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On the local closure of clones on countable sets

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Abstract. We consider clones on countable sets. If such a clone has quasigroup operations, is locally closed and countable, then there is a function $f\colon \mathbb{N} \to \mathbb{N}$ such that the n-ary part of C is equal to the n-ary part of $\operatorname{Pol}\operatorname{Inv}^{[f(n)]} C$, where $\operatorname{Inv}^{[f(n)]} C$ denotes the set of $f(n)$-ary invariant relations of C.

1. Results

We investigate clones on infinite sets [10, 11, 5]. For a clone C on A, its local closure $\overline{C}$ consists of all those finitary operations on A that can be interpolated at each finite subset of their domain by a function in C, and we have $\overline{C} = \operatorname{Pol}\operatorname{Inv} C$. Here, as in [10], $\operatorname{Inv} C$ denotes the set of those finitary relations on A that are preserved by all functions in C, and for a set R of relations on A, $\operatorname{Pol} R$ denotes the set of those finitary operations on A that preserve all relations in R. A clone is called *locally closed* if it is equal to its local closure. C is called a *clone with quasigroup operations* if there are three binary operations $\cdot, \backslash, / \in C$ such that $\langle A, \cdot, \backslash, / \rangle$ is a quasigroup [3, p. 24]. Theorem 1.1 states that a clone with quasigroup operations on a countable set is either locally closed, or its local closure $\operatorname{Pol}\operatorname{Inv} C$ is uncountable.

Theorem 1.1. *Let A be a set with $|A| = \aleph_0$, and let C be a clone with quasigroup operations on A. If $|\operatorname{Pol}\operatorname{Inv} C| \leq \aleph_0$, then $C = \operatorname{Pol}\operatorname{Inv} C$.*

This theorem does not hold for clones without quasigroup operations. We say that C is *constantive* if it contains all unary constant operations.

Theorem 1.2. *There exist a set A with $|A| = \aleph_0$ and a constantive clone C on A such that $|\operatorname{Pol}\operatorname{Inv} C| = \aleph_0$ and $C \neq \operatorname{Pol}\operatorname{Inv} C$.*

For a clone C on A, $\operatorname{Inv}^{[m]} C$ denotes the set of m-ary invariant relations of C. It is well known that a function f lies in $\operatorname{Pol}\operatorname{Inv}^{[m]} C$ if and only if it can be interpolated at every m-element subset of its domain by a function in C; this is discussed, e.g., in [9] and in [4, Lemma 7] and stated in Lemma 3.1. We write $C^{[n]}$ for the set of n-ary functions in C. Let B be any set, and let $F \subseteq A^B$. A subset D of B is a *base of equality* for F if for all $f, g \in F$ with $f|_D = g|_D$, we have $f = g$. Theorem 1.1 can be extended in the following way:

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Theorem 1.3. *Let A be a set with $|A| = \aleph_0$, and let C be a clone on A with quasigroup operations. Then the following are equivalent:*

(1) $|\text{Pol Inv}\, C| \leq \aleph_0$.
(2) *For each $n \in \mathbb{N}$, $C^{[n]}$ has a finite base of equality.*
(3) $|C| \leq \aleph_0$ *and* $\forall n \in \mathbb{N}\ \exists k \in \mathbb{N} : C^{[n]} = (\text{Pol Inv}^{[k]}\, C)^{[n]}$.
(4) $|C| \leq \aleph_0$ *and* $C = \text{Pol Inv}\, C$.

A weaker version of this result was proved in [1]. As an application, we obtain, e.g., that a countably infinite integral domain R cannot be affine complete: If it is affine complete, then the clone C of polynomial functions of R satisfies (3), and therefore the unary polynomials have a finite base of equality D. But $f(x) = 0$ and $g(x) = \prod_{d \in D}(x - d)$ show that this is not possible. In fact, Theorem 1.3 extracts a common idea of several "non-affine completeness" results [6, 8]. The proofs are given in Section 4.

