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Categorical equivalence and the Ramsey property for finite powers of a primal algebra

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Abstract. In this paper, we investigate the best known and most important example of a categorical equivalence in algebra, that between the variety of boolean algebras and any variety generated by a single primal algebra. We consider this equivalence in the context of Kechris-Pestov-Todorčević correspondence, a surprising correspondence between model theory, combinatorics and topological dynamics. We show that relevant combinatorial properties (such as the amalgamation property, Ramsey property and ordering property) carry over from a category to an equivalent category. We then use these results to show that the category whose objects are isomorphic copies of finite powers of a primal algebra $\mathcal{A}$ together with a particular linear ordering $<$, and whose morphisms are embeddings, is a Ramsey age (and hence a Fraïssé age). By the Kechris-Pestov-Todorčević correspondence, we then infer that the automorphism group of its Fraïssé limit is extremely amenable. This correspondence also enables us to compute the universal minimal flow of the Fraïssé limit of the class $\mathbf{V}_{\mathit{fin}}(\mathcal{A})$ whose objects are isomorphic copies of finite powers of a primal algebra $\mathcal{A}$ and whose morphisms are embeddings.

1. Introduction

In this paper, we investigate the best known and most important example of a categorical equivalence in algebra, that between the variety of boolean algebras and any variety generated by a single primal algebra [7, 8] (which is a finite algebra where all operations are term operations) but in the context of the Kechris-Pestov-Todorčević correspondence, a surprising correspondence between model theory, combinatorics and topological dynamics published in 2005 in [9]. For a locally finite countable ultrahomogeneous structure $\mathcal{F}$, the paper [9] establishes a correspondence between combinatorial properties of $\mathrm{Age}(\mathcal{F})$, the class of finite substructures of $\mathcal{F}$, and dynamical properties of $\mathrm{Aut}(\mathcal{F})$. The main result of [9] states that if $\mathcal{F}$ is a locally finite countable ultrahomogeneous structure, then $\mathrm{Aut}(\mathcal{F})$ is extremely amenable if and only if $\mathrm{Age}(\mathcal{F})$ consists of rigid objects and has the Ramsey property. In case $\mathrm{Aut}(\mathcal{F})$ is not extremely amenable, [9] offers a technique to compute its universal

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minimal flow in case the structure $\mathcal{F}$ can be expanded by a linear order $<$ in a particular way.

Our main result claims that the category $\mathbf{OV}_{\mathit{fin}}(\mathcal{A}, <)$, whose objects are (isomorphic copies of) finite powers of a primal algebra $\mathcal{A}$ together with a particular linear ordering $<$ and whose morphisms are embeddings, is categorically equivalent to the category of naturally ordered finite boolean algebras (defined later in the paper). As a consequence, we immediately get that $\mathbf{OV}_{\mathit{fin}}(\mathcal{A}, <)$ is a Ramsey age (and hence a Fraïssé age), whence follows, by the Kechris-Pestov-Todorčević correspondence, that the automorphism group of its Fraïssé limit is extremely amenable. The Kechris-Pestov-Todorčević correspondence also enables us to compute the universal minimal flow of the Fraïssé limit of the class $\mathbf{V}_{\mathit{fin}}(\mathcal{A})$ whose objects are (isomorphic copies of) finite powers of a primal algebra $\mathcal{A}$ and whose morphisms are embeddings. Note that this Fraïssé limit belongs to $\mathbf{V}(\mathcal{A})$, the variety generated by $\mathcal{A}$.

As our main tools come from category theory, we recall in Section 2 basic facts of category theory, structural Ramsey theory, Fraïssé theory and the Kechris-Pestov-Todorčević correspondence. We present the basic notions of structural Ramsey theory in the language of category theory as it is evident that the Ramsey property for a class of objects depends not only on the choice of objects, but also on the choice of morphisms involved (see [3, 11, 13, 15, 19, 20, 10]).

