\mathfrak{A})$. It has the form $u = x^{(i)}$ for some diversity atom x of $\mathfrak{A}$ and some $i < n$. We claim that the product of $\xi(x)$ with (at least) two elements in the trio a, b, c is $0'$, say $a;\xi(x) = b;\xi(x) = 0'$. This follows from (4.6) if $\xi(x)$ is a diversity atom distinct from

a, b, c, but if $\xi(x)$ is one of a, b, c, then it follows from (4.5). Choose an atom $a \geq y \in At(\mathfrak{A})$. Then $x^{(i)};J(a,n) \geq x^{(i)};y^{(n)} = 0'$ by $a;d = 0'$ and Lemma 3.4. Similarly $x^{(i)};J(b,n) = 0'$, so the atom $u = x^{(i)}$ multiplies to 0' with two of $J(a,n), J(b,n), J(c,n)$, as desired. This finishes the proof of (4.6) for $J(a,n)$, $J(b,n), J(c,n)$.

For (4.4), we will prove $J(a,n);J(a,n) = 1$ from $a;a = 1$. Assume $u = x^{(i)} \in At(C_{\mathfrak{E}}(\mathfrak{A})), 0' \geq x \in At(\mathfrak{A})$, and $i \in \omega$. Then $x \leq 1 = a;a = \sum\{y;z : a \geq y, z \in At(\mathfrak{A})\}$ so there are atoms $a \geq y, z \in At(\mathfrak{A})$ such that $x \leq y;z$. Note that $\xi(y) = \xi(z) = a$. Choose any j such that $\max(i,n) \leq j \in \omega$. Then $T(i,j,j)$ holds, so $[x^{(i)}, y^{(j)}, z^{(j)}]$ is a cycle of $C_{\mathfrak{E}}(\mathfrak{A})$ by (3.3), hence $u = x^{(i)} \leq y^{(j)};z^{(j)} \leq J(a,n);J(a,n)$. This shows $J(a,n);J(a,n) = 1$, and we obtain $J(b,n);J(b,n) = 1 = J(c,n);J(c,n)$ similarly from $b;b = c;c = 1$.

This completes the proof that $J(a,n), J(b,n), J(c,n)$ is a flexible trio of $\mathfrak{Z}_n$. By Theorem A.1 below, $\mathfrak{Z}_n$ has the 1-point extension property and is therefore representable. Every finitely generated subalgebra of $\mathfrak{R}$ is representable, hence $\mathfrak{R}$ is representable since RRA is a variety. Thus parts (1) and (2) hold.

For part (3), assume that $u;u \cdot u = 0$ whenever $1' \neq u \in At(\mathfrak{A})$. By Theorem 4.1(4), we need to show $C_{\mathfrak{E}}(\mathfrak{A})$ is not representable. Suppose that ρ is a representation of $C_{\mathfrak{E}}(\mathfrak{A})$ sending elements of $C_{\mathfrak{E}}(\mathfrak{A})$ to binary relations on U. Since $C_{\mathfrak{E}}(\mathfrak{A})$ is infinite, U must also be infinite. The diversity relation on U is partitioned into finitely many symmetric binary diversity relations, namely $\rho(J(a,0))$ with $1' \neq a \in At(\mathfrak{A})$. Because the relations are symmetric, this partition may be viewed as a partition of the two-element subsets of U into finitely many parts, so by Ramsey's Theorem, there is some diversity atom a of $\mathfrak{A}$ and some infinite subset $H \subseteq U$ such that all pairs of distinct elements of H are in $\rho(J(a,0))$. Since $\mathfrak{A}$ has no 1-cycles,

$$\rho(J(a,0))|\rho(;J(a,0)) \cap \rho(J(a,0)) = \rho(J(a,0);J(a,0) \cdot J(a,0)) = \rho(0) = \emptyset,$$

hence there cannot even be a three-element subset of U whose diversity pairs are all in $\rho(J(a,0))$, so we have a contradiction. □

Corollary 4.8. *If $\mathfrak{A}$ is a finite Monk algebra with six or more colors then $\mathfrak{A} \in$ SRRA^c. In fact, $\mathfrak{A}$ is a subalgebra of the completion of a representable relation algebra $\mathfrak{R} \in \mathsf{RRA}$ such that*

(1) *$\mathfrak{R}$ is a countable, atomic, symmetric, integral relation algebra that is generated by its atoms,*
(2) *every finitely generated subalgebra of $\mathfrak{R}$ is contained in a finite subalgebra of $\mathfrak{R}$ with the 1-point extension property,*
(3) *the completion of $\mathfrak{R}$ has the same atom structure as $\mathfrak{R}$, is isomorphic to the complex algebra of the atom structure of $\mathfrak{R}$, and is not representable.*

The smallest example to which these considerations apply is $\mathfrak{M}_7$. This algebra is a Monk algebra with six colors and no 1-cycles, obtained from itself by splitting. By the method of Comer [5], $\mathfrak{M}_7 \in \mathsf{RRA}$ because $\mathfrak{M}_7$ has representations on sets containing 97, 157, and 277 elements. But $\mathfrak{M}_7 = \mathfrak{M}_7^{\mathfrak{c}}$ since $\mathfrak{M}_7$ is finite, so $\mathfrak{M}_7 \in \mathsf{RRA}^c \subseteq \mathsf{SRRA}^c$. Thus the first conclusion of Corollary 4.8

holds for rather simple reasons, but from the rest of Corollary 4.8 we get a non-representable completion of a representable relation algebra.

Suppose the diversity atoms of $\mathfrak{M}_7$ are $\mathbf{e}_1, \ldots, \mathbf{e}_6$. Let $\mathsf{a}_1 = \mathbf{e}_1 + \mathbf{e}_2, \mathsf{a}_2 = \mathbf{e}_3 + \mathbf{e}_4$, and $\mathsf{a}_3 = \mathbf{e}_5 + \mathbf{e}_6$. Then $\mathfrak{A}$ is a special extension of the subalgebra $\mathfrak{E}$ whose atoms are $1', \mathsf{a}_1, \mathsf{a}_2, \mathsf{a}_3$, and $\mathsf{a}_1, \mathsf{a}_2, \mathsf{a}_3$ is a flexible trio of individually flexible atoms in $\mathfrak{E}$. $C_{\mathfrak{E}}(\mathfrak{M}_7)$ is the non-representable completion of the atomic representable subalgebra of $C_{\mathfrak{E}}(\mathfrak{M}_7)$ generated by $At\,(C_{\mathfrak{E}}(\mathfrak{M}_7))$. In the next section we compute the exact degree of non-representability of $C_{\mathfrak{E}}(\mathfrak{M}_7)$.