2. Finite bases of equality

Theorems 1.1 and 1.3 rely on the following observation. In a less general context, this observation appears in [1, Theorem 2], and large parts of its proof are verbatim copies from [1] and [2, pp.51-52].

Lemma 2.1. *Let A be a set with $|A| = \aleph_0$, let $m \in \mathbb{N}$, and let C be a clone on A with quasigroup operations. If $|(\text{Pol Inv}\, C)^{[m]}| \leq \aleph_0$, then $C^{[m]}$ has a finite base of equality.*

Proof. Let $\overline{C} := \text{Pol Inv}\, C$. In the case that $\overline{C}^{[m]}$ is finite, its subset $C^{[m]}$ is also finite. Then for every $f, g \in C^{[m]}$ with $f \neq g$, we choose $a_{(f,g)} \in A^m$ such that $f(a_{(f,g)}) \neq g(a_{(f,g)})$. Then $D := \{a_{(f,g)} \mid f, g \in C^{[m]}, f \neq g\}$ is a base of equality for $C^{[m]}$. Hence, we will from now on assume $|\overline{C}^{[m]}| = \aleph_0$. Let $a_0, a_1, a_2, \dots$ and $f_0, f_1, f_2, \dots$ be complete enumerations of A^m and $\overline{C}^{[m]}$, respectively. Furthermore, we abbreviate the set $\{a_i \mid i \leq r\}$ by $A(r)$. Seeking a contradiction, we suppose that there is no finite base of equality for $C^{[m]}$. We shall construct a sequence $(n_k)_{k \in \mathbb{N}_0}$ of non-negative integers and a sequence $(g_k)_{k \in \mathbb{N}_0}$ of elements of $C^{[m]}$ with the following properties:

(1) $\forall k \in \mathbb{N}_0 : g_k|_{A(n_k)} \neq f_k|_{A(n_k)}$,
(2) $\forall k \in \mathbb{N}_0 : n_{k+1} > n_k$,
(3) $\forall k \in \mathbb{N}_0 : g_{k+1}|_{A(n_k)} = g_k|_{A(n_k)}$.

We construct the sequences inductively. We choose $g_0 \in C^{[m]}$ such that $g_0 \neq f_0$, and $n_0 \in \mathbb{N}_0$ minimal with $g_0(a_{n_0}) \neq f_0(a_{n_0})$. If we have already constructed g_k and n_k, we construct g_{k+1} and n_{k+1} as follows: in the case that $g_k|_{A(n_k)} \neq f_{k+1}|_{A(n_k)}$, we set $g_{k+1} := g_k$ and $n_{k+1} := n_k + 1$. In the case $g_k|_{A(n_k)} = f_{k+1}|_{A(n_k)}$, we first show that there exists a function $h \in C^{[m]}$ with

$$g_k|_{A(n_k)} = h|_{A(n_k)} \text{ and } h \neq f_{k+1}. \tag{2.1}$$

Suppose that, on the contrary, every $h \in C^{[m]}$ with $g_k|_{A(n_k)} = h|_{A(n_k)}$ satisfies $h = f_{k+1}$. In this case, $g_k = f_{k+1}$, and therefore $f_{k+1} \in C^{[m]}$. We will show next that $A(n_k)$ is a base of equality of $C^{[m]}$. To this end, let $r, s \in C^{[m]}$ with $r|_{A(n_k)} = s|_{A(n_k)}$. We define $t(x) := r(x)\backslash(s(x) \cdot f_{k+1}(x))$. Then for every $x \in A(n_k)$, we have $t(x) = r(x)\backslash(r(x) \cdot f_{k+1}(x)) = f_{k+1}(x) = g_k(x)$. Hence, $t = f_{k+1}$. Therefore, for every $x \in A^m$, we have $r(x)\backslash(s(x) \cdot f_{k+1}(x)) = f_{k+1}(x)$, thus $s(x) \cdot f_{k+1}(x) = r(x) \cdot f_{k+1}(x)$, and therefore $(s(x) \cdot f_{k+1}(x))/f_{k+1}(x) = (r(x) \cdot f_{k+1}(x))/f_{k+1}(x))$, which implies $s(x) = r(x)$. Thus, $r = s$, which completes the proof that $A(n_k)$ is a base of equality of $C^{[m]}$, contradicting the assumption that no such base exists. Hence, there is $h \in C^{[m]}$ that satisfies (2.1). Continuing in the construction of g_{k+1}, we set $g_{k+1} := h$, and we choose n_{k+1} to be minimal with $h(a_{n_{k+1}}) \neq f_{k+1}(a_{n_{k+1}})$.