In Section 3, we show that relevant combinatorial properties carry over from a category to an equivalent category. More specifically, we prove that the Ramsey property is preserved under categorical equivalence (we consider two incarnations of the Ramsey property and prove that both are genuine categorical properties, but this line of thought will not be pursued further in this paper). As a corollary, we conclude that categorical equivalence preserves the property of being a Ramsey age. Finally, we show that the ordering property is preserved under a particular form of equivalence consisting of a pair of categorical equivalences: one for the category of base objects and one for the category of order expansions. These are the three ingredients that are required to infer the combinatorial and dynamical properties of the class of finite powers of a primal algebra and the automorphism group of its Fraïssé limit.

In Section 4, we apply the tools developed in Section 3 to the categorical equivalence between the variety of boolean algebras and any variety generated by a single primal algebra to obtain the main results of the paper, as discussed above.

We close the paper with two appendices. The first one (Section 5) is a spin-off of Section 3 and investigates the invariance of the Ramsey property under adjunctions. We show that right adjoints preserve the Ramsey property, while left adjoints preserve the dual Ramsey property.

The second appendix (Section 6) contains a discussion of Fraïssé limits with identical automorphism group. The principal motivation for this section is the following result from [9] which states that the Ramsey property is invariant under certain model-theoretic constructions and which is a special case of our results in Section 3:

Theorem 1.1. [9, Proposition 9.1 (i)] *Let* $\mathbf{K}_0$ *be a Fraïssé class in a signature* L_0*, let* $L = L_0 \cup \{<\}$ *and let* $\mathbf{K}$*,* $\mathbf{K}'$ *be reasonable Fraïssé order classes in* L *that are expansions of* $\mathbf{K}_0$*. Assume that* $\mathbf{K}$ *and* $\mathbf{K}'$ *are simply bi-definable. Then* $\mathbf{K}$ *satisfies the Ramsey property if and only if* $\mathbf{K}'$ *satisfies the Ramsey property.*

2. Preliminaries

2.1. Categories and structures. In order to specify a category $\mathbf{C}$, one has to specify a class of objects $\mathrm{Ob}(\mathbf{C})$, a set of morphisms $\hom_{\mathbf{C}}(\mathcal{A},\mathcal{B})$ for all $\mathcal{A},\mathcal{B} \in \mathrm{Ob}(\mathbf{C})$, an identity morphism $\mathrm{id}_{\mathcal{A}}$ for all $\mathcal{A} \in \mathrm{Ob}(\mathbf{C})$, and the composition of morphisms $\cdot$ so that

- $(f \cdot g) \cdot h = f \cdot (g \cdot h)$, and
- $\mathrm{id}_{\mathcal{B}} \cdot f = f \cdot \mathrm{id}_{\mathcal{A}}$ for all $f \in \hom_{\mathbf{C}}(\mathcal{A},\mathcal{B})$.

Let $\mathrm{Aut}(\mathcal{A})$ denote the set of all invertible morphisms $\mathcal{A} \to \mathcal{A}$. Recall that an object $\mathcal{A} \in \mathrm{Ob}(\mathbf{C})$ is *rigid* if $\mathrm{Aut}(\mathcal{A}) = \{\mathrm{id}_{\mathcal{A}}\}$.

For $\mathcal{A},\mathcal{B} \in \mathrm{Ob}(\mathbf{C})$ we write $\mathcal{A} \to \mathcal{B}$ to denote that $\hom_{\mathbf{C}}(\mathcal{A},\mathcal{B}) \neq \varnothing$. Note that morphisms in $\hom_{\mathbf{C}}(\mathcal{A},\mathcal{B})$ are not necessarily structure-preserving mappings from $\mathcal{A}$ to $\mathcal{B}$, and that the composition $\cdot$ in a category is not necessarily composition of mappings. We shall see examples later. Instead of $\hom_{\mathbf{C}}(\mathcal{A},\mathcal{B})$ we write $\hom(\mathcal{A},\mathcal{B})$ whenever $\mathbf{C}$ is obvious from the context.

