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On Boolean ranges of Banaschewski functions

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Dedicated to Jára Cimrman on the occasion of his 50th birthday.

Abstract. We construct a countable lattice $\mathcal{S}$ isomorphic to a bounded sublattice of the subspace lattice of a vector space with two non-isomorphic maximal Boolean sublattices. We represent one of them as the range of a Banaschewski function and we prove that this is not the case of the other. Hereby we solve a problem of F. Wehrung. We study coordinatizability of the lattice $\mathcal{S}$. We prove that although it does not contain a 3-frame, the lattice $\mathcal{S}$ is coordinatizable. We show that the two maximal Boolean sublattices correspond to maximal Abelian regular subalgebras of the coordinatizating ring.

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1. Introduction

In [20] Friedrich Wehrung defined a *Banaschewski function* on a bounded complemented lattice $\mathcal{L}$ as an antitone (i.e., order-reversing) map sending each element of $\mathcal{L}$ to one of its complements, being motivated by the earlier result of Bernhard Banaschewski that such a function exists on the lattice of all subspaces of a vector space [1]. Wehrung extended Banaschewski's result by proving that every countable complemented modular lattice has a Banaschewski function with Boolean range and that all the possible ranges of Banaschewski functions with Boolean range on $\mathcal{L}$ are isomorphic [20, Corollary 4.8].

Still in [20] Wehrung defined a ring-theoretic analogue of the Banaschewski function that, for a von Neuman regular ring $\boldsymbol{R}$, is closely connected to the lattice-theoretic Banaschewski function on the lattice $\mathcal{L}(\boldsymbol{R})$ of all finitely

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generated right ideals of $\boldsymbol{R}$. He made use of these ideas to construct a unit-regular ring $\boldsymbol{S}$ (in fact of bounded index 3) of size $\aleph_1$ with no Banaschewski function [21].

Furthermore in [20] Wehrung defined notions of a Banaschewski measure and a Banaschewski trace on sectionally complemented modular lattices and he proved that a sectionally complemented lattice which is either modular with a large 4-frame or Arguesian with a large 3-frame is coordinatizable (i.e. isomorphic to $\mathfrak{L}(\boldsymbol{R})$ for a possibly non-unital von Neumann regular ring $\boldsymbol{R}$) if and only if it has a Banaschewski trace. Applying those results, he constructed a non-coordinatizable sectionally complemented modular lattice, of size $\aleph_1$, with a large 4-frame [20, Theorem 7.5].

The aim of our paper is to solve the second problem from [20]:

Problem (*Problem 2 from* [20]). Is every maximal Boolean sublattice of an at most countable complemented modular lattice $\mathfrak{L}$ the range of some Banaschewski function on $\mathfrak{L}$? Are any two such Boolean sublattices isomorphic?

We construct a countable complemented modular lattice $\mathfrak{S}$ with two non-isomorphic maximal Boolean sublattices $\mathfrak{G}$ and $\mathfrak{H}$. We represent $\mathfrak{G}$ as the range of a Banaschewski function on $\mathfrak{S}$ and we prove that $\mathfrak{H}$ is not the range of any Banaschewski function. We represent the lattice $\mathfrak{S}$ as a bounded sublattice of the subspace lattice of a vector space over an arbitrary field. The lattice $\mathfrak{S}$ is constructed as a bounded sublattice of $\mathfrak{M}_3[\mathfrak{F}(\kappa)]$. We prove that there is no 3-frame in the lattice $\mathfrak{M}_3[\mathfrak{D}]$ for any distributive lattice $\mathfrak{D}$. As a consequence we get that there is no 3-frame in the lattice $\mathfrak{S}$. On the other hand we show that lattices $\mathfrak{M}_3[\mathfrak{B}]$ are cordinatizated by Boolean powers of the ring of 2×2 matrices over a two-element field $\mathbb{F}_2$ by a Boolean lattice $\mathfrak{B}$. We find a regular $\mathbb{F}_2$-algebra $\boldsymbol{S}$ such that $\mathfrak{S} \simeq \mathfrak{L}(\boldsymbol{S})$ and we show that the maximal Boolean sublattices $\mathfrak{G}$ and $\mathfrak{H}$ correspond to maximal Abelian regular subalgebras of the algebra $\boldsymbol{S}$.

2. Basic concepts

We start with recalling same basic notions as well as the precise definition of a Banaschewski function adopted from [20]. Next we outline Schmidt's $\mathfrak{M}_3[\mathfrak{L}]$ construction, which we then apply to define the bounded modular lattice $\mathfrak{S}$ containing a pair of non-isomorphic maximal Boolean sublattices.

2.1. Some standard notions, notation, and terminology

A lattice $\mathfrak{L}$ is *bounded* if it has both the least element and the greatest element, denoted by $0_{\mathfrak{L}}$ and $1_{\mathfrak{L}}$, respectively. A *bounded sublattice* of a bounded lattice is a sublattice containing the bounds. Given elements a, b, c of a lattice $\mathfrak{L}$ with zero, we will use the notation $c = a \oplus b$ when $a \wedge b = 0_{\mathfrak{L}}$ and $a \vee b = c$. A *complement* of an element a of a bounded lattice $\mathfrak{L}$ is an element a' of $\mathfrak{L}$ such that $a \oplus a' = 1_{\mathfrak{L}}$. A lattice $\mathfrak{L}$ is said to be *complemented* provided that it is bounded and each element of $\mathfrak{L}$ has a (not necessarily unique) complement. A lattice $\mathfrak{L}$ is *relatively complemented* if each of its closed intervals is complemented. Note that a relatively complemented lattice is not necessarily bounded.

We say that a lattice $\mathcal{L}$ is *uniquely complemented* if it is bounded and each element of $\mathcal{L}$ has a unique complement. By a *Boolean lattice* we mean a lattice reduct of a Boolean algebra, that is, a complemented distributive lattice. For the clarity, let us recall the formal definition of a Banaschewski function [20, Definition 3.1]:

Definition 2.1. Let $\mathcal{L}$ be a bounded lattice. A *Banaschewski function* on $\mathcal{L}$ is a map $\beta\colon \mathcal{L} \to \mathcal{L}$ such that both

(1) $x \leq y$ implies $\beta(x) \geq \beta(y)$, for all $x, y \in \mathcal{L}$, and
(2) $\beta(x) \oplus x = 1_{\mathcal{L}}$ for all $x \in \mathcal{L}$,

hold true.

2.2. The $\mathcal{M}_3[\mathcal{L}]$-construction.

Let $\mathcal{L}$ be a lattice. We will call a triple $\langle a, b, c\rangle \in \mathcal{L}^3$ *balanced*, if it satisfies

$$a \wedge b = a \wedge c = b \wedge c$$

and we denote by $\mathcal{M}_3[\mathcal{L}]$ the set of all balanced triples. It is readily seen that $\mathcal{M}_3[\mathcal{L}]$ is a meet-subsemilattice of the cartesian product $\mathcal{L}^3$. However, it is not necessarily a join-subsemilattice, for one easily observes that the componentwise join of balanced triples may not be balanced. The $\mathcal{M}_3[\mathcal{L}]$-construction was introduced by Schmidt [18,19] for a bounded distributive lattice $\mathcal{L}$. He proved [19, Lemma 1] that in this case $\mathcal{M}_3[\mathcal{L}]$ is a bounded modular lattice and that it is a congruence-preserving extension of the distributive lattice $\mathcal{L}$. This result was later extended by Grätzer and Schmidt in various directions [6,7]. In particular, in [6] they proved that every lattice with a non-trivial distributive interval has a proper congruence-preserving extension. This was further improved by Grätzer and Wehrung in [11], where they introduced a modification of the $\mathcal{M}_3[\mathcal{L}]$-construction, called $\mathcal{M}_3\langle\mathcal{L}\rangle$-construction. Using this new idea they proved that every non-trivial lattice admits a proper congruence-preserving extension.

The lattice constructions $\mathcal{M}_3[\mathcal{L}]$ and $\mathcal{M}_3\langle\mathcal{L}\rangle$ appeared in the series of papers by Grätzer and Wehrung [8,9,10,11,12,13,14] dealing with semilattice tensor product and its related structures, namely the box product and the lattice tensor product [10, Definition 2.1 and Definition 3.3]. Indeed, $\mathcal{M}_3 \boxtimes \mathcal{L} \simeq \mathcal{M}_3\langle\mathcal{L}\rangle$ for every lattice $\mathcal{L}$ and $\mathcal{M}_3 \otimes \mathcal{L} \simeq \mathcal{M}_3[\mathcal{L}]$ whenever $\mathcal{L}$ has a zero and $\mathcal{M}_3 \otimes \mathcal{L}$ is a lattice (see [14, Theorem 6.5] and [9, Corollary 6.3]). In particular, the latter is satisfied when the lattice $\mathcal{L}$ is modular with zero. Note also, that if $\mathcal{L}$ is a bounded distributive lattice both the constructions $\mathcal{M}_3[\mathcal{L}]$ and $\mathcal{M}_3\langle\mathcal{L}\rangle$ coincide. In our paper we get by with this simple case.

Let $\mathcal{L}$ be a distributive lattice. Given a triple $\langle a, b, c\rangle \in \mathcal{L}^3$, we define

$$\mu\langle a, b, c\rangle = (a \wedge b) \vee (a \wedge c) \vee (b \wedge c) \tag{2.1}$$

and we set

$$\overline{\langle a, b, c\rangle} = \langle a \vee \mu\langle a, b, c\rangle, b \vee \mu\langle a, b, c\rangle, c \vee \mu\langle a, b, c\rangle\rangle\,. \tag{2.2}$$

Using the distributivity of $\mathfrak{L}$ one easily sees that $\overline{\langle a,b,c\rangle}$ is the least balanced triple $\geq \langle a,b,c\rangle$ in $\mathfrak{L}^3$ and that the map $\overline{\langle - \rangle}\colon \mathfrak{L}^3 \to \mathfrak{L}^3$ determines a closure operator on the lattice $\mathfrak{L}^3$ (see [9, Lemma 2.3] for a refinement of this observation). It is also clear that

$$\begin{aligned} a \vee \mu\langle a,b,c\rangle &= a \vee (b \wedge c), \\ b \vee \mu\langle a,b,c\rangle &= b \vee (a \wedge c), \\ c \vee \mu\langle a,b,c\rangle &= c \vee (a \wedge b). \end{aligned}$$

A triple $\langle a,b,c\rangle \in \mathfrak{L}^3$ is closed with respect to the closure operator if and only if it is balanced. Therefore the set of all balanced triples, denoted by $\mathfrak{M}_3[\mathfrak{L}]$, forms a lattice [9, Lemma 2.1], where

$$\langle a,b,c\rangle \vee \langle a',b',c'\rangle = \overline{\langle a\vee a', b\vee b', c\vee c'\rangle} \tag{2.3}$$

and

$$\langle a,b,c\rangle \wedge \langle a',b',c'\rangle = \langle a\wedge a', b\wedge b', c\wedge c'\rangle\,. \tag{2.4}$$

By [9, Lemma 2.9] the lattice $\mathfrak{M}_3[\mathfrak{L}]$ is modular if and only if the lattice $\mathfrak{L}$ is distributive. The "if" part of the equivalence is included in the above mentioned [19, Lemma 1].

2.3. Coordinatizability

A ring $\boldsymbol{R}$ is *(von Neumann) regular* provided that for each element $x \in \boldsymbol{R}$, there is $y \in \boldsymbol{R}$ with $x = xyx$. This is equivalent to each (left) right finitely generated ideal of $\boldsymbol{R}$ being generated by an idempotent. An ideal $\boldsymbol{I}$ of a ring $\boldsymbol{R}$ is *regular* if for each element $x \in \boldsymbol{I}$, there is $y \in \boldsymbol{I}$ with $x = xyx$. By [4, Lemma 1.3], an ideal of a regular ring is regular.

Finitely generated right ideals of a regular ring $\boldsymbol{R}$ form a sectionally complemented modular lattice [4, Theorem 2.3]. We will denote this lattice by $\mathfrak{L}(\boldsymbol{R})$. Note that for a regular ring the map $e\boldsymbol{R} \mapsto \boldsymbol{R}(1-e)$ determines an anti-isomorphisms from the lattice $\mathfrak{L}(\boldsymbol{R})$, of all finitely generated right ideals of the ring $\boldsymbol{R}$, to the lattice of all finitely generated left ideals of the ring $\boldsymbol{R}$ (cf. [4, Theorem 2.5]).