Since for every $a \in A^m$, the sequence $(g_k(a))_{k \in \mathbb{N}_0}$ is eventually constant, we may define a function $l\colon A^m \to A$ by $l(a) := \lim_{k\to\infty} g_k(a)$. We will now show that $l \in \overline{C}^{[m]}$. The clone $\overline{C}$ contains exactly those functions that can be interpolated at every finite subset of their domain with a function in C. Hence, we show that l can be interpolated at every finite subset B of A^m by a function in C. Since $\bigcup_{i \in \mathbb{N}_0} A_i = A^m$, there is $k \in \mathbb{N}$ such that $B \subseteq A(n_k)$. Since $l|_{A(n_k)} = g_k|_{A(n_k)}$, the function $g_k \in C^{[m]}$ interpolates l at B. We conclude that the function l lies in $\overline{C}^{[m]}$. Thus, l is equal to f_k for some $k \in \mathbb{N}_0$. Since $l|_{A(n_k)} = g_k|_{A(n_k)}$ and $g_k|_{A(n_k)} \neq f_k|_{A(n_k)}$, we obtain $l|_{A(n_k)} \neq f_k|_{A(n_k)}$, a contradiction. Hence, $C^{[m]}$ has a finite base of equality. □

Lemma 2.2 (cf. [7, Lemma 1] and [1, Proposition 2]). *Let C be a clone on the set A, let $n \in \mathbb{N}$, let D be a finite base of equality for $C^{[n]}$, and let $k := |D|+1$. Then $C^{[n]} = (\mathrm{Pol}\,\mathrm{Inv}^{[k]}\,C)^{[n]}$.*

Proof. Let $l \in (\mathrm{Pol}\,\mathrm{Inv}^{[k]}\,C)^{[n]}$. Then l can be interpolated at every subset of A^n with at most k elements by a function in $C^{[n]}$. Hence, there is $f \in C^{[n]}$ such that $f|_D = l|_D$. If $f = l$, then $l \in C^{[n]}$. In the case $f \neq l$, we take $y \in A^n$ such that $f(y) \neq l(y)$. Now we choose $g \in C^{[n]}$ such that $g|_{D\cup\{y\}} = l|_{D\cup\{y\}}$. Then $f(y) \neq g(y)$ and $f|_D = g|_D$, contradicting the assumption that D is a base of equality for $C^{[n]}$. □

3. A compactness property for local interpolation

For two sets A and B, a set of functions $F \subseteq A^B$, and $k \in \mathbb{N}$, the set $\mathrm{Loc}_k\,F$ is defined as the set of those functions that can be interpolated at every subset of B with at most k elements by a function in F [9]. If C is a clone, and $F = C^{[m]}$ is its m-ary part, then $\mathrm{Loc}_k(C^{[m]})$ is the set of m-ary functions on A that preserve the k-ary relations in $\mathrm{Inv}\,C$.

Lemma 3.1 (cf. [9, p. 31, Theorem 4.1]). *Let C be a clone on the set A, and let $k, m \in \mathbb{N}$. Then $\mathrm{Loc}_k(C^{[m]}) = (\mathrm{Pol}\,\mathrm{Inv}^{[k]}\,C)^{[m]} = (\mathrm{Pol}\,\mathrm{Inv}^{[k]}(C^{[m]}))^{[m]}$.*

For countable sets A, we obtain the following result.