In this paper, we are mostly interested in categories of structures. A *structure* $\mathcal{A} = (A, \Delta)$ is a set A together with a set Δ of functions and relations on A, each having some finite arity. An embedding $f\colon \mathcal{A} \to \mathcal{B}$ is an injection $f\colon A \to B$ which respects the functions in Δ, and respects and reflects the relations in Δ. Surjective embeddings are isomorphisms. A structure $\mathcal{A}$ is a *substructure* of a structure $\mathcal{B}$ ($\mathcal{A} \leqslant \mathcal{B}$) if the identity map is an embedding of $\mathcal{A}$ into $\mathcal{B}$. Here is some further notation and terminology. A structure $\mathcal{A} = (A, \Delta)$ is *finite* if A is a finite set. The underlying set of a structure $\mathcal{A}, \mathcal{A}_1, \mathcal{A}^*, \ldots$ will always be denoted by its roman letter $A, A_1, A^*, \ldots$, respectively. We say that a structure $\mathcal{A} = (A, \Delta)$ is *ordered* if there is a binary relation $<$ in Δ which linearly orders A. Given a structure $\mathcal{A} = (A, \Delta)$ and a linear ordering $<$ on A, we write $\mathcal{A}_<$ for the structure $(A, \Delta, <)$. Moreover, we shall always write $\mathcal{A}$ to denote the unordered reduct of $\mathcal{A}_<$. Linear orders denoted by $<$, $\sqsubset$ etc. are irreflexive (strict linear orders), whereas by $\leqslant$, $\sqsubseteq$ etc. we denote the corresponding reflexive linear orders.

2.2. Adjunction, equivalence and isomorphism of categories. A pair of functors $F\colon \mathbf{C} \rightleftarrows \mathbf{D} : G$ is an *adjunction* provided there is a family of isomorphisms $\Phi_{\mathcal{C},\mathcal{D}}\colon \hom_{\mathbf{D}}(F(\mathcal{C}),\mathcal{D}) \cong \hom_{\mathbf{C}}(\mathcal{C}, G(\mathcal{D}))$ natural in both $\mathcal{C}$ and $\mathcal{D}$.

We say that F is *left adjoint* to G and G is *right adjoint* to F. Every adjunction $F\colon \mathbf{C} \rightleftarrows \mathbf{D}\colon G$ gives rise to two natural transformations $\eta\colon \mathrm{ID}_\mathbf{C} \to GF$ and $\varepsilon\colon FG \to \mathrm{ID}_\mathbf{D}$ referred to as *unit* and *counit*, respectively, satisfying the so-called *unit-counit identities* $\varepsilon F \cdot F\eta = \mathrm{id}_F$ and $G\varepsilon \cdot \eta G = \mathrm{id}_G$. If $f\colon F(\mathcal{C}) \to \mathcal{D}$ and $g\colon \mathcal{C} \to G(\mathcal{D})$ are morphisms in $\mathbf{D}$ and $\mathbf{C}$, respectively, then $\Phi(f) = G(f) \cdot \eta_\mathcal{C}$ and $\Phi^{-1}(g) = \varepsilon_\mathcal{D} \cdot F(g)$.

Categories $\mathbf{C}$ and $\mathbf{D}$ are *isomorphic* if there exist functors $E\colon \mathbf{C} \to \mathbf{D}$ and $H\colon \mathbf{D} \to \mathbf{C}$ such that H is the inverse of E. A functor $E\colon \mathbf{C} \to \mathbf{D}$ is isomorphism-dense if for every $D \in \mathrm{Ob}(\mathbf{D})$ there is a $C \in \mathrm{Ob}(\mathbf{C})$ such that $E(C) \cong D$.

Categories $\mathbf{C}$ and $\mathbf{D}$ are *equivalent* if there exist functors $E\colon \mathbf{C} \to \mathbf{D}$ and $H\colon \mathbf{D} \to \mathbf{C}$, and natural isomorphisms $\eta\colon \mathrm{ID}_\mathbf{C} \to HE$ and $\varepsilon\colon \mathrm{ID}_\mathbf{D} \to EH$. We say that H is a pseudoinverse of E and vice versa. It is a well-known fact that a functor $E\colon \mathbf{C} \to \mathbf{D}$ has a pseudoinverse if and only if it is full, faithful and isomorphism-dense. If E has a pseudoinverse then $\mathbf{C}$ and $\mathbf{D}$ are equivalent. Clearly, categorical equivalence is a particular form of adjunction.