5. Cylindric algebras

CA_n is the class of n-dimensional cylindric algebras; see [10, 1.1.1]. Given a cylindric algebra $\mathfrak{D} \in \mathsf{CA}_n$ of dimension $n \geq 3$, the *relation algebraic reduct* $\mathfrak{Ra}\,(\mathfrak{D})$ is defined in [11, Definition 5.3.7] and is a relation algebra if $n \geq 4$ by [11, Theorem 5.3.8]. For any class $K \subseteq \mathsf{CA}_n$ with $3 \leq n$, let $\mathsf{Ra}K$ be the class of relation algebraic reducts of subalgebras of neat 3-dimensional reducts of algebras in K:

$$\mathsf{Ra}K = \mathfrak{Ra}^*\mathbf{S}\mathsf{Nr}_3K. \tag{5.1}$$

By [11, 5.3.9, 5.3.16, 5.3.17], we have

$$\mathsf{RRA} = \bigcap_{n\in\omega} \mathsf{RaCA}_{n+4} \subseteq \cdots \subseteq \mathsf{RaCA}_5 \subseteq \mathsf{RaCA}_4 = \mathsf{RA}. \tag{5.2}$$

Every non-representable relation algebra lies somewhere on this chain. The location of the example $C_{\mathfrak{E}}(\mathfrak{M}_7)$ is determined by the main result in this section, which implies

$$C_{\mathfrak{E}}(\mathfrak{M}_7) \in \mathsf{RaCA}_7 \sim \mathsf{RaCA}_8. \tag{5.3}$$

Definition 5.1. Assume $\mathfrak{A} \in \mathsf{NA}$ is atomic and $k \leq \omega$. Two basic matrices μ and μ' in $B_k(\mathfrak{A})$ *agree up to* i if $\mu_{l,m} = \mu'_{l,m}$ whenever $i \neq l, m \in k$, and they *agree up to* i, j if $\mu_{l,m} = \mu'_{l,m}$ whenever $i, j \neq l, m \in k$. We say that $\mathcal{M} \subseteq B_k(\mathfrak{A})$ is an *k-dimensional relational basis for* $\mathfrak{A}$ if

1. (1) for every atom $a \in At\,(\mathfrak{A})$ there is a basic matrix $\mu \in \mathcal{M}$ such that $\mu_{0,1} = a$,
2. (2) if $\mu \in \mathcal{M}, i, j < k, x, y \in At\mathfrak{A}, \mu_{i,j} \leq x;y$, and $i, j \neq l < k$, then there is some $\mu' \in \mathcal{M}$ such that μ and μ' agree up to $l, \mu'_{i,l} = x$, and $\mu'_{l,j} = y$.

For any $i, j < k$ let

$$T_i^k(\mathfrak{A}) = \{\, \langle \mu, \mu' \rangle \in B_k(\mathfrak{A}) \times B_k(\mathfrak{A}) : \mu \text{ and } \mu' \text{ agree up to } i \,\},$$
$$E_{i,j}^k(\mathfrak{A}) = \{\, \mu \in B_k(\mathfrak{A}) : \mu_{i,j} \leq 1' \,\}.$$

We say that $\mathcal{M} \subseteq B_k\mathfrak{A}$ is a *k-dimensional cylindric basis for* $\mathfrak{A}$ if

3. (3) if $a, b, c \in At\,(\mathfrak{A})$, and $a \leq b;c$, then there is a basic matrix $\mu \in \mathcal{M}$ such that $\mu_{01} = a, \mu_{02} = b$, and $\mu_{21} = c$,
4. (4) if $\mu, \mu' \in \mathcal{M}$, $i, j < k$, $i \neq j$, and μ agrees with μ' up to i, j, then there is some $\mu'' \in \mathcal{M}$ such that μ'' agrees with μ up to i, and μ'' agrees with μ' up to j, *i.e.*, $\langle \mu'', \mu \rangle \in T_i^k(\mathfrak{A})$ and $\langle \mu'', \mu' \rangle \in T_j^k(\mathfrak{A})$,

(5) if $\mu \in \mathcal{M}$ and $i, j < k$ then $\mu[i/j] \in \mathcal{M}$, where $[i/j](m) = m$ if $i \neq m < k$, and $[i/j](i) = j$.

For every $\mathcal{M} \subseteq B_k(\mathfrak{A})$, let

$$\mathfrak{Ca}\,(\mathcal{M}) = \mathfrak{Cm}\left(\langle \mathcal{M}, T_i, E_{ij}\rangle_{i,j<k}\right) \tag{5.4}$$

be the complex algebra of the relational structure $\langle \mathcal{M}, T_i, E_{ij}\rangle_{i,j<k}$, as defined in [10, 2.7.33], where $E_{ij} = E^k_{i,j}(\mathfrak{A}) \cap \mathcal{M}$ and $T_i = T^k_i(\mathfrak{A}) \cap (\mathcal{M} \times \mathcal{M})$ for all $i, j < k$.

Theorem 5.2. *Assume $4 \leq r \in \omega, \mathfrak{A} = \mathfrak{M}_{r+3}$, and the atoms of $\mathfrak{A}$ are*

$$\mathbf{e}_0 = 1\text{'},\ \mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3,\ \mathbf{e}_4,\ \mathbf{e}_5,\ \mathbf{e}_6,\ \ldots,\ \mathbf{e}_{r+2}.$$

Then $\mathfrak{A}$ is a special extension of a subalgebra $\mathfrak{C}$ whose r atoms are

$$\begin{aligned} &\mathsf{a}_0 = \mathbf{e}_0 = 1\text{'}, \quad &\mathsf{a}_1 = \mathbf{e}_1 + \mathbf{e}_2, \quad &\mathsf{a}_2 = \mathbf{e}_3 + \mathbf{e}_4, \quad &\mathsf{a}_3 = \mathbf{e}_5 + \mathbf{e}_6,\\ & &\mathsf{a}_4 = \mathbf{e}_7, \quad &\ldots, \quad &\mathsf{a}_{r-1} = \mathbf{e}_{r+2}, \end{aligned}$$

and $\mathsf{a}_1, \mathsf{a}_2, \mathsf{a}_3$ is a flexible trio of $\mathfrak{C}$, so by Theorems 4.1 and 4.7, the atom-generated subalgebra of the complete atomic relation algebra $C_{\mathfrak{C}}(\mathfrak{A})$ is an atomic atom-generated symmetric integral representable relation algebra with finite finitely-generated subalgebras. Furthermore, if $3 \leq n \leq r+3$ then

(1) *$B_n(C_{\mathfrak{C}}(\mathfrak{A}))$ is an n-dimensional cylindric basis for $C_{\mathfrak{C}}(\mathfrak{A})$,*
(2) *$\mathfrak{Ca}\,(B_n(C_{\mathfrak{C}}(\mathfrak{A})))$ is a complete atomic n-dimensional cylindric algebra,*
(3) *$C_{\mathfrak{C}}(\mathfrak{A})$ is isomorphic to the relation algebraic reduct of $\mathfrak{Ca}\,(B_n(C_{\mathfrak{C}}(\mathfrak{A})))$ and $C_{\mathfrak{C}}(\mathfrak{A}) \in \mathsf{RaCA}_n$,*
(4) *$\mathfrak{Ca}\,(B_n(C_{\mathfrak{C}}(\mathfrak{A}))) \notin \mathsf{SNr}_n\mathsf{CA}_{r+4}$.*

Proof. By [32, Theorem 7], in order to prove $B_n(C_{\mathfrak{C}}(\mathfrak{A}))$ is a cylindric basis for $C_{\mathfrak{C}}(\mathfrak{A})$ it is enough to show, given $n-2$ pairs of diversity atoms $u_1, v_1, \ldots, u_{n-2}, v_{n-2}$ of $C_{\mathfrak{C}}(\mathfrak{A})$, that

$$\prod_{1 \leq i \leq n-2} u_i ; v_i \neq 0. \tag{5.5}$$

We will find a diversity atom w, such that w is included in every product $u_i;v_i, 1 \leq i \leq n-2$. Any product $u_i;v_i$ that is equal to 0' or 1 imposes no restriction on our choice of w. We therefore assume that none of the products is 0' or 1, *i.e.*, $0\text{'} \neq u_i;v_i \neq 1$ whenever $1 \leq i \leq n-2$. Consequently, for every product $u_i;v_i$ we know that there cannot be distinct atoms $a, b \in At\,(\mathfrak{C})$ such that $u_i \leq J(a, 0)$ and $v_i \leq J(b, 0)$, because we would obtain $u_i;v_i = J(a;b, 0)$ from $a \neq b$ by Lemma 3.4, and a computation in $\mathfrak{C}$ shows $a;b = 0\text{'}$ since $a \neq b$, forcing $u_i;v_i = 0\text{'}$ in $C_{\mathfrak{C}}(\mathfrak{A})$, contrary to our assumption that no product is 0' or 1. Therefore, there is a function $f\colon \{1, \ldots, n-2\} \to \{1, \ldots, r-1\}$ such that

$$u_i + v_i \leq J(\mathsf{a}_{f(i)}, 0) \qquad \text{for all } i \in \{1, \ldots, n-2\}. \tag{5.6}$$