Theorem 3.2. *Let A be a set with $|A| \leq \aleph_0$, and let C be a clone on A with quasigroup operations such that $|C^{[m]}| \leq \aleph_0$. If $\bigcap_{k\in\mathbb{N}} \mathrm{Loc}_k(C^{[m]}) = C^{[m]}$, then there exists $n \in \mathbb{N}$ such that $\mathrm{Loc}_n(C^{[m]}) = C^{[m]}$.*

Proof. By Lemma 3.1 and the assumptions,

$$C^{[m]} = \bigcap_{k\in\mathbb{N}} \mathrm{Loc}_k(C^{[m]}) = \bigcap_{k\in\mathbb{N}} (\mathrm{Pol}\,\mathrm{Inv}^{[k]}\, C)^{[m]} = (\mathrm{Pol}\,\mathrm{Inv}\, C)^{[m]}.$$

Now Lemma 2.1 yields a finite base of equality for $C^{[m]}$, and now by Lemma 2.2, there is $n \in \mathbb{N}$ such that $C^{[m]} = (\mathrm{Pol}\,\mathrm{Inv}^{[n]}\, C)^{[m]} = \mathrm{Loc}_n(C^{[m]})$. □

For an arbitrary m-ary operation f on the set A, we say that the property $I(f, n, C)$ holds if f can be interpolated by a function in C at each subset of A^m with at most n elements. Theorem 3.2 yields the following compactness property: if C is a countable clone with quasigroup operations, if A is countable, and if $\forall f \in A^{A^m} : ((\forall k \in \mathbb{N} : I(f, k, C)) \Rightarrow f \in C)$ holds, then there is a natural number $n \in \mathbb{N}$ such that $\forall f \in A^{A^m} : (I(f, n, C) \Rightarrow f \in C)$ holds.

4. Proofs of the theorems from Section 1

Proof of Theorem 1.3. (1)⇒(2): Let $n \in \mathbb{N}$. Since $(\mathrm{Pol}\,\mathrm{Inv}\, C)^{[n]} \subseteq \mathrm{Pol}\,\mathrm{Inv}\, C$, we have $|(\mathrm{Pol}\,\mathrm{Inv}\, C)^{[n]}| \leq \aleph_0$. Lemma 2.1 now yields a finite base of equality for $C^{[n]}$.

(2)⇒(3): Let $n \in \mathbb{N}$, and let $D \subseteq A^n$ be a finite base of equality for $C^{[n]}$. We set $k := |D| + 1$ and obtain $C^{[n]} = (\mathrm{Pol}\,\mathrm{Inv}^{[k]}\, C)^{[n]}$ from Lemma 2.2. The mapping $\varphi\colon C^{[n]} \to A^D$, $f \mapsto f|_D$ is injective, therefore $|C^{[n]}| \leq \aleph_0$. Since for every $n \in \mathbb{N}$, we have $|C^{[n]}| \leq \aleph_0$, we have $|C| \leq \aleph_0$.

(3)⇒(4): Let $n \in \mathbb{N}$, and let k be taken from (3). Then we have that $(\mathrm{Pol}\,\mathrm{Inv}\, C)^{[n]} \subseteq (\mathrm{Pol}\,\mathrm{Inv}^{[k]}\, C)^{[n]} = C^{[n]}$.

(4)⇒(1): This is obvious. □

Proof of Theorem 1.1. The statement of Theorem 1.1 is given by the implication (1)⇒(4) of Theorem 1.3. □