An *Abelian regular ring* is a ring $\boldsymbol{R}$ whose all idempotents are central. For various characterizations of Abelian regular rings see [4, Theorem 3.2]. A *maximal Abelian regular subalgebra* of a regular algebra $\boldsymbol{R}$ is a Abelian regular subalgebra of $\boldsymbol{R}$ that is not properly contained in any Abelian regular subalgebra of the ring $\boldsymbol{R}$.

A lattice, necessarily sectionally complemented modular, is *coordinatizable* if it is isomorphic to the lattice $\mathfrak{L}(\boldsymbol{R})$ for a regular ring $\boldsymbol{R}$. For a lucid introduction into the problem of coordinatizability of sectionally complemented modular lattice we refer to [5, Appendix D] and [21]. Here we will limit ourselves to Jónsson's coordinatization theorem [15], to our knowledge the most complete description of coordinatizable lattices.

We say a set X of non-zero elements of a lattice $\mathfrak{L}$ with zero is *independent* provided that for every finite $F, G \subseteq X$, the equality

$$\bigvee F \wedge \bigvee G = \bigvee (F \cap G)$$

holds true. If the lattice $\mathfrak{L}$ is modular then an n-element set $\{a_1, \ldots, a_n\}$ of distinct non-zero elements of $\mathfrak{L}$ is independent if and only if $a_{j+1} \wedge \bigvee_{i=1}^{j} a_i = 0$ for all $j = 1, \ldots, n-1$ (see [5, Theorem IV.1.11]). If the lattice $\mathfrak{L}$ is distributive, a subset $X \subseteq \mathfrak{L} \setminus \{0\}$ is independent if and only if $a \wedge b = 0$ for all distinct $a, b \in X$.

Elements a, b of a bounded lattice $\mathfrak{L}$ are *perspective* provided that there is $c \in \mathfrak{L}$ such that

$$1 = a \oplus c = b \oplus c. \tag{2.5}$$

The notation $a \sim_c b$ means that equalities (2.5) hold true. The notation $a \sim b$ means that $a \sim_c b$ for some $c \in \mathfrak{L}$, i.e. that a and b are perspective.

An element a of a lattice $\mathfrak{L}$ is *neutral* provided that the sublattice of $\mathfrak{L}$ generated by a triple $\{a, b, c\}$ is distributive for all $b, c \in \mathfrak{L}$ [5, Section III.2]. An ideal I of a lattice $\mathfrak{L}$ is *neutral* if it is a neutral element in the ideal lattice of $\mathfrak{L}$. An *n-frame* in a lattice $\mathfrak{L}$ is a pair

$$\langle \langle a_i \mid i = 0, \ldots, n-1 \rangle, \langle c_i \mid i = 1, \ldots, n-1 \rangle \rangle$$

of families of elements of $\mathfrak{L}$ such that the set $\{a_0, \ldots, a_{n-1}\}$ is independent and $a_0 \sim_{c_i} a_i$ for all $i = 1, \ldots, n-1$. An n-frame is *large* if the neutral ideal generated by a_0 is the entire $\mathfrak{L}$. In particular, an n-frame such that $\bigvee_{i=0}^{n-1} a_i = 1$ is large.

Theorem 2.2. (Jónsson's coordinatization theorem [15]) *A modular complemented lattice $\mathfrak{L}$ that has a large n-frame for some $n \geq 4$ or that is Arguesian and has a large n-frame with $n \geq 3$ is coordinatizable.*

2.4. Stone duality and Boolean powers

In this section we follow [2, Chapter IV,§§4-5]. For topological notions we refer to [3]. A *Boolean space* is a compact Hausdorff topological space with a basis consisting of clopen (i.e. closed and open) subsets. Let $\mathfrak{B}$ be a Boolean lattice. We denote by $\mathfrak{B}^*$ the collection of all ultrafilters on $\mathfrak{B}$. For each $a \in \mathfrak{B}$ we set

$$N_a := \{\mathfrak{u} \in \mathfrak{B}^* \mid a \in \mathfrak{u}\}. \tag{2.6}$$

The collection of all N_a, $a \in \mathfrak{B}$, is a basis of a topology on $\mathfrak{B}^*$, and $\mathfrak{B}^*$ equipped with this topology is a Boolean space called the *Stone space* of $\mathfrak{B}$.

All clopen subsets of a topological space $\mathfrak{T}$ form a sulattice, denoted by $\mathfrak{T}^*$, of the Boolean lattice of all subsets of $\mathfrak{T}$. Every Boolean lattice $\mathfrak{B}$ is isomorphic to $\mathfrak{B}^{**}$ via the map $a \mapsto N_a$ and every Boolean space $\mathfrak{T}$ is homeomorphic to $\mathfrak{T}^{**}$ via $x \mapsto \{N \in \mathfrak{T}^* \mid x \in N\}$.

Let $\boldsymbol{A}$ be an algebra and $\mathfrak{B}$ a Boolean lattice. We equip the set A with the discrete topology and we denote by $A[\mathfrak{B}]^*$ the set of all continuous functions from the Boolean space $\mathfrak{B}^*$ to A. By [2, Lemma IV.5.2], $A[\mathfrak{B}]^*$ is a subuniverse of the Cartesian power $\boldsymbol{A}^{\mathfrak{B}^*}$. We denote by $\boldsymbol{A}[\mathfrak{B}]^*$ the subalgebra of $\boldsymbol{A}^{\mathfrak{B}^*}$ with the universe $A[\mathfrak{B}]^*$ and we will call the subalgebra the *Boolean power* of $\boldsymbol{A}$ by $\mathfrak{B}$.

3. The lattice

Fix an infinite cardinal κ. As it is customary, we identify κ with the set of all ordinals of cardinality less than κ. Let us denote by $\mathfrak{P}(\kappa)$ the Boolean lattice of all subsets of κ and set

$$\mathfrak{F}(\kappa) := \{X \subseteq \kappa \mid X \text{ is finite or } \kappa\backslash X \text{ is finite}\}.$$

It is well-known that $\mathfrak{F}(\kappa)$ is a bounded Boolean sublattice of $\mathfrak{P}(\kappa)$.

Given sets X, Y, the notation $X \leq_{\text{fin}} Y$ means that $X\backslash Y$ is finite. Clearly $\leq_{\text{fin}}$ is a quasiorder on the class of all sets. We define

$$\mathfrak{E} = \{\langle A, B, C\rangle \in \mathfrak{F}(\kappa)^3 \mid C \leq_{\text{fin}} A \cup B\}.$$

Since for all A, A', B, B', C, C' we have that

$$(C \cup C')\backslash((A \cup A') \cup (B \cup B')) \subseteq (C\backslash(A \cup B)) \cup (C'\backslash(A' \cup B')), \tag{3.1}$$

the set $\mathfrak{E}$ is closed under finite joins. Both $0_{\mathfrak{F}(\kappa)^3} = \langle\emptyset, \emptyset, \emptyset\rangle$ and $1_{\mathfrak{F}(\kappa)^3} = \langle\kappa, \kappa, \kappa\rangle$ clearly belong to $\mathfrak{E}$, thus we conclude that $\mathfrak{E}$ forms a bounded join-subsemilattice of $\mathfrak{F}(\kappa)^3$.

Let $\mathfrak{S} := \mathfrak{E} \cap \mathfrak{M}_3[\mathfrak{F}(\kappa)]$ denote the set of all balanced triples from $\mathfrak{E}$. Since $A \cap C = B \cap C$ for every balanced triple $\langle A, B, C\rangle$, we have that

$$\begin{aligned} \mathfrak{S} &= \{\langle A, B, C\rangle \in \mathfrak{M}_3[\mathfrak{F}(\kappa)] \mid C \leq_{\text{fin}} A\} \\ &= \{\langle A, B, C\rangle \in \mathfrak{M}_3[\mathfrak{F}(\kappa)] \mid C \leq_{\text{fin}} B\}. \end{aligned} \tag{3.2}$$

Note that since for a balanced triple $\langle A, B, C\rangle$ the equality $A \cap C = \mu\langle A, B, C\rangle$ holds true, we get from (3.2) that

$$\mathfrak{S} = \{\langle A, B, C\rangle \in \mathfrak{M}_3[\mathfrak{F}(\kappa)] \mid C \leq_{\text{fin}} \mu\langle A, B, C\rangle\}. \tag{3.3}$$

Lemma 3.1. *The set $\mathfrak{S}$ forms a bounded sublattice of the lattice $\mathfrak{M}_3[\mathfrak{F}(\kappa)]$.*

Proof. Observe that

$$C\backslash(A \cup B) = (C \cup \mu\langle A, B, C\rangle)\backslash(A \cup B \cup \mu\langle A, B, C\rangle),$$

for all $\langle A, B, C\rangle \in \mathfrak{F}(\kappa)^3$. Therefore the join-semilattice $\mathfrak{E}$ is closed under the operation μ. It follows that $\mathfrak{S}$ forms a bounded join-subsemilattice of $\mathfrak{M}_3[\mathfrak{F}(\kappa)]$. It remains to prove that $\mathfrak{S}$ is closed under finite meets. However, this is a consequence of the inequality

$$(C \cap C')\backslash(A \cap A') \subseteq (C\backslash A) \cup (C'\backslash A'),$$

that holds for all sets A, A', C, C'. □

As discussed in Section 2, since the lattice $\mathfrak{F}(\kappa)$ is distributive, the lattice $\mathfrak{M}_3[\mathfrak{F}(\kappa)]$ is modular. Observe that the mapping $A \mapsto \langle A, A, A\rangle$ embeds $\mathfrak{F}(\kappa)$ into $\mathfrak{S}$, from which we deduce that

$$|\mathfrak{F}(\kappa)| \leq |\mathfrak{S}| \leq |\mathfrak{F}(\kappa)^3|.$$

Since the size of both $\mathfrak{F}(\kappa)$ and $\mathfrak{F}(\kappa)^3$ is κ, we get that $|\mathfrak{S}| = \kappa$. Let us sum up these observations in the following corollary to Lemma 3.1.

Corollary 3.2. *For $\kappa = \omega_0$, the lattice $\mathfrak{S}$ is countable infinite.*

Remark 3.3. Note that unlike $\mathcal{S}$, the lattice $\mathcal{E}$ is not a meet-subsemilattice of $\mathcal{F}(\kappa)^3$. Indeed, both $\langle\kappa, \emptyset, \kappa\rangle, \langle\emptyset, \kappa, \kappa\rangle \in \mathcal{E}$ while $\langle\kappa, \emptyset, \kappa\rangle \wedge \langle\emptyset, \kappa, \kappa\rangle = \langle\emptyset, \emptyset, \kappa\rangle \notin \mathcal{E}$.

4. A Banaschewski function on $\mathcal{S}$

In this section we define a Banaschewski function $\beta\colon \mathcal{S} \to \mathcal{S}$ and describe, element-wise, its range $\mathcal{G}$.

Lemma 4.1. *The map $\beta\colon \mathcal{S} \to \mathcal{S}$ defined by*

$$\beta\langle A, B, C\rangle := \langle\kappa\backslash A, \kappa\backslash(B \cup C), \kappa\backslash(A \cup B \cup C)\rangle, \tag{4.1}$$

for all $\langle A, B, C\rangle \in \mathcal{S}$, is a Banaschewski function on $\mathcal{S}$. Consequently, $\mathcal{S}$ is a complemented modular lattice.

Proof. First we prove that $\mathcal{S}$ contains the range of the map β. Observe that if we put $A' := \kappa\backslash A$ and $B' := \kappa\backslash(B \cup C)$, then $\beta\langle A, B, C\rangle = \langle A', B', A' \cap B'\rangle$. Since $\mathcal{F}(\kappa)$ is a Boolean lattice, the sets A', B' and $A' \cap B'$ all belong to $\mathcal{F}(\kappa)$. Furthermore, we have that

$$A' \cap B' = \mu\langle A', B', A' \cap B'\rangle = \mu\beta\langle A, B, C\rangle.$$

In particular, $(A' \cap B')\backslash\mu\beta\langle A, B, C\rangle = \emptyset$, whence $\beta\langle A, B, C\rangle \in \mathcal{S}$.