Suppose some index $j \in \{1, \ldots, r-1\}$ is not in the range of f. Consider any product $u_i;v_i$ with $1 \leq i \leq n-2$. Let $k = f(i)$ and note that $k \neq j$. There are atoms $x, y \in At\,(\mathfrak{A})$ such that

$$x + y \leq \mathsf{a}_k = \xi(x) = \xi(y), \qquad u_i \leq J(x, 0), \qquad v_i \leq J(y, 0). \tag{5.7}$$

Since $k \neq j, \overline{\mathsf{a}_k} \geq \mathsf{a}_j$. By Lemma 3.2(1), (3) and (5.7), $u_i ; v_i \geq J(0\text{'} \cdot \overline{\mathsf{a}_k} \cdot x ; y, 0)$. If $x \neq y$ then $x ; y = 0\text{'}$ in $\mathfrak{A}$, so

$$u_i ; v_i \geq J(0\text{'} \cdot \overline{\mathsf{a}_k} \cdot x ; y, 0) = J(0\text{'} \cdot \overline{\mathsf{a}_k}, 0) \geq J(\mathsf{a}_j, 0). \tag{5.8}$$

If $x = y$ then $x ; y = \overline{x} \geq \overline{\mathsf{a}_k}$ in $\mathfrak{A}$, so $\overline{\mathsf{a}_k} \cdot x ; y = \overline{\mathsf{a}_k}$, and again we have (5.8). By the way, we've shown

$$\text{if} \;\; j \neq f(i) \;\; \text{then} \;\; u_i ; v_i \geq J(\mathsf{a}_j, 0). \tag{5.9}$$

Since (5.8) holds for every i, we obtain much more than (5.5), in fact,

$$0 \neq J(\mathsf{a}_j, 0) \leq \prod_{i=1}^{n-2} u_i ; v_i. \tag{5.10}$$

We therefore assume that f is surjective.

Next we show that either (5.5) holds or f is actually maps two distinct indices onto each of the atoms $\mathsf{a}_1, \mathsf{a}_2$, and a_3, *i.e.*, those atoms of $\mathfrak{E}$ that are the join of two atoms of $\mathfrak{A}$. We prove this only for a_1. Since 1 is in the range of f, we'll suppose, for specificity and simplicity of notation, that $1 = f(1)$, *i.e.*, $u_1 + v_1 \leq J(\mathsf{a}_1, 0)$. We wish to show that $1 = f(i)$ for some $i \neq 1$, so we assume this does not happen, *i.e.*, assume $1 \neq f(i)$ for all $i \in \{2, \cdots, n-2\}$. Now $\mathsf{a}_1 = \mathbf{e}_1 + \mathbf{e}_2$, so there are atoms $x, y \in \{\mathbf{e}_1, \mathbf{e}_2\}$ and indices $k, l \in \omega$ such that $x + y \leq \mathsf{a}_1, u_1 = x^{(k)}$, and $v_1 = y^{(l)}$. Let $m = \max(k, l)$. Notice that $T(k, l, m)$ holds and $\mathfrak{M}_{r+3}$ contains the 2-cycles $[\mathbf{e}_1, \mathbf{e}_1, \mathbf{e}_2]$ and $[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_2]$. It follows by (3.3) that

$$\mathbf{e}_1^{(k)} ; \mathbf{e}_1^{(l)} \geq \mathbf{e}_2^{(m)},$$
$$\mathbf{e}_1^{(k)} ; \mathbf{e}_2^{(l)} = \mathbf{e}_2^{(k)} ; \mathbf{e}_1^{(l)} \geq \mathbf{e}_1^{(m)} + \mathbf{e}_2^{(m)},$$
$$\mathbf{e}_2^{(k)} ; \mathbf{e}_2^{(l)} \geq \mathbf{e}_1^{(m)}.$$

We may therefore let $w = \mathbf{e}_2^{(m)}$ if $x = y = \mathbf{e}_1, w = \mathbf{e}_1^{(m)}$ if $x = y = \mathbf{e}_2$, and either $w = \mathbf{e}_1$ or $w = \mathbf{e}_2$ if $x \neq y$. In every case, $u_1 ; v_1 \geq w$. For products other than $u_1 ; v_1$, note that if $2 \leq i \leq n-2$, then by our assumption we have $1 \neq f(i)$, hence by (5.9), $u_i ; v_i \geq J(\mathsf{a}_1, 0) \geq w$. This shows $w \leq \prod_{i=2}^{n-2} u_i ; v_i$, which, together with $w \leq u_1 ; v_1$, gives us (5.5).

At this point we know that either we are done as we have proved (5.5), or else f maps at least one index from $\{1, \ldots, n-2\}$ onto each of the indices in $\{4, \ldots, r-1\}$, and f maps at least two indices from $\{1, \ldots, n-2\}$ onto each of the indices in $\{1, 2, 3\}$. From $|\{1, \ldots, n-2\}| = n-2, |\{4, \ldots, r-1\}| = r-4$, and $|\{1, 2, 3\}| = 3$, we have $n - 2 \geq r - 4 + 2 \cdot 3 = r + 2$, but our restriction on r is $n \leq r + 3$, a contradiction. Therefore we do in fact know that (5.5) holds, as desired. This shows $B_n(C_{\mathfrak{E}}(\mathfrak{A}))$ is a cylindric basis for $C_{\mathfrak{E}}(\mathfrak{A})$ and completes the proof of part (1). Parts (2) and (3) follow from part (1) by [32, Theorem 10].

For part (4), assume to the contrary that $\mathfrak{Ca}\,(B_n(C_{\mathfrak{E}}(\mathfrak{A}))) \subseteq \mathfrak{Nr}_n\mathfrak{D}$ for some $\mathfrak{D} \in \mathsf{CA}_{r+4}$. We get a contradiction by finding a subalgebra $\mathfrak{F}$ of

$\mathfrak{Ca}\,(B_n(C_{\mathfrak{E}}(\mathfrak{A})))$ which is not in $\mathsf{SNr}_n\mathsf{CA}_{r+4}$. From Theorem B.1 with $p=r+3$ we get

$$\mathfrak{Ca}\,(B_3(\mathfrak{F})) \notin \mathsf{SNr}_3\mathsf{CA}_{p+1} = \mathsf{SNr}_3\mathsf{CA}_{r+4}. \tag{5.11}$$

For this we choose an arbitrary finite parameter $N \in \omega$ and make it big enough. For this fixed N there is a finite subalgebra $\mathfrak{F}$ of $C_{\mathfrak{E}}(\mathfrak{A})$ whose atoms are 1', $\mathbf{e}_i^{(j)}$ and $J(\mathbf{e}_i, N)$ for $1 \leq i \leq r+2$ and $j < N$. This finite subalgebra $\mathfrak{F}$ has a subalgebra isomorphic to $\mathfrak{M}_{r+3}$, whose atoms are 1' and $J(\mathbf{e}_i, 0)$ for $1 \leq i \leq r+2$. By Theorem B.1 below, a finite extension of $\mathfrak{M}_{r+3}$ with enough atoms satisfies (5.11). By choosing N large enough, the extension $\mathfrak{F}$ of $\mathfrak{M}_{r+3}$ has enough atoms. □

Acknowledgements

The most direct inspiration for this work is [15]; see also [12,13,16,19], and especially [14]. The method used in a proof of representability in [15] was embodied in the notion of flexible trio. A modification of a construction from [15] causes the finitely-generated subalgebras to be finite. Suggestions for the form and content of the Introduction came from the referee and the editors, to whom I express my thanks.