Proof of Theorem 1.2. Let $A := \mathbb{N}_0$, and let $p(x) := x \bmod 2$ for all $x \in \mathbb{N}_0$. For $a \in \mathbb{N}_0$, we define $g_a\colon \mathbb{N}_0 \to \mathbb{N}_0$ by

$$g_a(x) := \begin{cases} p(x) & \text{if } x < a, \\ x & \text{if } x \geq a, \end{cases}$$

with $c_a(x) := a$ for all $x \in \mathbb{N}_0$. Let $M := \{g_a \mid a \in \mathbb{N}_0\} \cup \{c_a(x) \mid a \in \mathbb{N}_0\}$. We will first show that $\langle M, \circ, g_0\rangle$ is a submonoid of $\langle {\mathbb{N}_0}^{\mathbb{N}_0}, \circ, \mathrm{id}_{\mathbb{N}_0}\rangle$. To this end, it is sufficient to show that $g_a \circ g_b \in M$ for all $a, b \in \mathbb{N}_0$. Since $g_0 = g_1 = g_2 = \mathrm{id}_{\mathbb{N}_0}$, we may assume $a \geq 3$ and $b \geq 3$. We will show

$$g_a(g_b(x)) = g_{\max(a,b)}(x) \text{ for all } x \in \mathbb{N}_0. \tag{4.1}$$

If $x < b$, then $g_a(g_b(x)) = g_a(p(x)) = p(p(x)) = p(x) = g_{\max(a,b)}(x)$. If $x \geq b$ and $x < a$, then $g_a(g_b(x)) = g_a(x)$; since in this case $b \leq a$, so $g_a(x) = g_{\max(a,b)}(x)$. If $x \geq a$ and $x \geq b$, then $g_a(g_b(x)) = g_a(x) = x = g_{\max(a,b)}(x)$. From (4.1), we deduce that M is closed under composition. Now let C be the clone on $\mathbb{N}_0$ that is generated by M; this clone consists of all functions $(x_1, \dots, x_n) \mapsto m(x_j)$ with $n, j \in \mathbb{N}$, $m \in M$ and $j \leq n$. Let $\overline{C} := \operatorname{Pol} \operatorname{Inv} C$. Next, we show

$$p \in \overline{C}. \tag{4.2}$$

To prove (4.2), we show that p can be interpolated at every finite subset B of $\mathbb{N}_0$ by a function in C. Let $a := \max(B)$. Then $g_{a+1}|_B = p|_B$. This completes the proof of (4.2). Now we show

$$\overline{C}^{[1]} = C^{[1]} \cup \{p\}. \tag{4.3}$$

We only have to establish $\subseteq$. It is helpful to write down the list of values of some of the functions in $M \cup \{p\}$.

c_3	333333...
c_2	222222...
c_1	111111...
c_0	000000...
p	010101...
id	012345...
g_3	010345...
g_4	010145...
g_5	010105...

Let $f \in \overline{C}^{[1]}$ with $f \neq p$, and let $k \in \mathbb{N}_0$ be minimal with $f(k) \neq p(k)$. Let $g \in C^{[1]}$ be such that $g|_{\{0,\dots,k\}} = f|_{\{0,\dots,k\}}$. We distinguish three cases.

Case $k = 0$: Then $g(0) \neq 0$, and therefore $g = c_{g(0)}$. If $f = c_{g(0)}$, we have $f \in C$. If $f \neq c_{g(0)}$, we let y be minimal with $f(y) \neq g(0)$. We interpolate f at $\{0, y\}$ by a function $h \in C$. This function h is not constant and satisfies $h(0) \neq 0$. Such a function does not exist in C, therefore the case $f \neq c_{g(0)}$ cannot occur.

Case $k = 1$: Then $g(1) \neq 1$. By examining the functions in M, we see that $g = c_0$. If $f = c_0$, we have $f \in C$. If $f \neq c_0$, we let y be minimal with $f(y) \neq 0$. Interpolating f at $\{0, 1, y\}$ by $h \in C$, we obtain a function $h \in C$ with $h(0) = h(1) = 0$ and $h(y) \neq 0$. Such a function does not exist in C; this contradiction shows $f = c_0$ and therefore $f \in C$.

Case $k \geq 2$: Then $g = g_k$. If $f = g_k$, then $f \in C$. If $f \neq g_k$, we choose y minimal with $f(y) \neq g_k(y)$ and interpolate f at $\{0, 1, \dots, k\} \cup \{y\}$ by a function $h \in C$. Again, such a function is not available in C, and therefore $f = g_k \in C$.