A *skeleton* of a category is a full, isomorphism-dense subcategory in which no two distinct objects are isomorphic. It is easy to see that (assuming (AC)) every category has a skeleton. It is also a well-known fact that two categories are equivalent if and only if they have isomorphic skeletons. Categories $\mathbf{C}$ and $\mathbf{D}$ are *dually equivalent* if $\mathbf{C}$ and $\mathbf{D}^{\mathrm{op}}$ are equivalent.

2.3. The Ramsey property for categories. We say $\mathcal{S} = \mathcal{M}_1 \cup \dots \cup \mathcal{M}_k$ is a k-*coloring* of $\mathcal{S}$ if $\mathcal{M}_i \cap \mathcal{M}_j = \varnothing$ whenever $i \neq j$. Equivalently, a k-coloring of $\mathcal{S}$ is a mapping $\chi\colon \mathcal{S} \to \{1, 2, \dots, k\}$. We shall use both points of view as we find appropriate.

Given a category $\mathbf{C}$ define $\sim_\mathcal{A}$ on $\hom(\mathcal{A}, \mathcal{B})$ by: for $f, f' \in \hom(\mathcal{A}, \mathcal{B})$ we let $f \sim_\mathcal{A} f'$ if $f' = f \cdot \alpha$ for some $\alpha \in \mathrm{Aut}(\mathcal{A})$. Then let

$$\binom{\mathcal{B}}{\mathcal{A}} = \hom(\mathcal{A}, \mathcal{B})/{\sim_\mathcal{A}}.$$

In case $\mathbf{C}$ is a category whose objects are structures and morphisms are embeddings, $\binom{\mathcal{B}}{\mathcal{A}}$ corresponds to all the subobjects of $\mathcal{B}$ isomorphic to $\mathcal{A}$ (see [11, 12]). For an integer $k \geqslant 2$ and $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathrm{Ob}(\mathbf{C})$ we write $\mathcal{C} \longrightarrow (\mathcal{B})^\mathcal{A}_k$ to denote that $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ and for every k-coloring $\binom{\mathcal{C}}{\mathcal{A}} = \mathcal{M}_1 \cup \dots \cup \mathcal{M}_k$, there is an $i \in \{1, \dots, k\}$ and a morphism $w\colon \mathcal{B} \to \mathcal{C}$ such that $w \cdot \binom{\mathcal{B}}{\mathcal{A}} \subseteq \mathcal{M}_i$. (Note that $w \cdot (f/{\sim_\mathcal{A}}) = (w \cdot f)/{\sim_\mathcal{A}}$ for $f/{\sim_\mathcal{A}} \in \binom{\mathcal{B}}{\mathcal{A}}$.)

We write $\mathcal{C} \overset{hom}{\longrightarrow} (\mathcal{B})^\mathcal{A}_k$ to denote that $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ in $\mathbf{C}$ and for every k-coloring $\hom(\mathcal{A}, \mathcal{C}) = \mathcal{M}_1 \cup \dots \cup \mathcal{M}_k$, there is an $i \in \{1, \dots, k\}$ and a morphism $w\colon \mathcal{B} \to \mathcal{C}$ such that $w \cdot \hom(\mathcal{A}, \mathcal{B}) \subseteq \mathcal{M}_i$.

A category $\mathbf{C}$ has the *Ramsey property for objects* if for every integer $k \geqslant 2$ and all $\mathcal{A}, \mathcal{B} \in \mathrm{Ob}(\mathbf{C})$ such that $\mathcal{A} \to \mathcal{B}$ there is a $\mathcal{C} \in \mathrm{Ob}(\mathbf{C})$ such that $\mathcal{C} \longrightarrow (\mathcal{B})^\mathcal{A}_k$. A category $\mathbf{C}$ has the *Ramsey property for morphisms* if for every integer $k \geqslant 2$ and all $\mathcal{A}, \mathcal{B} \in \mathrm{Ob}(\mathbf{C})$ such that $\mathcal{A} \to \mathcal{B}$ there is a $\mathcal{C} \in \mathrm{Ob}(\mathbf{C})$ such that $\mathcal{C} \overset{hom}{\longrightarrow} (\mathcal{B})^{\mathcal{A}}_k$.