Appendix A. A representation result

The next theorem was invoked in the proof of Theorem 4.7.

Theorem A.1. *Assume* $\mathfrak{A}$ *is an atomic symmetric integral relation algebra containing a flexible trio. Then* $\mathfrak{A}$ *has the 1-point extension property and* $\mathfrak{A} \in \mathsf{RRA}$.

Proof. Once it has been shown that $\mathfrak{A}$ has the 1-point extension property, it follows from some additional observations about the behavior of identity elements that the set $B_k(\mathfrak{A})$ of basic k-by-k matrices of atoms of $\mathfrak{A}$ is a relational basis for $\mathfrak{A}$ whenever $k \geq 3$; see Definition 5.1. Then $\mathfrak{A} \in \mathsf{RA}_k$ for all $k \geq 3$ because $\mathfrak{A}$ is atomic and has a k-dimensional relational basis, hence, $\mathfrak{A} \in \bigcap_{k\geq 3} \mathsf{RA}_k = \mathsf{RRA}$ by [33, Theorems 415, 418]. In fact, when $\mathfrak{A}$ is finite (as in the proof of Theorem 4.1) it is easy to prove directly from the 1-point extension property that $\mathfrak{A}$ has a representation on an infinite set.

Assume that a, b, c is a flexible trio of $\mathfrak{A}$. We show that there is a function f such that for any diversity atoms x and y, we have $f(x,y) \in \{a,b,c\}$ and

$$x;f(x,y) = y;f(x,y) \geq 0'. \tag{A.1}$$

For an arbitrary diversity atom x, consider the set

$$\Gamma_x = \{z : x;z \geq 0',\ 0' \geq z \in \mathit{At}\,(\mathfrak{A})\}. \tag{A.2}$$

If $x \in \{a,b,c\}$ then $a,b,c \in \Gamma_x$ by (4.5). If $x \notin \{a,b,c\}$ then by (4.6), $\{a,b,c\} \cap \Gamma_x$ has at least two elements. Consequently, if y is another, possibly different, diversity atom of $\mathfrak{A}$, then, since Γ_x and Γ_y are subsets of the 3-element set $\{a,b,c\}$ and they each contain at least two elements, they must intersect. We choose a value in the intersection as $f(x, y)$. There are several ways to do this. We pick this one—

(1) if $a \in \Gamma_x \cap \Gamma_y$ then $f(x, y) = a$,
(2) if $a \notin \Gamma_x \cap \Gamma_y$ and $b \in \Gamma_x \cap \Gamma_y$ then $f(x, y) = b$,
(3) if $a \notin \Gamma_x \cap \Gamma_y$ and $b \notin \Gamma_x \cap \Gamma_y$ then $f(x, y) = c$.

It is obvious from (A.2) that (A.1) holds in the first two cases. We need to show (A.1) also holds in the third case (3), *i.e.*, that under the assumptions $a \notin \Gamma_x \cap \Gamma_y$ and $b \notin \Gamma_x \cap \Gamma_y$ we have $c \in \Gamma_x \cap \Gamma_y$. But if $c \notin \Gamma_x \cap \Gamma_y$ then we would conclude that $\Gamma_x \cap \Gamma_y$ is empty, since it is a subset of $\{a, b, c\}$ that excludes each of a, b, and c by our assumptions, contrary to the observations made above.

For the 1-point extension property, assume $k \in \omega, \mu \in B_k(\mathfrak{A}), \mu$ satisfies the identity condition, x, y are diversity atoms of $\mathfrak{A}$, and $\mu_{i,j} \leq x;y$ for some fixed $i, j < k$. We will show that μ has a 1-point extension $\mu' \in B_{k+1}(\mathfrak{A})$ defined by the conditions $\mu \subseteq \mu', \mu'_{i,k} = x = \mu'_{k,i}, \mu'_{k,j} = y = \mu'_{j,k}, \mu'_{k,k} = 1\text{'}$, and if $k > l \neq i, j$ then

$$\mu'_{l,k} = \mu'_{k,l} = f(x, y).$$

First note that these conditions are consistent when $i = j$, for in that case we have $0 \neq \mu_{i,i} \leq 1\text{'} \cdot x;y$ by the identity condition, hence $0 \neq \breve{x};1\text{'} \cdot y = x \cdot y$, so $x = y$ since they are atoms. To show that μ' is a basic matrix in $B_{k+1}(\mathfrak{A})$, note that Definition 4.4(1) holds because $\mu \subseteq \mu'$ and $\mu'_{k,k} = 1\text{'}$, and Definition 4.4(2) holds trivially since $\mathfrak{A}$ is symmetric. For Definition 4.4(3), by definitions and (A.1) we have

$$x;\mu'_{k,l} = 0\text{'} = y;\mu'_{k,l}. \tag{A.3}$$

Having chosen $\mu'_{k,l}$ to be either a or b or c according to f, we must check for each $l < k$ whether the first two crucial cycle equations below hold, and finally whether the third equation holds for those points $l, m < k$ where $l \neq m$ and $\{l, m\} \cap \{i, j\} = \emptyset$.

$$\mu_{i,l} \leq \mu'_{i,k};\mu'_{k,l} \qquad i.e., \quad [\mu'_{i,k}, \mu'_{k,l}, \mu_{i,l}] \text{ is a cycle,}$$
$$\mu_{j,l} \leq \mu'_{j,k};\mu'_{k,l} \qquad i.e., \quad [\mu'_{j,k}, \mu'_{k,l}, \mu_{j,l}] \text{ is a cycle,}$$
$$\mu_{l,m} \leq \mu'_{l,k};\mu'_{k,m} \qquad i.e., \quad [\mu'_{l,k}, \mu'_{k,m}, \mu_{l,m}] \text{ is a cycle.}$$

The first two equations hold by (A.3) (their right sides are 0'). For the third equation, first note that $\mu'_{l,k} = \mu'_{k,m}$ because the value depends only on x and y, not on l or m. The right side of the third equation is therefore $a;a$ or $b;b$ or $c;c$, but $a;a = b;b = c;c = 1$, so the third equation holds. □

Appendix B. A non-representation result

The next theorem was invoked in the proof of Theorem 5.2(4).