It is clear from (4.1) that the map β is antitone. Finally, we check that

$$1_{\mathcal{S}} = \langle\kappa, \kappa, \kappa\rangle = \langle A, B, C\rangle \oplus \beta\langle A, B, C\rangle, \quad \text{for all } \langle A, B, C\rangle \in \mathcal{S}.$$

It follows immediately from the definition of β that

$$\langle A, B, C\rangle \wedge \beta\langle A, B, C\rangle = \langle\emptyset, \emptyset, \emptyset\rangle = 0_{\mathcal{S}}.$$

To prove that $\langle A, B, C\rangle \vee \beta\langle A, B, C\rangle = 1_{\mathcal{S}}$, let us verify that

$$\kappa = \mu\langle A \cup (\kappa\backslash A), B \cup (\kappa\backslash(B \cup C)), C \cup (\kappa\backslash(A \cup B \cup C))\rangle. \tag{4.2}$$

Note that each element of κ that is *not* contained in C belongs to $B \cup (\kappa\backslash(B \cup C))$. Together with $A \cup (\kappa\backslash A) = \kappa$, we get that (4.2) holds, which concludes the proof. □

Lemma 4.2. *Let $\mathcal{G}$ denote the range of the Banaschewski function $\beta\colon \mathcal{S} \to \mathcal{S}$. Then*

$$\mathcal{G} = \{\langle A, B, A \cap B\rangle \mid A, B \in \mathcal{F}(\kappa)\}$$

and the mapping

$$\langle A, B, A \cap B\rangle \mapsto \langle A, B\rangle \tag{4.3}$$

determines an isomorphism from $\mathcal{G}$ onto the Boolean lattice $\mathcal{F}(\kappa) \times \mathcal{F}(\kappa)$.

Proof. While proving Lemma 4.1, we have observed that

$$\begin{aligned} \mathcal{G} &\subseteq \{\langle A, B, C\rangle \in \mathcal{S} \mid C = A \cap B\} \\ &= \{\langle A', B', A' \cap B'\rangle \mid A', B' \in \mathcal{F}(\kappa)\}. \end{aligned} \tag{4.4}$$

It is straightforward that $\beta(\beta\langle A', B', A' \cap B'\rangle) = \langle A', B', A' \cap B'\rangle$, so the lattice $\mathcal{G}$ is equal to the right-hand side of (4.4). Finally, it is readily seen that the correspondence (4.3) determines an isomorphism $\mathcal{G} \to \mathcal{F}(\kappa) \times \mathcal{F}(\kappa)$. □

It was noted in [20] that if the range of a Banaschewski function on a lattice $\mathfrak{L}$ is Boolean, then it is a *maximal* Boolean sublattice of $\mathfrak{L}$. Thus we derive from Theorem 4.2 that $\mathfrak{G}$ is a maximal Boolean sublattice of $\mathfrak{S}$.

5. The counter-example

In the present section, we construct another maximal Boolean sublattice $\mathfrak{H}$ of the lattice $\mathfrak{S}$. We show that the lattices $\mathfrak{H}$ and $\mathfrak{G}$ are not isomorphic and we prove directly that the lattice $\mathfrak{H}$ is not the range of any Banaschewski function on $\mathfrak{S}$.

Lemma 5.1. *The assignment* $\langle A, C\rangle \mapsto g\langle A, C\rangle := \langle A, A\cap C, C\rangle$ *defines a bounded lattice embedding* $g\colon \mathfrak{F}(\kappa)\times\mathfrak{F}(\kappa) \to \mathfrak{M}_3[\mathfrak{F}(\kappa)]$*. In particular, the range of* g *is a bounded Boolean sublattice of* $\mathfrak{M}_3[\mathfrak{F}(\kappa)]$ *isomorphic to* $\mathfrak{F}(\kappa)\times\mathfrak{F}(\kappa)$*.*

Proof. It is clear from the definition of the map g that it is injective and that its range is included in $\mathfrak{M}_3[\mathfrak{F}(\kappa)]$. Further, for any $A, A', C, C' \subseteq \kappa$, the equality

$$g\langle A, C\rangle \wedge g\langle A', C'\rangle = g\langle A\cap A', C\cap C'\rangle$$

holds by (2.4), while

$$g\langle A, C\rangle \vee g\langle A', C'\rangle = g\langle A\cup A', C\cup C'\rangle \tag{5.1}$$

can be easily deduced from (2.2) to (2.3). Finally, observe that $g\langle \kappa, \kappa\rangle = \langle \kappa, \kappa, \kappa\rangle$ and $g\langle \emptyset, \emptyset\rangle = \langle \emptyset, \emptyset, \emptyset\rangle$, which concludes the proof. □

For any $A, C \in \mathfrak{F}(\kappa)$, we say that $\langle A, C\rangle$ is *finite* if both A and C are finite, and we say that $\langle A, C\rangle$ is *co-finite* if both $\kappa\backslash A$ and $\kappa\backslash C$ are finite. Let us write $A \approx C$ if $\langle A, C\rangle$ is either finite or co-finite. Note that there are pairs $A, C \in \mathfrak{F}(\kappa)$ such that $\langle A, C\rangle$ is neither finite nor co-finite; namely, $A \approx C$ if and only if the symmetric difference $(A\backslash C)\cup(C\backslash A)$ is finite.

Lemma 5.2. *The set*

$$\mathcal{A} = \{\langle A, C\rangle \in \mathfrak{F}(\kappa)\times\mathfrak{F}(\kappa) \mid A \approx C\}$$

forms a bounded Boolean sublattice of $\mathfrak{F}(\kappa)\times\mathfrak{F}(\kappa)$*.*

Proof. Let $\langle A, C\rangle, \langle A', C'\rangle \in \mathcal{A}$. If at least one of them is finite, then the pair $\langle A\cap A', C\cap C'\rangle$ is clearly finite as well. If both $\langle A, C\rangle$ and $\langle A', C'\rangle$ are co-finite, then so is $\langle A\cap A', C\cap C'\rangle$. In either case, $\langle A\cap A', C\cap C'\rangle \in \mathcal{A}$.

If at least one of $\langle A, C\rangle$, $\langle A', C'\rangle$ is co-finite, then $\langle A\cup A', C\cup C'\rangle$ is co-finite, while if both $\langle A, C\rangle$ and $\langle A', C'\rangle$ are finite, then so is $\langle A\cup A', C\cup C'\rangle$. In particular, we have that $\langle A\cup A', C\cup C'\rangle \in \mathcal{A}$ when $\langle A, C\rangle, \langle A', C'\rangle \in \mathcal{A}$.

We have shown that $\mathcal{A}$ is a sublattice of $\mathfrak{F}(\kappa)\times\mathfrak{F}(\kappa)$. To complete the proof, observe that $\langle \emptyset, \emptyset\rangle$ is finite and $\langle \kappa, \kappa\rangle$ is co-finite and that the unique complement in $\mathfrak{F}(\kappa)\times\mathfrak{F}(\kappa)$ of each $\langle A, C\rangle \in \mathcal{A}$, namely $\langle \kappa\backslash A, \kappa\backslash C\rangle$ belongs to $\mathcal{A}$. □

Lemma 5.3. *The* g*-image* $\mathfrak{H} = g(\mathcal{A})$ *of* $\mathcal{A}$ *is a bounded Boolean sublattice of* $\mathfrak{S}$*.*

Proof. Due to Lemmas 5.1 and 5.2, $\mathfrak{H}$ is a bounded Boolean sublattice of $\mathfrak{M}_3[\mathfrak{F}(\kappa)]$. Thus in view of Lemma 3.1, it suffices to verify that $\mathfrak{H} \subseteq \mathfrak{S}$, that is, that $C\backslash(A \cap C)$ is finite for every $\langle A, C\rangle \in \boldsymbol{\mathcal{A}}$. This is clear when $\langle A, C\rangle$ is finite. If $\langle A, C\rangle$ is co-finite, then $C\backslash(A \cap C) = C\backslash A \subseteq \kappa\backslash A$ is finite and we are done. $\square$

Observe that if $\langle A, B, C\rangle$ is a balanced triple then $B \subseteq A$ if and only if $B = A \cap B = A \cap C$. It follows that

$$\mathfrak{H} = \{\langle A, B, C\rangle \in \mathfrak{S} \mid A \approx C \text{ and } B \subseteq A\}. \tag{5.2}$$

Lemma 5.4. *Let $\langle A, B, C\rangle \in \mathfrak{S}\backslash\mathfrak{H}$ and let $\langle A', B', C'\rangle$ be a complement of $\langle A, B, C\rangle$ in $\mathfrak{S}$. If $B \subseteq A$, then $B' \nsubseteq A'$.*

Proof. Since $\langle A, B, C\rangle \notin \mathfrak{H}$ and $B \subseteq A$, it follows from (5.2) that $A \not\approx C$. Hence exactly one of the two sets A, C is finite. From $B \subseteq A$ and $C\backslash B$ being finite we conclude that C and $\kappa\backslash A$ are both finite. Furthermore from $B \subseteq A$ and $A \cap B = B \cap C$, we infer that $B = B \cap C$. It follows that the set B is finite as well.

Suppose now that $B' \subseteq A'$. Since $\langle A, B, C\rangle \wedge \langle A', B', C'\rangle = 0_{\mathfrak{S}}$, we have that $A \cap A' = \emptyset$, whence the set $A' \subseteq \kappa\backslash A$ is finite. A fortiori, the set B' is also finite due to the assumption that $B' \subseteq A'$. As $C'\backslash B' = C' \setminus (B' \cap A') = C'\backslash\mu\langle A', B', C'\rangle$ is also finite, we conclude that so is C'. But then

$$\mu\langle A \cup A', B \cup B', C \cup C'\rangle \subseteq B \cup B' \cup C \cup C'$$

is a finite set, which contradicts the assumption that $\langle A, B, C\rangle \vee \langle A', B', C'\rangle = \langle \kappa, \kappa, \kappa\rangle = 1_{\mathfrak{S}}$. $\square$

Corollary 5.5. *Every complemented bounded sublattice $\mathfrak{C}$ of $\mathfrak{S}$ such that $\mathfrak{H} \subsetneq \mathfrak{C}$ contains an element $\langle A, B, C\rangle$ with $B \nsubseteq A$.*

Proof. Let $\langle A, B, C\rangle \in \mathfrak{C}\backslash\mathfrak{H}$ and let $\langle A', B', C'\rangle$ be one of its complements in $\mathfrak{C}$. Applying Lemma 5.4, we get that either $B \nsubseteq A$ or $B' \nsubseteq A'$. $\square$

Proposition 5.6. *The lattice $\mathfrak{H}$ is a maximal Boolean sublattice of $\mathfrak{S}$.*

Proof. Let $\mathfrak{C}$ be a complemented bounded sublattice of $\mathfrak{S}$ satisfying $\mathfrak{H} \subsetneq \mathfrak{C}$. There is $\langle A, B, C\rangle \in \mathfrak{C}$ with $B \nsubseteq A$ by Corollary 5.5. We can pick a finite nonempty $F \subseteq B\backslash A$. Since the triple $\langle A, B, C\rangle$ is balanced,

$$\emptyset = F \cap A = F \cap B \cap A = F \cap B \cap C = F \cap C. \tag{5.3}$$

Now observe that both $g\,\langle F, \emptyset\rangle$ and $g\,\langle \emptyset, F\rangle$ are in $\mathfrak{H}$. Applying (5.1) and (5.3), we get that

$$\langle A, B, C\rangle \wedge \big(g\,\langle F, \emptyset\rangle \vee g\,\langle \emptyset, F\rangle\big) = \langle A, B, C\rangle \wedge g\,\langle F, F\rangle = \langle \emptyset, F, \emptyset\rangle\,, \tag{5.4}$$

while

$$\big(\,\langle A, B, C\rangle \wedge g\,\langle F, \emptyset\rangle\,\big) \vee \big(\,\langle A, B, C\rangle \wedge g\,\langle \emptyset, F\rangle\,\big) = \langle \emptyset, \emptyset, \emptyset\rangle\,. \tag{5.5}$$

It follows from (5.4) and (5.5) that the lattice $\mathfrak{C}$ is not distributive, *a fortiori* it is not Boolean. $\square$

Proposition 5.7. *The sublattice $\mathfrak{H}$ of $\mathfrak{S}$ is not the range of any Banaschewski function on $\mathfrak{S}$.*

Proof. The range of a Banaschewski function on $\mathcal{S}$ must contain a complement of each element of $\mathcal{S}$. We show that no complement of $\langle\kappa,\emptyset,\emptyset\rangle$ in $\mathcal{S}$ belongs to $\mathcal{H}$.