In a category of finite ordered structures, all the relations $\sim_{\mathcal{A}}$ are trivial and the two Ramsey properties coincide. Therefore, we say that a category of finite ordered structures and embeddings has the *Ramsey property* if it has the Ramsey property for morphisms.

Example 2.1. The category **FSI** of finite sets and injective maps has the Ramsey property for objects. This is just a reformulation of the Finite Ramsey Theorem [16]:

> For all positive integers k, a, m there is a positive integer n such that for every n-element set C and every k-coloring of the set $\binom{C}{a}$ of all a-element subsets of C there is an m-element subset B of C such that $\binom{B}{a}$ is monochromatic.

A category $\mathbf{C}$ has the *dual Ramsey property for objects (morphisms)* if $\mathbf{C}^{\mathrm{op}}$ has the Ramsey property for objects (morphisms).

Example 2.2. The category **FSS** of finite sets and surjective maps has the dual Ramsey property for objects. This is just a reformulation of the Finite Dual Ramsey Theorem [4]:

> For all positive integers k, a, m there is a positive integer n such that for every n-element set C and every k-coloring of the set $\begin{bmatrix} C \\ a \end{bmatrix}$ of all partitions of C with exactly a blocks there is a partition β of C with exactly m blocks such that the set of all partitions from $\begin{bmatrix} C \\ a \end{bmatrix}$ which are coarser than β is monochromatic.

We can show that the Ramsey property for objects and the Ramsey property for morphisms are closely related for categories where all the morphisms are monic (that is, left cancellable; compare with [20]). The assumption of rigidity below was pointed out in [11].

Proposition 2.3. *Let* $\mathbf{C}$ *be a category where morphisms are monic. If* $\mathbf{C}$ *has the Ramsey property for morphisms, then all the objects in* $\mathbf{C}$ *are rigid. Consequently, a category* $\mathbf{C}$ *has the Ramsey property for morphisms if and only if all the objects in* $\mathbf{C}$ *are rigid and* $\mathbf{C}$ *has the Ramsey property for objects.*

Proof. Assume that $\mathcal{A} \in \mathrm{Ob}(\mathbf{C})$ is not rigid and let $\alpha \in \mathrm{Aut}(\mathcal{A})$ be an automorphism of $\mathcal{A}$ such that $\alpha \neq \mathrm{id}_{\mathcal{A}}$. In order to show that $\mathbf{C}$ does not have the Ramsey property for morphisms, take any $\mathcal{C} \in \mathrm{Ob}(\mathbf{C})$ and let us show that $\mathcal{C} \not\longrightarrow (\mathcal{A})^{\mathcal{A}}_2$.

Let $\langle \alpha \rangle$ be the cyclic group generated by α. Then $|\langle \alpha \rangle| \geqslant 2$ because $\alpha \neq \mathrm{id}_{\mathcal{A}}$. Let $\langle \alpha \rangle$ act on $\hom(\mathcal{A}, \mathcal{C})$ by $h^\alpha = h \cdot \alpha$. The orbits of this action are of the form $h \cdot \langle \alpha \rangle$, where $h \in \hom(\mathcal{A}, \mathcal{C})$. It follows that $|h \cdot \langle \alpha \rangle| = |\langle \alpha \rangle| \geqslant 2$ because h is monic.