Theorem B.1. *Assume*

(1) $\mathfrak{C} \subseteq \mathfrak{A} \in \mathsf{NA}$,
(2) $\mathfrak{C}$ *is finite and symmetric,* $1\text{'} \in At(\mathfrak{C})$*, and* $\mathfrak{C}$ *has* $p \geq 3$ *atoms,*

(3) $\mathfrak{E}$ *has no 1-cycles:* $u;u \cdot u = 0$ *if* $0' \geq u \in At(\mathfrak{E})$,

(4) $\mathfrak{A}$ *is finite and symmetric,* $1' \in At(\mathfrak{A})$*, and some diversity atom of* $\mathfrak{E}$ *is the join of at least* p^{p-1} *atoms of* $\mathfrak{A}$*.*

Then $\mathfrak{A}$ *is not a subalgebra of the relation algebraic reduct of any* $(p+1)$*-dimensional cylindric algebra, i.e.,*

$$\mathfrak{A} \notin \mathsf{SRaCA}_{p+1}, \tag{B.1}$$

and the 3-dimensional cylindric algebra of $\mathfrak{A}$ *is not a subalgebra of the neat 3-reduct of any* $(p+1)$*-dimensional cylindric algebra, i.e.*

$$\mathfrak{Ca}\,(B_3(\mathfrak{A})) \notin \mathsf{SNr}_3\mathsf{CA}_{p+1}. \tag{B.2}$$

Proof of (B.1). Let the atoms of $\mathfrak{E}$ be $1' = \mathsf{a}_0, \mathsf{a}_1, \ldots, \mathsf{a}_{p-1}$, where $p \geq 3$ and a_1 is a diversity atom of $\mathfrak{E}$ which is the join of at least p^{p-1} atoms of $\mathfrak{A}$. Let $\xi(x)$ be the atom of $\mathfrak{E}$ containing $x \in At(\mathfrak{A})$. We refer to $\xi(x)$ as the color of x (or cover, as in the definition of splitting).

Assume, for the sake of obtaining a contradiction, that $\mathfrak{A} \subseteq \mathfrak{Ra}\,(\mathfrak{D})$ for some $\mathfrak{D} \in \mathsf{CA}_{p+1}$. All the elements of $\mathfrak{A}$, in particular all the atoms, are 2-dimensional elements of $\mathfrak{D}$, *i.e.*,

$$At(\mathfrak{A}) \subseteq Nr_2\mathfrak{D}. \tag{B.3}$$

If $1 < q \leq p+1$ and $x \in D$, we say x is *q-color-ordered* if $\xi(u) = \xi(v)$ whenever $u, v \in At(\mathfrak{A}), 0 \leq i < j < k < q$, and $x \leq \mathsf{s}^0_i\mathsf{s}^1_j u \cdot \mathsf{s}^0_i\mathsf{s}^1_k v$. The element $x \in D$ is *q-covered* if there are atoms $u_{i,j} \in At(\mathfrak{A})$ for $0 \leq i < j < q$ such that $x \leq \prod_{0 \leq i<j<q} \mathsf{s}^0_i\mathsf{s}^1_j u_{i,j}$, in which case the atoms $u_{i,j}$ are said to be a *q-covering* of x. The atoms in a q-covering of a non-zero $x \in D$ are unique, for if there are further atoms $v_{i,j} \in At(\mathfrak{A}), 0 \leq i < j < q$, such that $x \leq \prod_{0 \leq i<j<q} \mathsf{s}^0_i\mathsf{s}^1_j v_{i,j}$, then, since substitution is a complete Boolean endomorphism by [10, 1.5.3], we have

$$\begin{aligned} 0 \neq x &\leq \prod_{0 \leq i<j<q} \mathsf{s}^0_i\mathsf{s}^1_j u_{i,j} \cdot \prod_{0 \leq i<j<q} \mathsf{s}^0_i\mathsf{s}^1_j v_{i,j} \\ &= \prod_{0 \leq i<j<q} \mathsf{s}^0_i\mathsf{s}^1_j (u_{i,j} \cdot v_{i,j}), \end{aligned}$$

but if $u_{i,j} \neq v_{i,j}$ then, since distinct atoms are disjoint, a zero occurs with a contradiction ensuing. Thus $u_{i,j} = v_{i,j}$ whenever $0 \leq i < j < q$.

We will construct by induction for each dimension from $q = 2$ up to $q = p+1$ a set $S_q \subseteq Nr_q\mathfrak{D}$ such that

(1) S_q has at least p^{p+1-q} elements,

(2) every $x \in S_q$ is q-covered, q-color-ordered, and non-zero, and $x \leq \mathsf{s}^1_j\mathsf{a}_1$ for $0 < j < q$,

(3) $\mathsf{c}_{q-1}x = \mathsf{c}_{q-1}y$ if $x, y \in S_q$,

(4) $\xi(u) = \xi(v)$ if $u, v \in At(\mathfrak{A}), x, y \in S_q, x \leq \mathsf{s}^0_{q-2}\mathsf{s}^1_{q-1}u$, and $y \leq \mathsf{s}^0_{q-2}\mathsf{s}^1_{q-1}v$,

(5) $u \neq v$ if $u, v \in At(\mathfrak{A}), x, y \in S_q, x \leq \mathsf{s}^1_{q-1}u, y \leq \mathsf{s}^1_{q-1}v$, and $x \neq y$,

(6) $u \neq v$ if $0 < j < k < q, u, v \in At(\mathfrak{A}), x \in S_q$, and $x \leq \mathsf{s}^1_j u \cdot \mathsf{s}^1_k v$.

Let $S_2 = \{x : \mathsf{a}_1 \geq x \in At\,(\mathfrak{A})\}$.

Note that $S_2 \subseteq Nr_2\mathfrak{D}$ by (B.3). Obviously S_2 has property (1) since there are at least p^{p-1} atoms below a_1. Let $x \in S_2$. Then x is 2-covered by itself (take $u_{0,1} = x$), x is 2-color-ordered because the hypotheses in the definition of q-color-ordered are never met ($q = 2$ is too small), and x is not zero because it is an atom of $\mathfrak{A}$. For the last part of property (2), note that if $0 < j < q = 2$ then $j = 1$, and $x \leq \mathsf{a}_1$ by the definition of S_2, so $x \leq \mathsf{a}_1 = \mathsf{s}^1_1\mathsf{a}_1 = \mathsf{s}^1_j\mathsf{a}_1$. Therefore S_2 has property (2). Since $\mathfrak{A}$ is integral and $x \in S_2$ is non-zero, we have $x;1 = 1$, so

$$
\begin{aligned}
1 = x;1 &= \mathsf{c}_2(\mathsf{s}^1_2 x \cdot \mathsf{s}^0_2 1) && \text{definition of ; in } \mathfrak{Ra}\,(\mathfrak{D}) \\
&= \mathsf{c}_2\mathsf{s}^1_2 x && [10,\ 1.5.3] \\
&= \mathsf{c}_1\mathsf{s}^2_1 x && [10,\ 1.5.9(\mathrm{i})] \\
&= \mathsf{c}_1 x && [10,\ 1.5.8(\mathrm{i})],\ \ \mathsf{c}_2 x = x
\end{aligned}
$$

It follows that property (3) holds for S_2. For property (4), note that since $q = 2, \mathsf{s}^0_{q-2}\mathsf{s}^1_{q-1}$ is the identity mapping, hence the hypotheses are $u, v \in At\,(\mathfrak{A})\,, x, y \in S_2, x \leq u$, and $y \leq v$, which imply $x = u$ and $y = v$ since u, v, x, y are atoms. We wish to show $\xi(u) = \xi(v)$, *i.e.*, $\xi(x) = \xi(y)$, but this is true by the definition of S_2. Since $q = 2$, the substitution s^1_{q-1} is the identity mapping, hence the hypotheses of property (5) are $u, v \in At\,(\mathfrak{A})\,, x, y \in S_q, x \leq u, y \leq v$, and $x \neq y$. But these hypotheses imply $u = x \neq y = v$, so the conclusion holds trivially. Thus S_2 has property (5). Finally, S_2 has property (6) because the hypotheses cannot hold when $q = 2$.