Suppose the contrary, that is, that there is $\langle A,B,C\rangle = g\langle A,C\rangle \in \mathcal{H}$ satisfying $\langle\kappa,\emptyset,\emptyset\rangle \oplus \langle A,B,C\rangle = 1_{\mathcal{S}}$. Then $A = A\cap\kappa = \emptyset$, and by (5.2) also $B=\emptyset$. Then from $B=\emptyset$ and $\langle\kappa,\emptyset,\emptyset\rangle \vee \langle A,B,C\rangle = 1_{\mathcal{S}}$, one infers that $C=\kappa$. It follows that $\langle A,B,C\rangle \notin \mathcal{S}$; indeed, $C\backslash\mu\langle A,B,C\rangle = C\backslash\emptyset = \kappa$ is not finite. Thus $\langle A,B,C\rangle \notin \mathcal{H}$, which is a contradiction. □

Remark 5.8. Note that for the particular case of $\kappa = \aleph_0$, the assertion of Proposition 5.7 follows from Proposition 5.9 together with [20, Corollary 4.8], which states that the ranges of two Boolean Banaschewski functions on a *countable* complemented modular lattice are isomorphic.

Proposition 5.9. *The lattices $\mathcal{H}$ and $\mathcal{G}$ are not isomorphic.*

Proof. In $\mathcal{H}$, every *finite* element $g\langle A,C\rangle$ is a join of a finite set of atoms, namely

$$g\langle A,C\rangle = \left(\bigvee_{\alpha\in A} g\langle\{\alpha\},\emptyset\rangle\right) \vee \left(\bigvee_{\gamma\in C} g\langle\emptyset,\{\gamma\}\rangle\right),$$

and, dually, every co-finite element is a meet of a finite set of co-atoms. On the other hand, there are elements in $\mathcal{F}(\kappa)\times\mathcal{F}(\kappa)$ that are neither finite joins of atoms nor finite meets of co-atoms. Recall that in Lemma 4.2, we have observed that the lattice $\mathcal{G}$ is isomorphic to $\mathcal{F}(\kappa)\times\mathcal{F}(\kappa)$. Therefore the lattices $\mathcal{H}$ and $\mathcal{G}$ are not isomorphic. □

6. Representing $\mathcal{S}$ in a subspace lattice

Although the construction in the three previous sections was performed for an *infinite* cardinal κ, the results of the present section on embedding the lattice $\mathcal{M}_3[\mathcal{P}(\kappa)]$ into $\mathrm{Sub}(\boldsymbol{\mathcal{V}})$ (namely Theorem 6.4) work just as well for κ finite. In particular, Proposition 6.5 (an enhancement of [9, Lemma 2.9]) holds for lattices of any cardinality.

Let $\mathbb{F}$ be an arbitrary field and let $\boldsymbol{V}$ denote the vector space over the field $\mathbb{F}$ presented by generators $x_\alpha, y_\alpha, z_\alpha$, $\alpha\in\kappa$, and relations $x_\alpha+y_\alpha+z_\alpha=0$. For a subset X of the vector space $\boldsymbol{V}$ we denote by $\mathrm{Span}(X)$ the subspace of $\boldsymbol{V}$ generated by X. Given subspaces of $\boldsymbol{V}$, say $\boldsymbol{X}$ and $\boldsymbol{Y}$, we will use the notation $\boldsymbol{X}+\boldsymbol{Y} = \mathrm{Span}(\boldsymbol{X}\cup\boldsymbol{Y})$. Let $\mathrm{Sub}(\boldsymbol{V})$ denote the lattice of all subspaces of the vector space $\boldsymbol{V}$.

For all $A,B,C\subseteq\kappa$ we put $\boldsymbol{X}_A = \mathrm{Span}(\{x_\alpha \mid \alpha\in A\})$, $\boldsymbol{Y}_B = \mathrm{Span}(\{y_\beta \mid \beta\in B\})$, and $\boldsymbol{Z}_C = \mathrm{Span}(\{z_\gamma \mid \gamma\in C\})$.

We define the map $F\colon \mathcal{P}(\kappa)^3 \to \mathrm{Sub}(\boldsymbol{V})$ by the correspondence

$$\langle A,B,C\rangle \mapsto \boldsymbol{X}_A+\boldsymbol{Y}_B+\boldsymbol{Z}_C. \tag{6.1}$$

Each of the sets $\{x_\alpha \mid \alpha\in\kappa\}$, $\{y_\beta \mid \beta\in\kappa\}$, and $\{z_\gamma \mid \gamma\in\kappa\}$ is clearly linearly independent. It follows that $\boldsymbol{X}_{A\cup A'} = \boldsymbol{X}_A + \boldsymbol{X}_{A'}$ for all $A,A'\subseteq\kappa$ and, similarly, $\boldsymbol{Y}_{B\cup B'} = \boldsymbol{Y}_B + \boldsymbol{Y}_{B'}$ and $\boldsymbol{Z}_{C\cup C'} = \boldsymbol{Z}_C + \boldsymbol{Z}_{C'}$ for all $B, B', C, C' \subseteq \kappa$. A straightforward computation gives the following lemma:

Lemma 6.1. *The map* $F\colon \mathfrak{P}(\kappa)^3 \to \operatorname{Sub}(\boldsymbol{V})$ *is a bounded join-homomorphism.*

Proof. Clearly $F\langle\emptyset,\emptyset,\emptyset\rangle = \mathbf{0}$ and $F\langle\kappa,\kappa,\kappa\rangle = \boldsymbol{V}$. Following the definitions, we compute $F(\langle A,B,C\rangle) + F(\langle A',B',C'\rangle) = \boldsymbol{X}_A + \boldsymbol{Y}_B + \boldsymbol{Z}_C + \boldsymbol{X}_{A'} + \boldsymbol{Y}_{B'} + \boldsymbol{Z}_{C'}$ $= \boldsymbol{X}_{A\cup A'} + \boldsymbol{Y}_{B\cup B'} + \boldsymbol{Z}_{C\cup C'} = F(\langle A\cup A', B\cup B', C\cup C'\rangle)$. □

Let $G\colon \operatorname{Sub}(\boldsymbol{V}) \to \mathfrak{P}(\kappa)^3$ be a map defined by

$$\boldsymbol{W} \mapsto \langle\{\alpha \mid x_\alpha \in \boldsymbol{W}\}, \{\beta \mid y_\beta \in \boldsymbol{W}\}, \{\gamma \mid z_\gamma \in \boldsymbol{W}\}\rangle\,,$$

for all $\boldsymbol{W} \in \operatorname{Sub}(\boldsymbol{V})$.

It is straightforward that G is a bounded meet-homomorphism and that it is the right adjoint of F (i.e., replacing the lattice $\operatorname{Sub}(\boldsymbol{V})$ with its dual, the maps F and G form a Galois correspondence [17]). Indeed, one readily sees that

$$F\langle A,B,C\rangle \subseteq \boldsymbol{W} \text{ iff } \langle A,B,C\rangle \le G(\boldsymbol{W}).$$

The maps F and G induce a closure operator GF on $\mathfrak{P}(\kappa)^3$.

Lemma 6.2. *The composition* $GF\colon \mathfrak{P}(\kappa)^3 \to \mathfrak{P}(\kappa)^3$ *is precisely the closure operator* $\overline{\langle - \rangle}$ *on* $\mathfrak{P}(\kappa)^3$ *defined by* (2.2).

Proof. We shall prove that $GF\langle A,B,C\rangle = \overline{\langle A,B,C\rangle}$ for every $\langle A,B,C\rangle \in \mathfrak{P}(\kappa)^3$. By symmetry, it suffices to prove that

$$\{\alpha \in \kappa \mid x_\alpha \in F\langle A,B,C\rangle\} = A \cup (B \cap C).$$

Let $\alpha \in A \cup (B\cap C)$. If $\alpha \in A$, then $x_\alpha \in F\langle A,B,C\rangle$ by the definition (6.1), while if $\alpha \in B \cap C$, then $x_\alpha = -y_\alpha - z_\alpha \in F\langle A,B,C\rangle$ by (6.1) and the defining relations of $\boldsymbol{V}$. It follows that $A\cup(B\cap C) \subseteq \{\alpha\in\kappa \mid x_\alpha \in F\langle A,B,C\rangle\}$.

In order to prove the opposite inclusion, take any $\xi \in \kappa\setminus A$ satisfying $x_\xi \in F\langle A,B,C\rangle$; if there is none, there is nothing to prove. We need to show that then $\xi \in B\cap C$. Certainly

$$x_\xi = \sum_{\alpha\in A} a_\alpha x_\alpha + \sum_{\beta\in B} b_\beta y_\beta + \sum_{\gamma\in C} c_\gamma z_\gamma \tag{6.2}$$

for suitable a_α, b_β, and $c_\gamma \in \mathbb{F}$ such that all but finitely many of them are zero. We set $a_\alpha = 0$ for $\alpha \notin A$, $b_\beta = 0$ for $\beta \notin B$, and $c_\gamma = 0$ for $\gamma \notin C$. Since $z_\gamma + x_\gamma + y_\gamma = 0$ for every $\gamma\in\kappa$, it follows from (6.2) that

$$x_\xi = \left(\sum_{\alpha\in A} a_\alpha x_\alpha - \sum_{\gamma\in C} c_\gamma x_\gamma\right) + \left(\sum_{\beta\in B} b_\beta y_\beta - \sum_{\gamma\in C} c_\gamma y_\gamma\right). \tag{6.3}$$

It easily follows from the defining relations of $\boldsymbol{V}$ that $\{x_\alpha, y_\alpha \mid \alpha\in\kappa\}$ forms a basis of $\boldsymbol{V}$. Thus, applying (6.3) we get that

$$a_\xi - c_\xi = 1 \text{ and } b_\xi - c_\xi = 0. \tag{6.4}$$

Since by our assumption $\xi \notin A$, we get from (6.2) that $a_\xi = 0$. Substituting to (6.4) we get that $b_\xi = c_\xi = -1$, hence $\xi \in B\cap C$. This concludes the proof that $A\cup(B\cap C) \supseteq \{\alpha\in\kappa \mid x_\alpha \in F\langle A,B,C\rangle\}$. □

The next lemma shows that $F \restriction \mathbf{M}_3[\mathbf{P}(\kappa)]$ preserves meets. Note that with Lemma 6.1, this means that $F \restriction \mathbf{M}_3[\mathbf{P}(\kappa)]$ is a lattice embedding of $\mathbf{M}_3[\mathbf{P}(\kappa)]$ into the lattice $\mathrm{Sub}(\boldsymbol{V})$.

Lemma 6.3. *Let $\langle A, B, C\rangle, \langle A', B', C'\rangle \in \mathbf{M}_3[\mathbf{P}(\kappa)]$ be balanced triples. Then*

$$F\langle A, B, C\rangle \cap F\langle A', B', C'\rangle = F\langle A\cap A', B\cap B', C\cap C'\rangle .$$

Proof. Since, by Lemma 6.1, F is a join-homomorphism, it is monotone, whence $F\langle A\cap A', B\cap B', C\cap C'\rangle \subseteq F\langle A, B, C\rangle \cap F\langle A', B', C'\rangle$. Thus it remains to prove the opposite inclusion.