Let $\chi \colon \hom(\mathcal{A}, \mathcal{C}) \to 2$ be any coloring of $\hom(\mathcal{A}, \mathcal{C})$ such that χ assumes both colors on each orbit of the action of $\langle \alpha \rangle$ on $\hom(\mathcal{A}, \mathcal{C})$. Then for every

$w\colon \mathcal{A} \to \mathcal{C}$ we have that $|\chi(w \cdot \hom_{\mathbf{C}}(\mathcal{A}, \mathcal{A}))| \geqslant |\chi(w \cdot \langle\alpha\rangle)| = 2$ because $w \cdot \langle\alpha\rangle \subseteq w \cdot \hom(\mathcal{A}, \mathcal{A})$ and χ assumes both colors on each orbit. □

The following are easy lemmas:

Lemma 2.4. *(a) If* $\mathcal{C} \overset{hom}{\longrightarrow} (\mathcal{B})^{\mathcal{A}}_k$ *and* $\mathcal{B}_1 \to \mathcal{B}$*, then* $\mathcal{C} \overset{hom}{\longrightarrow} (\mathcal{B}_1)^{\mathcal{A}}_k$*.*
(b) If $\mathcal{C} \longrightarrow (\mathcal{B})^{\mathcal{A}}_k$ *and* $\mathcal{B}_1 \to \mathcal{B}$*, then* $\mathcal{C} \longrightarrow (\mathcal{B}_1)^{\mathcal{A}}_k$*.*
(c) If $\mathcal{C} \overset{hom}{\longrightarrow} (\mathcal{B})^{\mathcal{A}}_k$ *and* $\mathcal{C} \to \mathcal{D}$*, then* $\mathcal{D} \overset{hom}{\longrightarrow} (\mathcal{B})^{\mathcal{A}}_k$*.*
(d) If $\mathcal{C} \longrightarrow (\mathcal{B})^{\mathcal{A}}_k$ *and* $\mathcal{C} \to \mathcal{D}$*, then* $\mathcal{D} \longrightarrow (\mathcal{B})^{\mathcal{A}}_k$*.*

Lemma 2.5. *Let* $\mathbf{C}$ *be a category whose morphisms are monic. If* $\mathbf{C}$ *has the Ramsey property for morphisms (objects) and* $\mathbf{D}$ *is a full subcategory of* $\mathbf{C}$ *such that* $\mathrm{Ob}(\mathbf{D})$ *is cofinal in* $\mathrm{Ob}(\mathbf{C})$*, then* $\mathbf{D}$ *has the Ramsey property for morphisms (objects).*

2.4. Fraïssé theory. For a countable structure $\mathcal{M}$, the class of all finitely generated substructures of $\mathcal{M}$ is is denoted by $\mathrm{Age}(\mathcal{M})$ and is called the *age* of $\mathcal{M}$. A class $\mathbf{K}$ of finite structures is an *age* if there is countable structure $\mathcal{M}$ such that $\mathbf{K} = \mathrm{Age}(\mathcal{M})$. It is a well-known result that a class $\mathbf{K}$ of finite structures is an age if and only if the following hold:

- $\mathbf{K}$ is an abstract class (that is, it is closed for isomorphisms);
- there are at most countably many pairwise nonisomorphic structures in $\mathbf{K}$,
- $\mathbf{K}$ has the *hereditary property (HP)*: if $\mathcal{A} \in \mathbf{K}$ and $\mathcal{B} \hookrightarrow \mathcal{A}$ then $\mathcal{B} \in \mathbf{K}$; and
- $\mathbf{K}$ has the *joint embedding property (JEP)*: for all $\mathcal{A}, \mathcal{B} \in \mathbf{K}$ there is a $\mathcal{C} \in \mathbf{K}$ such that $\mathcal{A} \hookrightarrow \mathcal{C}$ and $\mathcal{B} \hookrightarrow \mathcal{C}$.

An age $\mathbf{K}$ is a *Fraïssé age* (= Fraïssé class = amalgamation class) if $\mathbf{K}$ satisfies the *amalgamation property (AP)*: for all $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbf{K}$ and embeddings $f\colon \mathcal{A} \hookrightarrow \mathcal{B}$ and $g\colon \mathcal{A} \hookrightarrow \mathcal{C}$, there exist $\mathcal{D} \in \mathbf{K}$ and embeddings $f'\colon \mathcal{B} \hookrightarrow \mathcal{D}$ and $g'\colon \mathcal{C} \hookrightarrow \mathcal{D}$ such that $f' \circ f = g' \circ g$.