Suppose we have a set $S_q \subseteq Nr_q\mathfrak{D}$ with properties (1)–(6) such that $q \geq 2$. Choose an arbitrary but fixed $w \in S_q$, and let $S^w_q = S_q \sim\{w\}$. We will obtain a function h that sends every $x \in S^w_q$ to a $(q+1)$-dimensional element $h(x) \in Nr_{q+1}\mathfrak{D}$, and will choose S_{q+1} to be a subset of the range of h.

For every $x \in S^w_q$, we have $\mathsf{c}_q x = x$ and $\mathsf{c}_q w = w$ since $x, w \in S_q \subseteq Nr_q\mathfrak{D}$, so

$$
\begin{aligned}
0 &\neq w && \text{property (2)} \\
&= w \cdot \mathsf{c}_{q-1} w && [10,\ 1.1.1(\mathrm{C}_2)] \\
&= w \cdot \mathsf{c}_{q-1} x && \text{property (3)} \\
&= w \cdot \mathsf{c}_{q-1}\mathsf{s}^q_{q-1} x && [10,\ 1.5.8(\mathrm{i})],\ \mathsf{c}_q x = x \\
&= w \cdot \mathsf{c}_q\mathsf{s}^{q-1}_q x && [10,\ 1.5.9(\mathrm{i})] \\
&= \mathsf{c}_q(w \cdot \mathsf{s}^{q-1}_q x) && [10,\ 1.1.1(\mathrm{C}_3)],\ \mathsf{c}_q w = w \\
&= \mathsf{c}_q\left(w \cdot \mathsf{s}^{q-1}_q x \cdot \mathsf{s}^0_{q-1}\mathsf{s}^1_q\,(1)\right) && [10,\ 1.5.3] \\
&= \mathsf{c}_q\left(w \cdot \mathsf{s}^{q-1}_q x \cdot \mathsf{s}^0_{q-1}\mathsf{s}^1_q\left(\sum_{y \in At(\mathfrak{A})} y\right)\right) && \mathfrak{A} \text{ is finite} \\
&= \sum_{y \in At(\mathfrak{A})} \mathsf{c}_q\left(w \cdot \mathsf{s}^{q-1}_q x \cdot \mathsf{s}^0_{q-1}\mathsf{s}^1_q(y)\right) && [10,\ 1.5.3,\ 1.2.6]
\end{aligned}
$$

The distributive law holds in all Boolean algebras whenever all the joins and meets involved are finite, so

$$0 \neq w = \prod_{x \in S_q^w} \left(\sum_{y \in At(\mathfrak{A})} \mathsf{c}_q \left(w \cdot \mathsf{s}_q^{q-1} x \cdot \mathsf{s}_{q-1}^0 \mathsf{s}_q^1(y) \right) \right)$$
$$= \sum_{f \colon S_q^w \to At(\mathfrak{A})} \left(\prod_{x \in S_q^w} \mathsf{c}_q \left(w \cdot \mathsf{s}_q^{q-1} x \cdot \mathsf{s}_{q-1}^0 \mathsf{s}_q^1(f(x)) \right) \right).$$

Consequently there must be some function $f \colon S_q^w \to At(\mathfrak{A})$ such that

$$0 \neq \prod_{x \in S_q^w} \mathsf{c}_q \left(w \cdot \mathsf{s}_q^{q-1} x \cdot \mathsf{s}_{q-1}^0 \mathsf{s}_q^1(f(x)) \right). \tag{B.4}$$

Let f be such a function. From our chosen f we define additional functions $g, h \colon S_q^w \to D$ and an element $z \in D$ as follows.

$$g(x) = w \cdot \mathsf{s}_q^{q-1} x \cdot \mathsf{s}_{q-1}^0 \mathsf{s}_q^1(f(x)) \qquad \text{for all } x \in S_q^w, \tag{B.5}$$
$$z = \prod_{x \in S_q^w} \mathsf{c}_q (g(x)), \tag{B.6}$$
$$h(x) = g(x) \cdot z \qquad \text{for all } x \in S_q^w. \tag{B.7}$$

Let $R = \{h(x) : x \in S_q^w\}$. We will show that R itself has properties (2), (3), (5), and (6). Consequently every subset of R also has these properties. We will partition R into disjoint subsets that have property (4) and prove that at least one of them must be large enough to also have property (1). We take S_{q+1} to be any such subset of R.

To see that R has property (3), we observe that $\mathsf{c}_q h(x) = \mathsf{c}_q h(y)$ for all $x, y \in S_q^w$, because

$$\begin{aligned} \mathsf{c}_q h(x) &= \mathsf{c}_q (g(x) \cdot z) && \text{(B.7)} \\ &= \mathsf{c}_q (g(x)) \cdot z && \text{[10, 1.1.1(C}_3\text{)], } \mathsf{c}_q z = z \text{ by (B.6)} \\ &= z && \text{(B.6)} \end{aligned}$$

It follows that $h(x) \neq 0$ for every $x \in S_q$, since $z \neq 0$ by (B.4). This is part of property (2). For the last part of property (2), we want to show $h(x) \leq \mathsf{s}_j^1(\mathsf{a}_1)$ whenever $0 < j < q+1$ and $x \in S_q$. We have $h(x) \leq g(x) \leq w \cdot \mathsf{s}_q^{q-1} x$ by definitions (B.7) and (B.5), so there are two cases. First, assume $0 < j < q$. From $w \in S_q$ and property (2) for S_q we get $w \leq \mathsf{s}_j^1 \mathsf{a}_1$, so $h(x) \leq \mathsf{s}_j^1 \mathsf{a}_1$. Suppose $j = q$. In this case we have $x \leq \mathsf{s}_k^1 \mathsf{a}_1$ for $0 < k < q$ by property (2) for S_q since $x \in S_q$. In particular, $x \leq \mathsf{s}_{q-1}^1 \mathsf{a}_1$. If $q > 2$ then $\mathsf{c}_{q-1} \mathsf{a}_1 = \mathsf{a}_1$ since a_1 is 2-dimensional, so $h(x) \leq \mathsf{s}_q^{q-1} \mathsf{s}_{q-1}^1 \mathsf{a}_1 = \mathsf{s}_q^1 \mathsf{a}_1$ by [10, 1.5.11(i)], while if $q = 2$, then $h(x) \leq \mathsf{s}_q^{q-1} \mathsf{s}_{q-1}^1 \mathsf{a}_1 = \mathsf{s}_2^1 \mathsf{s}_1^1 \mathsf{a}_1 = \mathsf{s}_q^1 \mathsf{a}_1$. We get the rest of property (2) by showing $h(x)$ is $(q+1)$-color-ordered and $(q+1)$-covered for every $x \in S_q^w$. From $x \in S_q^w$ and property (2) for S_q we know x is q-covered, so there are atoms $x_{i,j} \in At(\mathfrak{A})$ such that

$$x \leq \prod_{0 \leq i < j < q} \mathsf{s}^0_i \mathsf{s}^1_j (x_{i,j}). \tag{B.8}$$

Of course, we also know $w \in S_q$, so there is a q-covering $w_{i,j} \in At\,(\mathfrak{A})\,, 0 \leq i < j < q$, of w as well, where

$$w \leq \prod_{0 \leq i < j < q} \mathsf{s}^0_i \mathsf{s}^1_j (w_{i,j}). \tag{B.9}$$

Let

$$t_{i,j} = \begin{cases} w_{i,j} & \text{if} \quad 0 \leq i < j < q \\ x_{i,q-1} & \text{if} \quad 0 \leq i < q-1 \text{ and } j = q \\ f(x) & \text{if} \quad i = q-1 \text{ and } j = q \end{cases} \tag{B.10}$$