Let $v \in F\langle A, B, C\rangle \cap F\langle A', B', C'\rangle$ be a non-zero vector. Then v can be expressed as

$$\begin{aligned} v &= \sum_{\alpha\in A} a_\alpha x_\alpha + \sum_{\beta\in B} b_\beta y_\beta + \sum_{\gamma\in C} c_\gamma z_\gamma \\ &= \sum_{\alpha\in A'} a'_\alpha x_\alpha + \sum_{\beta\in B'} b'_\beta y_\beta + \sum_{\gamma\in C'} c'_\gamma z_\gamma . \end{aligned} \tag{6.5}$$

Consider such an expression of v with

$$|\{\alpha \mid a_\alpha \neq 0\}| + |\{\beta \mid b_\beta \neq 0\}| + |\{\gamma \mid c_\gamma \neq 0\}| \tag{6.6}$$

minimal possible. Put $a_\alpha = 0$ for $\alpha \notin A$, $b_\beta = 0$ for $\beta \notin B$, and $c_\gamma = 0$ for $\gamma \notin C$. By symmetry, we can assume that $a_\alpha \neq 0$ for some $\alpha \in A$. Suppose for a contradiction that $\alpha \notin A'$. Since the triple $\langle A', B', C'\rangle$ is balanced, $B'\cap C' \subseteq A'$, whence $\alpha \notin B'\cap C'$. Without loss of generality we can assume that $\alpha \notin B'$. If all a_α, b_α, and c_α were non-zero, we could replace $c_\alpha z_\alpha$ with $-c_\alpha x_\alpha - c_\alpha y_\alpha$ and reduce the value of the expression in (6.6) which is assumed minimal possible. Thus either $b_\alpha = 0$ or $c_\alpha = 0$ (recall that we assume that $a_\alpha \neq 0$). We will deal with these two cases separately. If $b_\alpha = 0$, then the equality

$$a_\alpha x_\alpha + c_\alpha z_\alpha = c'_\alpha z_\alpha \tag{6.7}$$

must hold true. Since x_α and z_α are linearly independent, it follows from (6.7) that $a_\alpha = 0$ which contradicts our choice of α. The remaining case is when $c_\alpha = 0$. Under this assumption we have that

$$a_\alpha x_\alpha + b_\alpha y_\alpha = c'_\alpha z_\alpha .$$

It follows that

$$a_\alpha x_\alpha = c'_\alpha z_\alpha - b_\alpha y_\alpha = -c'_\alpha x_\alpha - (c'_\alpha + b_\alpha) y_\alpha . \tag{6.8}$$

Since x_α and y_α are linearly independent, we infer from (6.8) that $a_\alpha = -c'_\alpha = b_\alpha$. Then we could reduce the value of (6.6) by replacing $a_\alpha x_\alpha + b_\alpha y_\alpha$ with $c'_\alpha z_\alpha$ in (6.5). This contradicts the minimality of (6.6). □

Combining Lemmas 6.1, 6.2, and 6.3, we conclude:

Theorem 6.4. *The restrictions $F \restriction \mathbf{M}_3[\mathbf{P}(\kappa)] \colon \mathbf{M}_3[\mathbf{P}(\kappa)] \to \mathrm{Sub}(\boldsymbol{V})$ and,* a fortiory*, $F \restriction \mathbf{S} \colon \mathbf{S} \to \mathrm{Sub}(\boldsymbol{V})$ are bounded lattice embeddings. In particular, the lattice $\mathbf{S}$ is isomorphic to a bounded sublattice of the subspace lattice of a vector space.*

It is well-known that a distributive lattice $\mathcal{L}$ embeds (via a bounds-preserving lattice embedding) into the lattice $\mathcal{P}(\kappa)$, where κ is the cardinality of the set of all maximal ideals of $\mathcal{L}$. Such embedding induces an embedding $\mathcal{M}_3[\mathcal{L}] \hookrightarrow \mathcal{M}_3[\mathcal{P}(\kappa)]$ (cf. Lemma 3.1). By Theorem 6.4, the lattice $\mathcal{M}_3[\mathcal{P}(\kappa)]$ embeds into the lattice $\mathrm{Sub}(\boldsymbol{V})$ for a suitable vector space $\boldsymbol{V}$ (note again that we now also admit finite κ). Since the lattice $\mathrm{Sub}(\mathcal{V})$ is Arguesian, so are $\mathcal{M}_3[\mathcal{P}(\kappa)]$ and $\mathcal{M}_3[\mathcal{L}]$.

On the other hand, [9, Lemma 2.9] states that a lattice L is distributive if and only if $\mathcal{M}_3[\mathcal{L}]$ is modular. Hence, if $\mathcal{M}_3[\mathcal{L}]$ is modular, it follows that $\mathcal{L}$ is distributive, and, by the above argument, $\mathcal{M}_3[\mathcal{L}]$ is even Arguesian. We have thus proven the following strengthening of [9, Lemma 2.9]:

Proposition 6.5. *Let L be a lattice. Then L is distributive iff the lattice $\mathcal{M}_3[\mathcal{L}]$ is modular iff $\mathcal{M}_3[\mathcal{L}]$ is Arguesian. If this is the case, then $\mathcal{M}_3[\mathcal{L}]$ can be embedded into the lattice of all subspaces of a suitable vector space over any given field.*

7. Non existence of 3-frames

In this section we prove that there is no 3-frame in the lattice $\mathcal{M}_3[\mathcal{D}]$ for any distributive lattice $\mathcal{D}$. As a consequence, we cannot apply the Jónsson's coordinatization theorem in order to prove coordinatizability of any of these lattices, in particular, of the lattices $\mathcal{M}_3[\mathcal{F}(\kappa)]$ and $\mathcal{S}$.

Lemma 7.1. *Let $\mathcal{D}$ be a distributive lattice. Then for each $\langle a_1, a_2, a_3\rangle \in \mathcal{D}^3$, the equality*

$$\mu\overline{\langle a_1, a_2, a_3\rangle} = \mu\langle a_1, a_2, a_3\rangle.$$

holds true.

Proof. First observe that for all $1 \le k < l \le 3$ we have that

$$a_k \wedge a_l \le \bigvee_{1\le i<j\le 3} (a_i \wedge a_j) = \mu\langle a_1, a_2, a_3\rangle. \tag{7.1}$$

By (2.2) we have the equalities

$$\begin{aligned}\mu\overline{\langle a_1, a_2, a_3\rangle} &= \mu\langle a_1 \vee \mu\langle a_1, a_2, a_3\rangle, a_2 \vee \mu\langle a_1, a_2, a_3\rangle, a_3 \vee \mu\langle a_1, a_2, a_3\rangle\rangle \\ &= \bigvee_{1\le i<j\le 3} ((a_i \vee \mu\langle a_1, a_2, a_3\rangle) \wedge (a_j \vee \mu\langle a_1, a_2, a_3\rangle)).\end{aligned}$$

Since the lattice $\mathcal{D}$ is distributive,

$$(a_i \vee \mu\langle a_1, a_2, a_3\rangle) \wedge (a_j \vee \mu\langle a_1, a_2, a_3\rangle) = (a_i \wedge a_j) \vee \mu\langle a_1, a_2, a_3\rangle,$$

for all $1 \le i < j \le 3$. Applying (7.1), we conclude that

$$\mu\overline{\langle a_1, a_2, a_3\rangle} = \bigvee_{1\le i<j\le 3} ((a_i \wedge a_j) \vee \mu\langle a_1, a_2, a_3\rangle) = \mu\langle a_1, a_2, a_3\rangle. \qquad \square$$

With regard to (2.3), we conclude from Lemma 7.1 that

Corollary 7.2. *If $\mathfrak{D}$ is a distributive lattice, then*

$$\mu(\boldsymbol{a} \vee \boldsymbol{b}) = \mu\langle a_1 \vee b_1, a_2 \vee b_2, a_3 \vee b_3\rangle,$$

for all $\boldsymbol{a} = \langle a_1, a_2, a_3\rangle, \boldsymbol{b} = \langle b_1, b_2, b_3\rangle \in \mathfrak{M}_3[\mathfrak{D}]$.

Lemma 7.3. *Let $\mathfrak{D}$ be a distributive lattice and $\boldsymbol{a} = \langle a_1, a_2, a_3\rangle$ and $\boldsymbol{b} = \langle b_1, b_2, b_3\rangle$ elements of $\mathfrak{M}_3[\mathfrak{D}]$. If $\boldsymbol{a} \wedge \boldsymbol{b} = \boldsymbol{0}$, then*

$$\mu(\boldsymbol{a} \vee \boldsymbol{b}) = \mu\boldsymbol{a} \vee \mu\boldsymbol{b} \vee \left(\left(\bigvee_{i=1}^{3} a_i\right) \wedge \left(\bigvee_{j=1}^{3} b_j\right)\right).$$

Proof. Applying Corollary 7.2 and using the distributivity of $\mathfrak{D}$, we compute that

$$\begin{aligned}\mu(\boldsymbol{a} \vee \boldsymbol{b}) &= \mu\langle a_1 \vee b_1, a_2 \vee b_2, a_3 \vee b_3\rangle = \bigvee_{1\le i<j\le 3} ((a_i \vee b_i) \wedge (a_j \vee b_j)) \\ &= \bigvee_{1\le i<j\le 3} ((a_i \wedge a_j) \vee (b_i \wedge b_j) \vee (a_i \wedge b_j) \vee (a_j \wedge b_i)).\end{aligned}$$

Since $\boldsymbol{a}$ and $\boldsymbol{b}$ are balanced triples, $\mu\boldsymbol{a} = a_i \wedge a_j$ and $\mu\boldsymbol{b} = b_i \wedge b_j$ for all $1 \le i < j \le 3$. Thus

$$\begin{aligned}\mu(\boldsymbol{a} \vee \boldsymbol{b}) &= \bigvee_{1\le i<j\le 3} (\mu\boldsymbol{a} \vee \mu\boldsymbol{b} \vee (a_i \wedge b_j) \vee (a_j \wedge b_i)) \\ &= \mu\boldsymbol{a} \vee \mu\boldsymbol{b} \vee \bigvee_{1\le i<j\le 3} ((a_i \wedge b_j) \vee (a_j \wedge b_i)).\end{aligned} \tag{7.2}$$

From $\boldsymbol{a} \wedge \boldsymbol{b} = \boldsymbol{0}$ we get that $a_i \wedge b_i = 0$, for all $i = 1, 2, 3$. Substituting to (7.2) we get that

$$\mu(\boldsymbol{a} \vee \boldsymbol{b}) = \mu\boldsymbol{a} \vee \mu\boldsymbol{b} \vee \bigvee_{1\le i\le j\le 3} ((a_i \wedge b_j) \vee (a_j \wedge b_i)) = \mu\boldsymbol{a} \vee \mu\boldsymbol{b} \vee \bigvee_{i=1}^{3}\bigvee_{j=1}^{3} (a_i \wedge b_j).$$

Applying the distributivity of $\mathfrak{D}$ again we conclude that

$$\mu(\boldsymbol{a} \vee \boldsymbol{b}) = \mu\boldsymbol{a} \vee \mu\boldsymbol{b} \vee \left(\left(\bigvee_{i=1}^{3} a_i\right) \wedge \left(\bigvee_{j=1}^{3} b_j\right)\right). \qquad \square$$

Lemma 7.4. *Let $\mathfrak{D}$ be a bounded distributive lattice and $\boldsymbol{a} = \langle a_1, a_2, a_3\rangle, \boldsymbol{b} = \langle b_1, b_2, b_3\rangle \in \mathfrak{M}_3[\mathfrak{D}]$. If $\boldsymbol{a} \oplus \boldsymbol{b} = \boldsymbol{1}$, then*

$$\mu\boldsymbol{a} \oplus \bigvee_{j=1}^{3} b_j = 1.$$

Proof. Since trivially

$$\mu\boldsymbol{b} \vee \left(\left(\bigvee_{i=1}^{3} a_i\right) \wedge \left(\bigvee_{j=1}^{3} b_j\right)\right) \le \bigvee_{j=1}^{3} b_j,$$

we infer from Lemma 7.3 that

$$1 = \mu(\boldsymbol{a} \oplus \boldsymbol{b}) = \mu\boldsymbol{a} \vee \mu\boldsymbol{b} \vee \left(\left(\bigvee_{i=1}^{3} a_i \right) \wedge \left(\bigvee_{j=1}^{3} b_j \right) \right) \leq \mu\boldsymbol{a} \vee \bigvee_{j=1}^{3} b_j \leq 1. \tag{7.3}$$

Since $\boldsymbol{a} \wedge \boldsymbol{b} = \boldsymbol{0}$, we have that $\mu\boldsymbol{a} \leq a_i \leq b_i$, for all $i = 1, 2, 3$. Since the lattice $\mathfrak{D}$ is distributive, we conclude that

$$0 = \bigvee_{j=1}^{3} (\mu\boldsymbol{a} \wedge b_j) = \mu\boldsymbol{a} \wedge \bigvee_{j=1}^{3} b_j. \tag{7.4}$$

Combining (7.3) and (7.4) we get the desired equality $\mu\boldsymbol{a} \oplus \bigvee_{j=1}^{3} b_j = 1$. □

Lemma 7.5. *Let $\mathfrak{D}$ be a bounded distributive lattice and $\boldsymbol{a} = \langle a_1, a_2, a_3 \rangle$, $\boldsymbol{a}' = \langle a_1', a_2', a_3' \rangle$ perspective elements of $\mathfrak{M}_3[\mathfrak{D}]$. If $\boldsymbol{a} \wedge \boldsymbol{a}' = \boldsymbol{0}$, then*

$$\mu\boldsymbol{a} = \mu\boldsymbol{a}' \quad \textit{and} \quad \mu(\boldsymbol{a} \vee \boldsymbol{a}') = \bigvee_{i=1}^{3} a_i = \bigvee_{i=1}^{3} a_i'.$$