A countable structure $\mathcal{M}$ is *ultrahomogeneous* if partial isomorphisms between finite substructures lift to an automorphism of the entire structure. In other words, for any tuple $\overline{a}$ from $\mathcal{M}$, the orbit of $\overline{a}$ under $\mathrm{Aut}(\mathcal{M})$ is defined by the quantifier-free type of $\overline{a}$. An $L_{\omega_1,\omega}$ formula is a formula built out of basic relations, countable conjunctions/disjunctions and negations (called a *simple* formula in [9].) In general, the orbit of a finite tuple is defined by a quantifier-free $L_{\omega_1,\omega}$-formula, using Scott sentences.

For every Fraïssé age $\mathbf{K}$, there is a unique (up to isomorphism) countable ultrahomogeneous structure $\mathcal{A}$ such that $\mathbf{K} = \mathrm{Age}(\mathcal{A})$. We say that $\mathcal{A}$ is the *Fraïssé limit* of $\mathbf{K}$, denoted $\mathrm{Flim}\,\mathbf{K}$. For further model theoretic background, see [6].

If $\mathbf{K}$ is a Ramsey class of finite ordered structures which is closed under isomorphisms and taking substructures, and has the joint embedding property, then **K** is a Fraïssé age [11]. In that case, we say that **K** is a *Ramsey age*. So, every Ramsey age is a Fraïssé age.

2.5. The Kechris-Pestov-Todorčević correspondence. Let G be a topological group. Its *action* on X is a mapping $\cdot : G \times X \to X$ such that $1 \cdot x = x$ and $g \cdot (f \cdot x) = (gf) \cdot x$. We also say that G *acts* on X. A *G-flow* is a continuous action of a topological group G on a topological space X. A *subflow* of a G-flow $\cdot : G \times X \to X$ is a continuous map $*\colon G \times Y \to Y$ where $Y \subseteq X$ is a closed subspace of X and $g * y = g \cdot y$ for all $g \in G$ and $y \in Y$. A G-flow $G \times X \to X$ is *minimal* if it has no proper closed subflows. A G-flow $u\colon G \times X \to X$ is *universal* if every compact minimal G-flow $G \times Z \to Z$ is a factor of u. It is a well-known fact that for a compact Hausdorff space X there is, up to isomorphism of G-flows, a unique universal minimal G-flow, usually denoted by $G \curvearrowright M(G)$.

A topological group G is *extremely amenable* if every G-flow $\cdot\colon G \times X \to X$ on a compact Hausdorff space X has a fixed point, that is, there is an $x_0 \in X$ such that $g \cdot x_0 = x_0$ for all $g \in G$. Since $\mathrm{Sym}(A)$ carries naturally the topology of pointwise convergence, permutation groups can be thought of as topological groups. For example, it was shown in [14] that $\mathrm{Aut}(\mathbb{Q}, <)$ is extremely amenable while $\mathrm{Sym}(A)$, the group of all permutations on A, is not, for a countably infinite set A. In [9], the authors show the following.

Theorem 2.6. [9, Theorem 4.7] *Let G be a closed subgroup of* $\mathrm{Sym}(F)$ *for a countable set F. Then G is extremely amenable if and only if $G = \mathrm{Aut}(\mathcal{F})$ for a countable homogeneous structure $\mathcal{F}$ whose age has the Ramsey property and consists of rigid elements.*

Let $\mathrm{LO}(A)$ be the set of all linear orders on A and let G be a closed subgroup of $\mathrm{Sym}(A)$. The set $\mathrm{LO}(A)$ with the standard product topology is a compact Hausdorff space and the action of G on $\mathrm{LO}(A)$ given by $x <^g y$ if and only if $g^{-1}(x) < g^{-1}(y)$ is continuous. This action is usually referred to as the *logical action of G on* $\mathrm{LO}(A)$.