We shall see that $t_{i,j}$ is a $(q+1)$-covering of $h(x)$. To begin, we prove

$$\mathsf{s}^{q-1}_q x \leq \prod_{0 \leq i < q-1} \mathsf{s}^0_i \mathsf{s}^1_q (x_{i,q-1}). \tag{B.11}$$

Suppose $0 \leq i < q-1$. Then $x \leq \mathsf{s}^0_i \mathsf{s}^1_{q-1}(x_{i,q-1})$ by (B.8). If $q = 2$ then $i = 0$, and $x \leq x_{0,1}$ since x is q-covered, so

$$\mathsf{s}^{q-1}_q x = \mathsf{s}^1_2 x \leq \mathsf{s}^1_2(x_{0,1}) = \mathsf{s}^0_0 \mathsf{s}^1_2 (x_{0,1}) = \mathsf{s}^0_i \mathsf{s}^1_q (x_{i,q-1}),$$

while if $q > 2$ then $\mathsf{c}_{q-1} x_{i,q-1} = x_{i,q-1}$ since $x_{i,q-1}$ is 2-dimensional, so

$$\begin{aligned} \mathsf{s}^{q-1}_q x &\leq \mathsf{s}^{q-1}_q \mathsf{s}^0_i \mathsf{s}^1_{q-1} (x_{i,q-1}) && [10,\ 1.5.3] \\ &= \mathsf{s}^0_i \mathsf{s}^{q-1}_q \mathsf{s}^1_{q-1} (x_{i,q-1}) && [10,\ 1.5.10\text{(iii)}] \\ &= \mathsf{s}^0_i \mathsf{s}^1_q (x_{i,q-1}) && [10,\ 1.5.11\text{(i)}]. \end{aligned}$$

Then we have

$$h(x) \leq \prod_{0 \leq i < j < q+1} \mathsf{s}^0_i \mathsf{s}^1_j (t_{i,j}) \tag{B.12}$$

because

$$\begin{aligned} h(x) &\leq g(x) = w \cdot \mathsf{s}^{q-1}_q x \cdot \mathsf{s}^0_{q-1} \mathsf{s}^1_q (f(x)) && \text{(B.7), (B.5)} \\ &\leq \prod_{0 \leq i < j < q} \mathsf{s}^0_i \mathsf{s}^1_j (w_{i,j}) \cdot \prod_{0 \leq i < q-1} \mathsf{s}^0_i \mathsf{s}^1_q (x_{i,q-1}) \cdot \mathsf{s}^0_{q-1} \mathsf{s}^1_q (f(x)) && \text{(B.9), (B.11)} \\ &= \prod_{0 \leq i < j < q} \mathsf{s}^0_i \mathsf{s}^1_j (t_{i,j}) \cdot \prod_{0 \leq i < q-1} \mathsf{s}^0_i \mathsf{s}^1_q (t_{i,q}) \cdot \mathsf{s}^0_{q-1} \mathsf{s}^1_q (t_{q-1,q}) && \text{(B.10)} \\ &= \prod_{0 \leq i < j < q+1} \mathsf{s}^0_i \mathsf{s}^1_j (t_{i,j}) \end{aligned}$$

so $h(x)$ is $(q+1)$-covered.

To show $h(x)$ is $(q+1)$-color-ordered, we assume $0 \leq i < j < k < q+1$ and must show $\xi(t_{i,j}) = \xi(t_{i,k})$. If $i < j < k < q$ then the first case in (B.10) applies to both $t_{i,j}$ and $t_{i,k}$, hence $t_{i,j} = w_{i,j}$ and $w_{i,k} = t_{i,k}$, but $\xi(w_{i,j}) = \xi(w_{i,k})$ because w is q-color-ordered, so we have $\xi(t_{i,j}) = \xi(t_{i,k})$. We may therefore assume $k = q$.

We need to observe before going on that if $q > 2$, then

$$\begin{aligned} w \le \mathsf{c}_{q-1} w = \mathsf{c}_{q-1} x & & \text{property (3) of } S_q \\ \le \mathsf{c}_{q-1}\left(\prod_{0\le i<j<q-1} \mathsf{s}^0_i \mathsf{s}^1_j (x_{i,j})\right) & & [10, 1.2.6], (B.8) \\ = \prod_{0\le i<j<q-1} \mathsf{s}^0_i \mathsf{s}^1_j (x_{i,j}) & & \mathsf{c}_{q-1} x_{i,j} = x_{i,j}, q-1 \ge 2 \end{aligned}$$

By the uniqueness of coverings this tells us that

$$w_{i,j} = x_{i,j} \text{ if } 0 \le i < j < q-1. \tag{B.13}$$

If, in addition to $k = q$, we have $i < j < q-1$, then $q > 2$ and the first and second cases of (B.10) apply, so we have $t_{i,j} = w_{i,j}$ and $t_{i,k} = t_{i,q} = x_{i,q-1}$. But $w_{i,j} = x_{i,j}$ by (B.13). Also, x is color-ordered, so $\xi(x_{i,j}) = \xi(x_{i,q-1})$, which is equivalent to $\xi(t_{i,j}) = \xi(t_{i,k})$ by the previous equations.

The final case is that $i < j = q-1$ and $k = q$. The possibilities for i divide into two sub-cases, i is smaller than $q-2$, and i is equal to $q-2$. If $0 \le i < q-2$ then $i < q-2 < q-1$, so $\xi(t_{i,j}) = \xi(w_{i,q-1}) = \xi(w_{i,q-2})$ since w is q-color-ordered by property (2) of S_q, and $\xi(x_{i,q-2}) = \xi(x_{i,q-1})$ since x is q-color-ordered, but $w_{i,q-2} = x_{i,q-2}$ by (B.13), so

$$\begin{aligned} \xi(t_{i,j}) &= \xi(w_{i,q-1}) & & j = q-1, \text{(B.10)} \\ &= \xi(w_{i,q-2}) & & w \text{ is } q\text{-color-ordered} \\ &= \xi(x_{i,q-2}) & & \text{(B.13)} \\ &= \xi(x_{i,q-1}) & & x \text{ is } q\text{-color-ordered} \\ &= \xi(t_{i,k}) & & q = k, \text{(B.10)} \end{aligned}$$

We are reduced to assuming $i = q-2$, hence

$$\begin{aligned} \xi(t_{i,j}) &= \xi(w_{q-2,q-1}) & & i = q-2, j = q-1, \text{(B.10)} \\ &= \xi(x_{q-2,q-1}) & & \text{property (4) of } S_q \\ &= \xi(t_{q-2,q}) & & \text{(B.10)} \\ &= \xi(t_{i,k}) & & i = q-2, q = k \end{aligned}$$

We have shown that every $h(x)$ constructed from some $x \in S^w_q$ is non-zero, $(q+1)$-covered, and $(q+1)$-color-ordered. Thus R and all its subsets have property (2).

To prove property (5) for R (and its subsets), we assume $x, y \in S^w_q, h(x) \ne h(y), u, v \in At(\mathfrak{A}), h(x) \le \mathsf{s}^1_q u, h(y) \le \mathsf{s}^1_q v$. We must show $u \ne v$. If we have a q-covering of x as in (B.8), then by (B.12) we get $u = x_{0,q-1}$ from $h(x) \le \mathsf{s}^1_q u$, and, similarly, $v = y_{0,q-1}$ from $h(y) \le \mathsf{s}^1_q v$ for some q-covering $y_{i,j}$ of y. Hence $x \le \mathsf{s}^1_{q-1}(x_{0,q-1})$ and $y \le \mathsf{s}^1_{q-1}(y_{0,q-1})$, so, by property (5) for S_q, we know $x_{0,q-1} \ne y_{0,q-1}$, *i.e.*, $u \ne v$, as desired.