Thus, every $f \in \overline{C}^{[1]}$ with $f \neq p$ is an element of C. By its definition, C contains all constant unary operations in $\mathbb{N}_0$. Since C preserves the relation

$\rho = \{(a,b,c,d) \in A^4 \mid a = b \text{ or } c = d\}$, $\overline{C}$ also preserves ρ. Therefore, by [10, Lemma 1.3.1(a)], every function in $\overline{C}$ is essentially unary and hence of the form $l(x_1,\dots,x_n) = f(x_j)$ with $n \in \mathbb{N}$, $j \in \{1,\dots,n\}$, and $f \in \overline{C}^{[1]} = M \cup \{p\}$. This implies that $\overline{C}$ is countable. The function p witnesses $C \neq \overline{C}$. □

5. Constantive clones

In constantive clones, a finite base of equality for the functions of arity m yields finite bases of equality for all other arities. This will allow us to refine Theorem 1.3.

Lemma 5.1. *Let C be a clone on the set A, let $m \in \mathbb{N}$, and let $D \subseteq A^m$ be a base of equality for $C^{[m]}$. Then the projection of D to the first component $\pi_1(D)$ is a base of equality for $C^{[1]}$.*

Proof. Let $f, g \in C^{[1]}$ with $f|_{\pi_1(D)} = g|_{\pi_1(D)}$. Let $f_1(x_1,\dots,x_m) := f(x_1)$ and $g_1(x_1,\dots,x_m) := g(x_1)$. Then for every $(d_1,\dots,d_m) \in D$, we have $f_1(d_1,\dots,d_m) = f(d_1) = g(d_1) = g_1(d_1,\dots,d_m)$, and therefore $f_1 = g_1$, which implies $f = g$. □

Lemma 5.2. *Let A be a set, let C be a constantive clone on A, and let $D \subseteq A$ be a base of equality for $C^{[1]}$. Then for every $n \in \mathbb{N}$, D^n is a base of equality for $C^{[n]}$.*

Proof. We proceed by induction on n. If $n = 1$, $D^1 = D$ is a base of equality of $C^{[1]}$ by assumption. For the induction step, let $n \geq 2$, and suppose that D^{n-1} is a base of equality for $C^{[n-1]}$. Let $f, g \in C^{[n]}$ and assume $f|_{D^n} = g|_{D^n}$. We first show

$$f|_{A\times D^{n-1}} = g|_{A\times D^{n-1}}. \tag{5.1}$$

Let $(a, d_2,\dots,d_n) \in A \times D^{n-1}$, and define $f_1(x) := f(x, d_2,\dots,d_n)$ and $g_1(x) := g(x, d_2,\dots,d_n)$ for $x \in A$. Then $f_1, g_1 \in C^{[1]}$ and $f_1|_D = g_1|_D$. Hence, $f_1 = g_1$, and thus $f(a, d_2,\dots,d_n) = f_1(a) = g_1(a) = g(a, d_2,\dots,d_n)$, which completes the proof of (5.1).

We prove $f = g$. Let $(b_1,\dots,b_n) \in A^n$; for all $x_2,\dots,x_n \in A$, define $f_2(x_2,\dots,x_n) := f(b_1, x_2,\dots,x_n)$ and $g_2(x_2,\dots,x_n) := g(b_1, x_2,\dots,x_n)$. By (5.1), $f_2|_{D^{n-1}} = g_2|_{D^{n-1}}$, and thus by the induction hypothesis, $f_2 = g_2$. Thus, $f(b_1,\dots,b_n) = g(b_1,\dots,b_n)$. □

Hence, for constantive clones we can give the following slight refinement of Theorem 1.3.