Proof. Let $\boldsymbol{b} = \langle b_1, b_2, b_3 \rangle$ be a common complement of $\boldsymbol{a}$ and $\boldsymbol{a}'$. It follows from Lemma 7.4 that both $\mu\boldsymbol{a}$ and $\mu\boldsymbol{a}'$ are complements of $\bigvee_{j=1}^{3} b_j$. Since complements in a distributive lattice are unique, we get that $\mu\boldsymbol{a} = \mu\boldsymbol{a}'$. Similarly we get that both $\bigvee_{i=1}^{3} a_i$ and $\bigvee_{i=1}^{3} a_i'$ are complements of $\mu\boldsymbol{b}$, hence they are equal. From these equalities we infer that

$$\mu\boldsymbol{a} = \mu\boldsymbol{a}' \leq \bigvee_{i=1}^{3} a_i' = \bigvee_{i=1}^{3} a_i.$$

Applying Lemma 7.3 we conclude that

$$\mu(\boldsymbol{a} \vee \boldsymbol{a}') = \bigvee_{i=1}^{3} a_i = \bigvee_{i=1}^{3} a_i'.$$ □

Proposition 7.6. *There is no 3-frame in the lattice $\mathfrak{M}_3[\mathfrak{D}]$, for any bounded distributive lattice $\mathfrak{D}$.*

Proof. Suppose that there are elements $\boldsymbol{a} = \langle a_1, a_2, a_3 \rangle$, $\boldsymbol{a}' = \langle a_1', a_2', a_3' \rangle$, and $\boldsymbol{a}'' = \langle a_1'', a_2'', a_3'' \rangle$ of $\mathfrak{M}_3[\mathfrak{D}]$ such that $\boldsymbol{a} \sim \boldsymbol{a}'$, $\boldsymbol{a} \sim \boldsymbol{a}''$ and the family $\langle \boldsymbol{a}, \boldsymbol{a}', \boldsymbol{a}'' \rangle$ is independent. Then $\mu(\boldsymbol{a} \vee \boldsymbol{a}') = \bigvee_{i=1}^{3} a_i = \bigvee_{i=1}^{3} a_i''$ due to Lemma 7.5. It follows that $\boldsymbol{a} \vee \boldsymbol{a}' \geq \boldsymbol{a}''$ which contradicts the independence of the family $\langle \boldsymbol{a}, \boldsymbol{a}', \boldsymbol{a}'' \rangle$. □

Corollary 7.7. *There is no 3-frame in the lattice $\mathfrak{M}_3[\mathfrak{B}]$, for any Boolean lattice $\mathfrak{B}$. In particular, neither the lattices $\mathfrak{M}_3[\mathfrak{F}(\kappa)]$ nor the lattice $\mathfrak{S}$ has a 3-frame.*

Remark 7.8. This remark is due to the anonymous referee. He pointed out that the main results of Sections 6 and 7 can be obtained by a simpler argument using the representation of a distributive lattice as a subdirect product of the two-element lattice **2**. Namely, it is well-known that a distributive lattice $\mathfrak{D}$ is a subdirect power of **2**. In particular, there is an index set I and an embedding

$\varphi\colon \mathfrak{D} \hookrightarrow \mathbf{2}^I$ such that the composition $\pi_i \circ \varphi\colon \mathfrak{D} \to \mathbf{2}$ with the canonical projection $\pi_i\colon \mathbf{2}^I \to \mathbf{2}$ is a surjective homomorphism for all $i \in I$. The map φ induces the embedding $\mathfrak{M}_3[\mathfrak{D}] \to \mathfrak{M}_3[\mathbf{2}^I]$ given by $\langle a, b, c\rangle \mapsto \langle \varphi(a), \varphi(b), \varphi(c)\rangle$. Observing that $\mathfrak{M}_3[\mathbf{2}] \simeq \mathfrak{M}_3$ we get isomorphisms $\mathfrak{M}_3[\mathbf{2}^I] \simeq \mathfrak{M}_3[\mathbf{2}]^I \simeq {\mathfrak{M}_3}^I$. Thus we have an embedding $\Phi\colon \mathfrak{M}_3[\mathfrak{D}] \hookrightarrow {\mathfrak{M}_3}^I$. It is straightforward to see that the composition of Φ with the i*th* canonical projection ${\mathfrak{M}_3}^I \to \mathfrak{M}_3$ is a surjective homomorphism $\mathfrak{M}_3[\mathfrak{D}] \to \mathfrak{M}_3$. Therefore $\mathfrak{M}_3[\mathfrak{D}]$ is a subdirect power of $\mathfrak{M}_3$. The lattice $\mathfrak{M}_3$ embeds into the subspace lattice of the 2-dimensional vectors space $\boldsymbol{V}$ over an arbitrary field. Let $\psi\colon \mathfrak{M}_3 \hookrightarrow \mathrm{Sub}\boldsymbol{V}$ be such an embedding. Then $\mathfrak{M}_3^I$ embeds into $\mathrm{Sub}\boldsymbol{V}^{(I)}$ (here $\boldsymbol{V}^{(I)}$ denotes the direct sum of copies of $\boldsymbol{V}$) via the mapping $(a_i)_{i\in I} \mapsto \bigoplus_{i\in I}\psi(a_i)$. The restriction of the map to $\mathfrak{M}_3[\mathfrak{D}]$ is an embedding of $\mathfrak{M}_3[\mathfrak{D}]$ into $\mathrm{Sub}\boldsymbol{V}^{(I)}$. Clearly, if $\mathfrak{D}$ is bounded, the embedding can be chosen bounds-preserving. This gives the main results of Section 6.

Let $\mathfrak{D}$ be a bounded lattice. Observe that the embedding $\Phi\colon \mathfrak{M}_3[\mathfrak{D}] \hookrightarrow {\mathfrak{M}_3}^I$ preserves the bounds. It follows that the Φ-image of a 3-frame would be a 3-frame in ${\mathfrak{M}_3}^I$. Let $i \in I$ and $\pi_i\colon {\mathfrak{M}_3}^I \to \mathfrak{M}_3$ be the corresponding canonical projection. The π_i image of the 3-frame in ${\mathfrak{M}_3}^I$ would be a 3-frame in $\mathfrak{M}_3$. However, it is easy to see that there is no 3-frame in $\mathfrak{M}_3$. Consequently, there is no 3-frame in $\mathfrak{M}_3[\mathfrak{D}]$. Thus we get Proposition 7.6.

8. Coordinatizability

We prove that despite of non-existence of 3-frames, the lattice $\mathfrak{M}_3[\mathfrak{B}]$ is coordinatizated for any Boolean lattice $\mathfrak{B}$. It is isomorphic to $\mathfrak{L}(\boldsymbol{M}[\mathfrak{B}]^*)$, the lattice of all finitely generated right ideals of the Boolean power of the ring $\boldsymbol{M}$, the ring of 2×2 matrices over the two-element field, by the Boolean lattice $\mathfrak{B}$. Modifying this construction we show that the lattice $\mathfrak{S}$ introduced in Section 3 is coordinatizable as well.

Let the notation $\boldsymbol{M}$ stand for the ring of all 2×2-matrices over the two-element field $\mathbb{F}_2$. It is well known that the matrix ring over a regular ring is regular, in particular, the ring $\boldsymbol{M}$ is regular (cf. [4, Theorem 1.7]). We put

$$e_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_2 := \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \quad \text{and} \quad e_3 := \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

There are exactly eight idempotents in the ring $\boldsymbol{M}$, namely $0, 1, e_1, e_2, e_3, 1-e_1, 1-e_2$, and $1-e_3$, and there are exactly three proper non-zero right ideals of $\boldsymbol{M}$, namely $e_1\boldsymbol{M} = (1-e_3)\boldsymbol{M}$, $e_2\boldsymbol{M} = (1-e_1)\boldsymbol{M}$, and $e_3\boldsymbol{M} = (1-e_2)\boldsymbol{M}$. Thus the lattice $\mathfrak{L}(\boldsymbol{M})$ is isomorphic to the five-element modular non-distributive lattice $\mathfrak{M}_3$ (see Figure 1).

We denote by $\mathrm{Idemp}(\boldsymbol{R})$ the set of all idempotents of a ring $\boldsymbol{R}$. We are going to make use of the next elementary lemma.

Lemma 8.1. *Let $\boldsymbol{R}$ be a ring and $e, f \in \mathrm{Idemp}(\boldsymbol{R})$. Then*

$$ef = f \iff f\boldsymbol{R} \subseteq e\boldsymbol{R}.$$

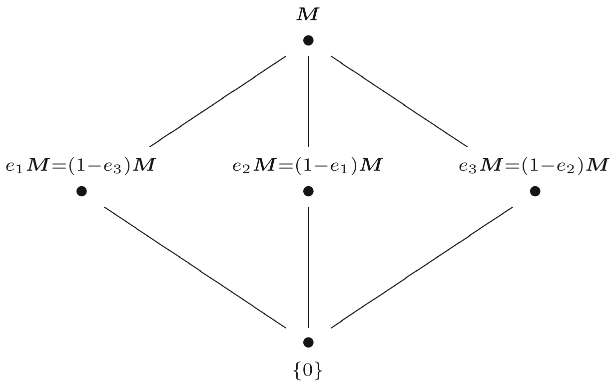

FIGURE 1. The lattice $\mathcal{L}(\boldsymbol{M})$

Proof. ($\Leftarrow$) If $f\boldsymbol{R} \subseteq e\boldsymbol{R}$, then $f \in e\boldsymbol{R}$ and so $f = er$ for some $r \in \boldsymbol{R}$. It follows that $ef = eer = er = f$. ($\Rightarrow$) Conversely, $ef = f$ implies that $f \in e\boldsymbol{R}$. We get readily that $f\boldsymbol{R} \subseteq e\boldsymbol{R}$. □

We equip the set $\mathrm{Idemp}(\boldsymbol{R})$ with a quasi-order $\leq_{\mathrm{e}}$ defined as follows: $f \leq_{\mathrm{e}} e$ provided that $ef = f$, for all $e, f \in \mathrm{Idemp}(\boldsymbol{R})$. Further, we denote by $\equiv_{\mathrm{e}}$ the corresponding equivalence relation on $\mathrm{Idemp}(\boldsymbol{R})$, i.e., $e \equiv_{\mathrm{e}} f$ if and only if both $e \leq_{\mathrm{e}} f$ and $f \leq_{\mathrm{e}} e$, for all $e, f \in \mathrm{Idemp}(\boldsymbol{R})$.

Suppose that $\boldsymbol{R}$ is a regular ring. Let $\iota_{\boldsymbol{R}}\colon \mathrm{Idemp}(\boldsymbol{R}) \to \mathcal{L}(\boldsymbol{R})$ be the map given by the correspondence $e \mapsto e\boldsymbol{R}$. It follows from Lemma 8.1 that the kernel of the map $\iota_{\boldsymbol{R}}$ coincides with the the equivalence relation $\equiv_{\mathrm{e}}$ and the quotient $\mathrm{Idemp}(\boldsymbol{R})/\equiv_{\mathrm{e}}$ is order-isomorphic to the set $\mathcal{L}(\boldsymbol{R})$ ordered by inclusion. Since $\mathcal{L}(\boldsymbol{R})$ is a lattice, $\mathrm{Idemp}(\boldsymbol{R})/\equiv_{\mathrm{e}}$ has finite suprema and infima, and the lattices $\mathcal{L}(\boldsymbol{R})$ and $\mathrm{Idemp}(\boldsymbol{R})/\equiv_{\mathrm{e}}$ are isomorphic.

The following lemma is a trivial consequence of the preceding two paragraphs. We leave the details of the proof to the reader.