To prove property (6) for R (and its subsets), we assume $0 < j < k < q+1, u, v \in At(\mathfrak{A}), x \in S^w_q$, and $h(x) \le \mathsf{s}^1_j u \cdot \mathsf{s}^1_k v$. If $k < q$, then $u = t_{0,j} = w_{0,j}$ and $v = t_{0,k} = w_{0,k}$ by (B.12) and (B.10), but $w \in S_q$, so by property (6) for S_q, we have $w_{0,j} \neq w_{0,k}$, hence $u \neq v$. Suppose that $k = q$. In this case, by (B.12) and (B.10), we again have $u = t_{0,j} = w_{0,j}$ but this time $v = t_{0,q} = x_{0,q-1}$. Hence $w \leq \mathsf{s}^1_j u$ and $x \leq \mathsf{s}^1_{q-1} v$ by (B.8) and (B.9). If $j = q-1$ we note that $w \neq x$ since $x \in S^w_q$, hence $u \neq v$ by property (5) for S_q, which gives us $t_{0,j} \neq t_{0,q}$, *i.e.*, $u \neq v$. If $j < q-1$ then $v = t_{0,q} = x_{0,q-1} \neq x_{0,j}$ for $0 < j < q-1$ by property (6) for S_q, applied this time to x. But $x_{0,j} = w_{0,j} = t_{0,j}$ by (B.13) and (B.10), so again we have $t_{0,q} \neq t_{0,j}$.

We have proved R has properties (2), (3), (5), and (6), and wish to show that h is one-to-one on S^w_q. Assume $x, y \in S^w_q$ and $x \neq y$. We want to show $h(x) \neq h(y)$. By property (2) for S_q, x and y have q-coverings that include atoms $x_{0,q-1}, y_{0,q-1} \in At\,(\mathfrak{A})$ satisfying $x \leq \mathsf{s}^1_{q-1}(x_{0,q-1})$ and $y \leq \mathsf{s}^1_{q-1}(y_{0,q-1})$. By (B.10) and (B.12) these last two equations imply $h(x) \leq \mathsf{s}^1_q(x_{0,q-1})$ and $h(y) \leq \mathsf{s}^1_q(y_{0,q-1})$. From $x \neq y$ we conclude by property (5) for S_q that $x_{0,q-1} \neq y_{0,q-1}$. Distinct atoms are disjoint, so if $h(x) = h(y)$ then

$$\begin{aligned} h(x) &= h(x) \cdot h(y) \\ &\leq \mathsf{s}^1_q(x_{0,q-1}) \cdot \mathsf{s}^1_q(y_{0,q-1}) \\ &= \mathsf{s}^1_q(x_{0,q-1} \cdot y_{0,q-1}) \\ &= \mathsf{s}^1_q(0) = 0, \end{aligned}$$

contradicting property (2) (which has already been shown).

Now we want to choose a subset S_{q+1} of R with property (4) that contains at least $p^{p+1-(q+1)}$ elements. We partition R and let S_{q+1} be the largest piece. Recall from (B.12) that $h(x) \leq \mathsf{s}^0_{q-1}\mathsf{s}^1_q(f(x))$ for every $x \in S^w_q$, and $f(x)$ has color $\xi(f(x)) \in At\,(\mathfrak{C})$. For every color a_i we get a piece of R, namely

$$R_i = \{h(x) : x \in S^w_q,\ \xi(f(x)) = \mathsf{a}_i\}.$$

Note that R is the disjoint union of the pieces, the number of pieces is at most p, and R has at least p^{p+1-q} elements because h is one-to-one and S^w_q has more than p^{p+1-q} elements. Consequently some piece has at least $p^{p+1-q}/p = p^{p-q}$ elements in it, and we let S_{q+1} be any such piece. Thus S_{q+1} has property (1). Every piece has property (4), so in particular S_{q+1} has this property. Finally, as a subset of R, S_{q+1} has all the other properties. This completes the construction of the sets S_q.

Consider what happens when $q = p+1$. We may choose some $x \in S_{p+1}$ because S_{p+1} has at least one element, by property (1). Then x is $(p+1)$-covered, $(p+1)$-color-ordered, and non-zero by property (2). Let x have $(p+1)$-covering $x_{i,j} \in At\,(\mathfrak{A})$ for $0 \leq i < j < p+1$.

Consider the set $\{\xi(x_{i,p}) : 0 \leq i < p\} \subseteq At\,(\mathfrak{C})$. Note that $\xi(x_{0,p}) = \mathsf{a}_1 \neq$ 1' since $x_{0,p} \leq \mathsf{a}_1$ by property (2). We can also show $\xi(x_{i,p}) \neq$ 1' for $0 < i < p$ because we have, by the covering of $x, x \leq \mathsf{s}^1_i(x_{0,i}) \cdot \mathsf{s}^0_i\mathsf{s}^1_p(x_{i,p}) \cdot \mathsf{s}^1_p(x_{0,p})$ so it follows by [17, Lemma 10] that $[x_{0,i}, x_{i,p}, x_{0,p}]$ is a cycle, *i.e.*, $x_{0,i} ; x_{i,p} \geq x_{0,p}$. If $\xi(x_{i,p}) =$ 1' then $x_{i,p} =$ 1' and we would get $x_{0,i} = x_{0,p}$, contradicting property (6), which says $x_{0,i} \neq x_{0,p}$ for $0 < i < p$. Thus we know $\xi(x_{i,p})$ is a diversity atom of $\mathfrak{C}$ for $0 \leq i < p$.

The number of diversity atoms in $\mathfrak{E}$ is $p-1$, but the size of the index set $\{i : 0 \leq i < p\}$ is p. Therefore some atom is repeated, *i.e.*, there are $0 \leq i < j < p$ such that $\xi(x_{i,p}) = \xi(x_{j,p})$. By the $(p+1)$-color-ordering of $x, \xi(x_{i,j}) = \xi(x_{i,p})$. Let $u = \xi(x_{i,j}) = \xi(x_{i,p}) = \xi(x_{j,p})$. We proved above that $u \neq 1$'. By the covering of x and property (2) we have $0 \neq x \leq \mathsf{s}^0_i\mathsf{s}^1_j(x_{i,j}) \cdot \mathsf{s}^0_j\mathsf{s}^1_p(x_{j,p}) \cdot \mathsf{s}^0_i\mathsf{s}^1_p(x_{i,p})$, hence by the definition of u and [17, Lemma 10] we have $0 \neq u;u \cdot u$. Since $u \neq 1$', this contradicts the assumption that $\mathfrak{E}$ has no such diversity atom as the u we have found. □

Proof of (B.2). Assume to the contrary that $\mathfrak{Ca}\,(B_3(\mathfrak{A})) \in \mathbf{S}\mathsf{Nr}_3\mathsf{CA}_{p+1}$. Then, by (5.1),

$$\mathfrak{A} \cong \mathfrak{Ra}\,(\mathfrak{Ca}\,(B_3(\mathfrak{A}))) \in \mathfrak{Ra}^*\mathbf{S}\mathsf{Nr}_3\mathsf{CA}_{p+1} = \mathsf{RaCA}_{p+1},$$

hence $\mathfrak{A} \in \mathbf{S}\mathsf{RaCA}_{p+1}$, contradicting part (B.1). □

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