Theorem 5.3. *Let A be a set with $|A| = \aleph_0$, let C be a constantive clone on A with quasigroup operations, and let $m \in \mathbb{N}$. Then the following are equivalent:*

(1) $|(\mathrm{Pol}\,\mathrm{Inv}\,C)^{[1]}| \leq \aleph_0$.
(2) *$C^{[1]}$ has a finite base of equality.*
(3) *$C^{[m]}$ has a finite base of equality.*

(4) $|C| \leq \aleph_0$ *and* $\exists d \in \mathbb{N}\ \forall n \in \mathbb{N} : C^{[n]} = (\mathrm{Pol\ Inv}^{[d^n+1]} C)^{[n]}$.
(5) $|C| \leq \aleph_0$ *and* $\forall n \in \mathbb{N}\ \exists k \in \mathbb{N} : C^{[n]} = (\mathrm{Pol\ Inv}^{[k]} C)^{[n]}$.
(6) $|C| \leq \aleph_0$ *and* $C = \mathrm{Pol\ Inv}\, C$.

Proof. (1)⇒(2): This is Lemma 2.1.

(2)⇒(3): This is Lemma 5.2.

(3)⇒(2): This is Lemma 5.1.

(2)⇒(4): Let D be a finite base of equality for $C^{[1]}$. Let $n \in \mathbb{N}$, and set $k := |D|^n + 1$. By Lemma 5.2, D^n is a base of equality for $C^{[n]}$, and Lemma 2.2 yields $C^{[n]} = (\mathrm{Pol\ Inv}^{[k]} C)^{[n]}$. Since D^n is a finite base of equality, the mapping $f \mapsto f|_{D^n}$ is an injective mapping from $C^{[n]}$ to A^{D^n}, making $C^{[n]}$ countable. Since $C^{[n]}$ is countable for every $n \in \mathbb{N}$, we obtain $|C| \leq \aleph_0$.

(4)⇒(5): Set $k := d^n + 1$.

(5)⇒(6): Let $n \in \mathbb{N}$, and let k be produced by (5). Then we have that $(\mathrm{Pol\ Inv}\, C)^{[n]} \subseteq (\mathrm{Pol\ Inv}^{[k]} C)^{[n]} = C^{[n]}$.

(6)⇒(1): We have $(\mathrm{Pol\ Inv}\, C)^{[1]} \subseteq \mathrm{Pol\ Inv}\, C \subseteq C$. □

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References

[1] Aichinger, E.: Local polynomial functions on the integers. Riv. Mat. Univ. Parma (5) **6**, 169–177 (1997)
[2] Aichinger, E.: The structure of composition algebras. Ph.D. thesis, Johannes Kepler Universität Linz (1998). Available at `www.algebra.uni-linz.ac.at/~erhard/Diss/`
[3] Burris, S., Sankappanavar, H.P.: A course in universal algebra. Springer New York Heidelberg Berlin (1981)
[4] Eigenthaler, G.: Einige Bemerkungen über Clones und interpolierbare Funktionen auf universellen Algebren. Beiträge Algebra Geom. **15**, 121–127 (1983)
[5] Goldstern, M., Pinsker, M.: A survey of clones on infinite sets. Algebra Universalis **59**, 365–403 (2008)
[6] Hall, R.R.: On pseudo-polynomials. Mathematika **18**, 71–77 (1971)
[7] Hule, H., Nöbauer, W.: Local polynomial functions on universal algebras. Anais da Acad. Brasiliana de Ciencias **49**, 365–372 (1977)
[8] Kaarli, K.: Affine complete abelian groups. Math.Nachr. **107**, 235–239 (1982)
[9] Pöschel, R.: A general Galois theory for operations and relations and concrete characterization of related algebraic structures, Report 1980, vol. 1. Akademie der Wissenschaften der DDR, Institut für Mathematik, Berlin (1980)
[10] Pöschel, R., Kalužnin, L.A.: Funktionen- und Relationenalgebren, Mathematische Monographien, vol. 15. VEB Deutscher Verlag der Wissenschaften, Berlin (1979)
[11] Szendrei, Á.: Clones in universal algebra. Séminaire de Mathématiques Supérieures [Seminar on Higher Mathematics], vol. 99. Presses de l'Université de Montréal, Montreal, QC (1986)

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