Lemma 8.2. *Let $\mathcal{L}$ be a lattice and $\boldsymbol{R}$ a regular ring. Suppose that there is a surjective map $\varepsilon\colon \mathrm{Idemp}(\boldsymbol{R}) \to \mathcal{L}$ such that*

$$e \leq_{\mathrm{e}} f \iff \varepsilon(e) \leq \varepsilon(f), \quad \textit{for all } e, f \in \mathrm{Idemp}(\boldsymbol{R}). \tag{8.1}$$

Then $\ker \varepsilon = \ker \iota_R$ is equal to $\equiv_{\mathrm{e}}$ and the lattice $\mathcal{L}$ is isomorphic to $\mathcal{L}(\boldsymbol{R})$ via the composition[1] *$\iota_R \circ \varepsilon^{-1}\colon \mathcal{L} \to \mathcal{L}(\boldsymbol{R})$.*

Note that in the ring $\boldsymbol{M}$ introduced above, we have $e_1 \equiv_{\mathrm{e}} 1 - e_3$, $e_2 \equiv_{\mathrm{e}} 1 - e_1$, and $e_3 \equiv_{\mathrm{e}} 1 - e_2$, and the idempotents e_1, e_2, and e_3 are pairwise incomparable. Recall from Section 2.4 that the Boolean power $\boldsymbol{M}[\mathcal{B}]^*$ of the ring $\boldsymbol{M}$ by a Boolean lattice $\mathcal{B}$ is the set of all continuous functions from the Stone space of $\mathcal{B}$ to $\boldsymbol{M}$ equipped with the discrete topology.

[1] Purists would object that the composition $\iota_R \circ \varepsilon^{-1}$ sends an element $a \in \mathcal{L}$ to a singleton set $\{e\boldsymbol{R}\}$, where e is any idempotent from the $\equiv_{\mathrm{e}}$-block $\varepsilon^{-1}(a)$. For the sake of simplicity we identify the singleton set $\{e\boldsymbol{R}\}$ with its element $e\boldsymbol{R}$.

Lemma 8.3. *Let $\mathfrak{B}$ be a Boolean lattice. If a ring $\boldsymbol{R}$ is regular, then the Boolean power $\boldsymbol{R}[\mathfrak{B}]^*$ is regular as well.*

Proof. For each $a \in \boldsymbol{R}$ we pick an element $a^* \in \boldsymbol{R}$ such that $a = aa^*a$. Given $\boldsymbol{x} \in \boldsymbol{R}[\mathfrak{B}]^*$, we define a map $\boldsymbol{x}^* \colon \mathfrak{B}^* \to \boldsymbol{R}$ by the correspondence $\mathfrak{u} \mapsto \boldsymbol{x}(\mathfrak{u})^*$, $\mathfrak{u} \in \mathfrak{B}^*$. The $\boldsymbol{x}^*$-preimage of an element $b \in \boldsymbol{R}$ is $\bigcup\{\boldsymbol{x}^{-1}(a) \mid a^* = b\}$, which is a union of open sets. It follows that the map $\boldsymbol{x}^*$ is continuous and clearly $\boldsymbol{x} = \boldsymbol{x}\boldsymbol{x}^*\boldsymbol{x}$. Therefore $\boldsymbol{R}[\mathfrak{B}]^*$ is a regular ring. □

Given elements a, b of a Boolean lattice $\mathfrak{B}$, we set $a - b := a \wedge b'$, where b' is a unique complement of b. Note that an element $\boldsymbol{x} \in \boldsymbol{M}[\mathfrak{B}]^*$ is an idempotent if and only if $\boldsymbol{x}(\mathfrak{u}) \in \operatorname{Idemp}(\boldsymbol{M})$ for every $\mathfrak{u} \in \mathfrak{B}^*$. For each $\boldsymbol{e} \in \operatorname{Idemp}(\boldsymbol{M}[\mathfrak{B}]^*)$ we set $\varepsilon(\boldsymbol{e}) := \langle a_1, a_2, a_3 \rangle$, where[2]

$$\begin{aligned} N_{a_1} &= \{\mathfrak{u} \mid \boldsymbol{e}(\mathfrak{u}) \in \{1, e_1, 1 - e_3\}\}, \\ N_{a_2} &= \{\mathfrak{u} \mid \boldsymbol{e}(\mathfrak{u}) \in \{1, e_2, 1 - e_1\}\}, \\ N_{a_3} &= \{\mathfrak{u} \mid \boldsymbol{e}(\mathfrak{u}) \in \{1, e_3, 1 - e_2\}\}. \end{aligned} \tag{8.2}$$

It is clear that $\varepsilon(\boldsymbol{e})$ is a balanced triple with $N_{\mu\varepsilon(\boldsymbol{e})} = \{\mathfrak{u} \mid \boldsymbol{e}(\mathfrak{u}) = 1\}$. Therefore (8.2) defines a map $\varepsilon \colon \operatorname{Idemp}(\boldsymbol{M}[\mathfrak{B}]^*) \to \mathfrak{M}_3[\mathfrak{B}]$.

Lemma 8.4. *Let $\mathfrak{B}$ be a Boolean lattice. Then the map $\varepsilon \colon \operatorname{Idemp}(\boldsymbol{M}[\mathfrak{B}]^*) \to \mathfrak{M}_3[\mathfrak{B}]$ defined by (8.2) satisfies property (8.1).*

Proof. The implications $\boldsymbol{e} \leq_{\mathrm{e}} \boldsymbol{f} \implies \varepsilon(\boldsymbol{e}) \leq \varepsilon(\boldsymbol{f})$, $\boldsymbol{e}, \boldsymbol{f} \in \operatorname{Idemp}(\boldsymbol{M}[\mathfrak{B}]^*)$, are seen readily from (8.2). Let $\boldsymbol{e}, \boldsymbol{f} \in \operatorname{Idemp}(\boldsymbol{M}[\mathfrak{B}]^*)$ with $\varepsilon(\boldsymbol{e}) = \langle a_1, a_2, a_3 \rangle$ and $\varepsilon(\boldsymbol{f}) = \langle b_1, b_2, b_3 \rangle$. Suppose that $\varepsilon(\boldsymbol{e}) \leq \varepsilon(\boldsymbol{f})$ and let $\mathfrak{u} \in \mathfrak{B}^*$. The inequality implies that $\mu\varepsilon(\boldsymbol{e}) \leq \mu\varepsilon(\boldsymbol{f})$, hence $\boldsymbol{e}(\mathfrak{u}) = 1 \implies \boldsymbol{f}(\mathfrak{u}) = 1$. From $a_1 \leq b_1$ we infer that $\boldsymbol{e}(\mathfrak{u}) \in \{e_1, 1 - e_3\} \implies \boldsymbol{f}(\mathfrak{u}) \in \{1, e_1, 1 - e_3\}$. Similarly, from $a_2 \leq b_2$ we get that $\boldsymbol{e}(\mathfrak{u}) \in \{e_2, 1 - e_1\} \implies \boldsymbol{f}(\mathfrak{u}) \in \{1, e_2, 1 - e_1\}$ and from $a_3 \leq b_3$ we conclude that $\boldsymbol{e}(\mathfrak{u}) \in \{e_3, 1 - e_2\} \implies \boldsymbol{f}(\mathfrak{u}) \in \{1, e_3, 1 - e_2\}$. Therefore $\boldsymbol{e} \leq_{\mathrm{e}} \boldsymbol{f}$. □

Theorem 8.5. *Let $\mathfrak{B}$ be a Boolean lattice. The ring $\boldsymbol{M}[\mathfrak{B}]^*$ is regular and*

$$\mathfrak{L}(\boldsymbol{M}[\mathfrak{B}]^*) \simeq \mathfrak{M}_3[\mathfrak{B}].$$

Proof. The ring $\boldsymbol{M}[\mathfrak{B}]^*$ is regular due to Lemma 8.3.

Let $\boldsymbol{b} = \langle b_1, b_2, b_3 \rangle \in \mathfrak{M}_3[\mathfrak{B}]$. Note that since $\boldsymbol{b}$ is a balanced triple, each ultrafilter on $\mathfrak{B}$ contains at most one element from $\{b_i - \mu\boldsymbol{b} \mid i = 1, 2, 3\} \cup \{\mu\boldsymbol{b}\}$. Thus we can define $\boldsymbol{e} \in \operatorname{Idemp}(\boldsymbol{M}[\mathfrak{B}]^*)$ by

$$\boldsymbol{e}(\mathfrak{u}) := \begin{cases} 1: & \text{if} \quad \mu\boldsymbol{b} \in \mathfrak{u}, \\ e_i: & \text{if} \quad b_i - \mu\boldsymbol{b} \in \mathfrak{u}, \\ 0: & \text{otherwise}, \end{cases}$$

for all $\mathfrak{u} \in \mathfrak{B}^*$. It follows from (8.2) that $\varepsilon(\boldsymbol{e}) = \boldsymbol{b}$, and so ε is a projection.

By Lemma 8.4, the map $\varepsilon \colon \mathfrak{L}(\boldsymbol{M}[\mathfrak{B}]^*) \to \mathfrak{M}_3[\mathfrak{B}]$ satisfies (8.1), and so it is an isomorphism due to Lemma 8.2. □

[2] Recall definition (2.6).

Corollary 8.6. *Let $\boldsymbol{\mathcal{L}}$ be a bounded lattice. The lattice $\mathbf{\mathcal{M}}_3[\boldsymbol{\mathcal{L}}]$ is coordinatizable if and only if the lattice $\boldsymbol{\mathcal{L}}$ is Boolean.*

Proof. If $\boldsymbol{\mathcal{L}}$ is Boolean, then the lattice $\mathcal{M}_3[\boldsymbol{\mathcal{L}}]$ is coordinatizable by Theorem 8.5. In order to prove the opposite implication, suppose that the lattice $\mathcal{M}_3[\boldsymbol{\mathcal{L}}]$ is modular and complemented. We will prove that $\boldsymbol{\mathcal{L}}$ is Boolean. By [9, Lemma 2.9] the lattice $\mathcal{M}_3[\boldsymbol{\mathcal{L}}]$ is modular if and only if the lattice $\boldsymbol{\mathcal{L}}$ is distributive. Thus the lattice $\boldsymbol{\mathcal{L}}$ must be distributive. It follows from Lemma 7.4 that $\boldsymbol{\mathcal{L}}$ is complemented. Therefore it is a Boolean lattice. □

Let us now turn our attention to the lattice $\boldsymbol{\mathcal{S}}$ introduced in Section 3. Let κ be an infinite cardinal. There are exactly κ principal ultrafilters on $\boldsymbol{\mathfrak{F}}(\kappa)$, each corresponding to an ordinal $\alpha \in \kappa$, namely $\mathfrak{u}_\alpha = \{X \in \boldsymbol{\mathfrak{F}}(\kappa) \mid \alpha \in X\}$. Besides there is the only non-principal ultrafilter, $\mathfrak{f}$, consisting of all cofinite subsets of κ. The topological space $\boldsymbol{\mathfrak{F}}(\kappa)^*$ is the one-point compactification of the discrete space $\{\mathfrak{u}_\alpha \mid \alpha \in \kappa\}$. In particular, the singleton sets $\{\mathfrak{u}_\alpha\}$, $\alpha \in \kappa$, are open, and neighborhoods of $\mathfrak{f}$ are of the form $\boldsymbol{\mathfrak{F}}(\kappa)\setminus\{\mathfrak{u}_\alpha \mid \alpha \in F\}$, where F runs through all finite subsets of κ.

We put

$$\boldsymbol{S} := \{\boldsymbol{x} \in \boldsymbol{M}[\boldsymbol{\mathfrak{F}}(\kappa)]^* \mid \boldsymbol{x}(\mathfrak{f}) \in \{0, 1, e_1, 1 - e_1\}\}.$$

Theorem 8.7. *The ring $\boldsymbol{S}$ is regular and $\boldsymbol{\mathcal{L}}(\boldsymbol{S}) \simeq \boldsymbol{\mathcal{S}}$.*

Proof. Observe that the $\boldsymbol{I} := \{\boldsymbol{x} \in \boldsymbol{S} \mid \boldsymbol{x}(\mathfrak{f}) = 0\}$ is an ideal of the ring $\boldsymbol{M}[\boldsymbol{\mathfrak{F}}(\kappa)]^*$. Since the ring $\boldsymbol{M}[\boldsymbol{\mathfrak{F}}(\kappa)]^*$ is regular due to Lemma 8.3, we get from [4, Lemma 1.3] that $\boldsymbol{I}$ is a regular ideal. Thus $\boldsymbol{I}$ is a regular ideal of the ring $\boldsymbol{S}$ and it is easy to see that $\boldsymbol{S}/\boldsymbol{I} \simeq \mathbb{F}_2 \times \mathbb{F}_2$. Applying [4, Lemma 1.3] again, we conclude that the ring $\boldsymbol{S}$ is regular.

Let $\varepsilon\colon \boldsymbol{M}[\boldsymbol{\mathfrak{F}}(\kappa)]^* \to \mathcal{M}_3[\boldsymbol{\mathfrak{F}}(\kappa)]$ be the map defined by (8.2). The map ε satisfies (8.1) due to Lemma 8.4. To conclude that it is an isomorphism, it remains to prove that $\varepsilon(\mathrm{Idemp}(\boldsymbol{S})) = \boldsymbol{\mathcal{S}}$ (cf. Lemma 8.2).

Let $\boldsymbol{e} \in \mathrm{Idemp}(\boldsymbol{S})$. Then $\boldsymbol{e}(\mathfrak{f}) \in \{0, 1, e_1, 1 - e_1\}$. Since the function $\boldsymbol{e}\colon \boldsymbol{\mathfrak{F}}(\kappa)^* \to \boldsymbol{M}$ is by definition continuous, it is constant on some neighborhood of $\mathfrak{f}$. It follows that the set $\{\alpha \mid \boldsymbol{e}(\mathfrak{u}_\alpha) \in \{e_3, 1 - e_2\}\}$ is finite. We infer from (8.2) that this set is in fact $C\setminus\mu\langle A, B, C\rangle$, hence the set $C\setminus\mu\langle A, B, C\rangle$ is finite. Thus $\varepsilon(\mathrm{Idemp}(\boldsymbol{S})) \subseteq \boldsymbol{\mathcal{S}}$.

It now remains to prove the opposite inclusion. Given $\langle A, B, C\rangle \in \boldsymbol{\mathcal{S}}$, we define an idempotent $\boldsymbol{e} \in \boldsymbol{M}[\boldsymbol{\mathfrak{F}}(\kappa)]^*$ by

$$\boldsymbol{e}(\mathfrak{u}) := \begin{cases} 1 & \text{if} \quad \mu\langle A, B, C\rangle \in \mathfrak{u}, \\ e_1 & \text{if} \quad A\setminus\mu\langle A, B, C\rangle \in \mathfrak{u}, \\ 1 - e_1 & \text{if} \quad B\setminus\mu\langle A, B, C\rangle \in \mathfrak{u}, \\ e_3 & \text{if} \quad C\setminus\mu\langle A, B, C\rangle \in \mathfrak{u}, \\ 0 & \text{otherwise}, \end{cases}$$

for all $\mathfrak{u} \in \boldsymbol{\mathfrak{F}}(\kappa)^*$. Since $\langle A, B, C\rangle \in \boldsymbol{\mathcal{S}}$, the set $C\setminus\mu\langle A, B, C\rangle$ is finite by (3.3), hence $C\setminus\mu\langle A, B, C\rangle \notin \mathfrak{f}$. It follows that $\boldsymbol{e}(\mathfrak{f}) \in \{0, 1, e_1, 1 - e_1\}$, and so $\boldsymbol{e} \in \boldsymbol{S}$. We infer that $\boldsymbol{\mathcal{S}} \subseteq \varepsilon(\mathrm{Idemp}(\boldsymbol{S}))$. This concludes the proof. □

9. Maximal Abelian regular subalgebras

We prove that the maximal Boolean sublattices $\mathcal{G}$ and $\mathcal{H}$ of the lattice $\mathcal{S}$ from Sections 4 and 5, respectively, correspond to maximal Abelian regular subalgebras (over the field $\mathbb{F}_2$) of $\boldsymbol{S}$.

Observe that the diagonal matrices, namely 0, 1, e_1, and $1 - e_1$, form a subalgebra of $\boldsymbol{M}$, which we denote by $\boldsymbol{G}$. It is easy to compute by hand that the elements from $\boldsymbol{M}$ commuting with e_1 are exactly the diagonal matrices. It follows that $\boldsymbol{G}$ is a maximal Abelian regular subalgebra of the $\mathbb{F}_2$-algebra $\boldsymbol{M}$(cf. [16, Section 4.4]).

Proposition 9.1. *Let $\mathcal{B}$ be a Boolean lattice and* $\varepsilon\colon \operatorname{Idemp}(\boldsymbol{M}[\mathcal{B}]^*) \to \mathcal{M}_3[\mathcal{B}]$ *the map defined by* (8.2). *Then $\boldsymbol{G}[\mathcal{B}]^*$ is a maximal Abelian regular subalgebra of $\boldsymbol{M}[\mathcal{B}]^*$, it is commutative, and*

$$\varepsilon(\operatorname{Idemp}(\boldsymbol{G}[\mathcal{B}]^*)) = \{\langle a, b, a \wedge b\rangle \mid a, b \in \mathcal{B}\}. \tag{9.1}$$

Proof. The ring $\boldsymbol{G}[\mathcal{B}]^*$ is regular due to Lemma 8.3. (Observe that the equality $\operatorname{Idemp}(\boldsymbol{G}[\mathcal{B}]^*) = \boldsymbol{G}[\mathcal{B}]^*$ holds true.)

Since $\boldsymbol{G}$ is commutative, the Boolean power $\boldsymbol{G}[\mathcal{B}]^*$ is commutative as well. As observed above, $\boldsymbol{G} = \{a \in \boldsymbol{M} \mid ae_1 = e_1a\}$. Thus the range of each $\boldsymbol{x} \in \boldsymbol{M}[\mathcal{B}]^*$ commuting with the constant map $\mathcal{B}^* \to \{e_1\}$ must be included in $\boldsymbol{G}$. It follows that $\boldsymbol{G}[\mathcal{B}]^*$ is a maximal Abelian regular subalgebra of $\boldsymbol{M}[\mathcal{B}]^*$.

It is clear from (8.2) that $\varepsilon(\boldsymbol{e}) \in \{\langle a, b, a \wedge b\rangle \mid a, b \in \mathcal{B}\}$ for every $\boldsymbol{e} \in \operatorname{Idemp}(\boldsymbol{G}[\mathcal{B}]^*)$. Conversely, given $a, b \in \mathcal{B}$ and an ultrafilter $\mathfrak{u}$ on $\mathcal{B}$, we set

$$\boldsymbol{e}(\mathfrak{u}) := \begin{cases} 1 & \text{if} \quad a \wedge b \in \mathfrak{u}, \\ e_1 & \text{if} \quad a - b \in \mathfrak{u}, \\ 1 - e_1 & \text{if} \quad b - a \in \mathfrak{u}, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\boldsymbol{e} \in \operatorname{Idemp}(\boldsymbol{G}[\mathcal{B}]^*)$ and $\varepsilon(\boldsymbol{e}) = \langle a, b, a \wedge b\rangle$. This proves (9.1). □

In the case that $\mathcal{B} = \mathcal{F}(\kappa)$, we have $\boldsymbol{G}[\mathcal{F}(\kappa)]^* \subseteq \mathcal{S}$. Thus it follows from Proposition 9.1 that

Corollary 9.2. *The ring $\boldsymbol{G}[\mathcal{F}(\kappa)]^*$ is commutative, forms a maximal Abelian regular subalgebra of $\mathcal{S}$, and $\varepsilon(\operatorname{Idemp}(\boldsymbol{G}[\mathcal{F}(\boldsymbol{\kappa})]^*)) = \mathcal{G}$, where $\mathcal{G}$ is the Boolean lattice introduced in Section 4.*

Put

$$m := \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \in \boldsymbol{M}$$

and observe $e_3 = me_1m^{-1}$. It follows that the subalgebra $\boldsymbol{H} = \{0, 1, e_3, 1 - e_3\}$ of $\boldsymbol{M}$ is the image of $\boldsymbol{G}$ under the inner automorphism of $\boldsymbol{M}$ given by $x \mapsto mxm^{-1}$, $x \in \boldsymbol{M}$. Consequently, $\boldsymbol{H}$ is a maximal Abelian regular subalgebra of $\boldsymbol{M}$ and also $\boldsymbol{H}[\mathcal{B}]^*$ is a maximal Abelian regular subalgebra of $\boldsymbol{M}[\mathcal{B}]^*$ for every Boolean lattice $\mathcal{B}$.

Proposition 9.3. *The intersection* $\boldsymbol{H'} := \boldsymbol{H}[\boldsymbol{\mathfrak{F}(\kappa)}]^* \cap \boldsymbol{S}$ *is a maximal Abelian regular subalgebra of* $\boldsymbol{S}$*, it is commutative, and* $\varepsilon(\mathrm{Idemp}(\boldsymbol{H'})) = \mathcal{H}$*, where* $\mathcal{H}$ *is the Boolean lattice introduced in Section 5.*

Proof. Clearly $\boldsymbol{H}$, and so also $\boldsymbol{H'}$, are commutative. Put $\boldsymbol{J} = \{\boldsymbol{x} \in \boldsymbol{H'} \mid \boldsymbol{x}(\mathfrak{f}) = 0\}$ and observe that $\boldsymbol{J}$ is isomorphic to a direct sum of copies of $\mathbb{F}_2$. In particular, $\boldsymbol{J}$ is a regular ideal of $\boldsymbol{H'}$. Since $\boldsymbol{H'}/\boldsymbol{J} \simeq \mathbb{F}_2$, the algebra $\boldsymbol{H'}$ is regular due to [4, Lemma 1.3].

Given a principal ultrafilter $\mathfrak{u} \in \boldsymbol{\mathfrak{F}(\kappa)}^*$, set

$$\boldsymbol{e}_{\mathfrak{u}}(\mathfrak{v}) := \begin{cases} e_3 & \text{if } \mathfrak{v} = \mathfrak{u}, \\ 0 & \text{whenever } \mathfrak{v} \neq \mathfrak{u}, \end{cases}$$

for all $\mathfrak{v} \in \boldsymbol{\mathfrak{F}(\kappa)}^*$. Observe that since $\boldsymbol{e}_{\mathfrak{u}}(\mathfrak{f}) = 0$, we have $\boldsymbol{e}_{\mathfrak{u}} \in \boldsymbol{H'}$. Let $\boldsymbol{x} \in \boldsymbol{S}$ be commuting with every element of $\boldsymbol{H'}$. Since $\boldsymbol{x}$ commutes with all $\boldsymbol{e}_{\mathfrak{u}}$ and $\boldsymbol{H}$ is a maximal Abelian regular subalgebra of $\boldsymbol{M}$, we have that $\boldsymbol{x}(\mathfrak{u}) \in \boldsymbol{H}$ for all principal ultrafilters $\mathfrak{u}$. Since the map $\boldsymbol{x}$ is continuous, it is constant on some neighborhood of $\mathfrak{f}$, and so $\boldsymbol{x}(\mathfrak{f}) \notin \{e_1, 1 - e_1\}$. We conclude that $\boldsymbol{x} \in \boldsymbol{H'}$. Therefore $\boldsymbol{H'}$ is a maximal Abelian regular subalgebra of $\boldsymbol{S}$.

Let $\boldsymbol{e} \in \mathrm{Idemp}(\boldsymbol{H'})$ (note that $\mathrm{Idemp}(\boldsymbol{H'}) = \boldsymbol{H'}$) and put $\langle A, B, C\rangle := \varepsilon(\boldsymbol{e})$. We get readily from (8.2) that $B \subseteq A$. From $\boldsymbol{e}(\mathfrak{f}) \in \{0, 1\}$ and $\boldsymbol{e}$ being constant on some neighborhood of $\mathfrak{f}$, we conclude that $A \approx C$. Therefore $\langle A, B, C\rangle \in \mathcal{H}$ due to (5.2). Thus we have proved that $\varepsilon(\mathrm{Idemp}(\boldsymbol{H'})) \subseteq \mathcal{H}$.

Given $\langle A, B, C\rangle \in \mathcal{H}$, we define an idempotent $\boldsymbol{e} \in \boldsymbol{H}[\boldsymbol{\mathfrak{F}(\kappa)}]^*$ by

$$\boldsymbol{e}(\mathfrak{u}) := \begin{cases} 1 & \text{if} \quad B \in \mathfrak{u}, \\ 1 - e_3 & \text{if} \quad A \backslash B \in \mathfrak{u}, \\ e_3 & \text{if} \quad C \backslash B \in \mathfrak{u}, \\ 0 & \text{otherwise}, \end{cases}$$

for every ultrafilter $\mathfrak{u}$ on $\mathfrak{F}(\kappa)$. Since $\langle A, B, C\rangle \in \mathcal{H}$, both $A\backslash B$ and $C\backslash B$ are finite, and so $\boldsymbol{e}(\mathfrak{f}) \in \{0, 1\}$. It follows that $\boldsymbol{e} \in \boldsymbol{S}$, and so $\boldsymbol{e} \in \boldsymbol{H'}$. Therefore $\mathcal{H} \subseteq \varepsilon(\mathrm{Idemp}(\boldsymbol{H'}))$. □

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