Let **C** be a category of finite structures and embeddings, and $\mathbf{C}^*$ a category of finite ordered structures and embeddings. We say that $\mathbf{C}^*$ *is an order expansion of* **C** (cf. [9]) if

- for every structure $\mathcal{A}_< = (A, \Delta, <) \in \mathrm{Ob}(\mathbf{C}^*)$, we have that $\mathcal{A} = (A, \Delta) \in \mathrm{Ob}(\mathbf{C})$, and
- the forgetful functor $U\colon \mathbf{C}^* \to \mathbf{C}$, which acts on objects by $U(A, \Delta, <) = (A, \Delta)$ and on morphisms by $U(f) = f$, is surjective on objects.

An order expansion $\mathbf{C}^*$ of **C** is *reasonable* (cf. [9]) if for all $\mathcal{A}, \mathcal{B} \in \mathrm{Ob}(\mathbf{C})$, every embedding $f\colon \mathcal{A} \hookrightarrow \mathcal{B}$ and every $\mathcal{A}_< \in \mathrm{Ob}(\mathbf{C}^*)$ such that $U(\mathcal{A}_<) = \mathcal{A}$, there is a $\mathcal{B}_\sqsubset \in \mathrm{Ob}(\mathbf{C}^*)$ such that $U(\mathcal{B}_\sqsubset) = \mathcal{B}$ and f is an embedding of $\mathcal{A}_<$ into $\mathcal{B}_\sqsubset$. It is easy to show that if $\mathbf{C}^*$ is a reasonable expansion of **C** and $\mathbf{C}^*$ has (HP), resp. (JEP) or (AP), then **C** has (HP), resp. (JEP) or (AP) (cf. [9]); consequently, if $\mathrm{Ob}(\mathbf{C}^*)$ is a Fraïssé age, then so is $\mathrm{Ob}(\mathbf{C})$.

Let $\mathbf{C}^*$ be an order expansion of $\mathbf{C}$. We say that $\mathbf{C}^*$ has the *ordering property over* $\mathbf{C}$ if the following holds: for every $\mathcal{A} \in \mathrm{Ob}(\mathbf{C})$, there is a $\mathcal{B} \in \mathrm{Ob}(\mathbf{C})$ such that $\mathcal{A}_< \hookrightarrow \mathcal{B}_\sqsubset$ for all $\mathcal{A}_<, \mathcal{B}_\sqsubset \in \mathrm{Ob}(\mathbf{C}^*)$ such that $U(\mathcal{A}_<) = \mathcal{A}$ and $U(\mathcal{B}_\sqsubset) = \mathcal{B}$. We say that $\mathcal{B}$ is a *witness of the ordering property for* $\mathcal{A}$.

Theorem 2.7. [9, Theorem 10.8] *Let* $\mathbf{K}^*$ *be a Fraïssé age which is a reasonable order expansion of a Fraïssé age* $\mathbf{K}$. *Let* $\mathcal{F}$ *be the Fraïssé limit of* $\mathbf{K}$, *let* $\mathcal{F}_\sqsubset$ *be the Fraïssé limit of* $\mathbf{K}^*$, *let* $G = \mathrm{Aut}(\mathcal{F})$ *and* $X^* = \overline{G \cdot \sqsubset}$ *(in the logical action of* G *on* $\mathrm{LO}(F)$*). Then the logical action of* G *on* X^* *is the universal minimal flow of* G *if and only if the class* $\mathbf{K}^*$ *has the Ramsey property, as well as the ordering property with respect to* $\mathbf{K}$.

3. The Ramsey property, Fraïssé classes and order expansions under categorical equivalence

In this section, we set the stage for the results in Section 4. First, we prove that the Ramsey property for objects as well as the Ramsey property for morphisms are preserved under categorical equivalence (proving thus that both are genuine categorical properties, but this line of thought will not be pursued further in this paper). As a corollary, we conclude that categorical equivalence preserves the property of being a Ramsey age. Finally, we show that the ordering property is preserved under a particular form of equivalence consisting of a pair of categorical equivalences: one for the category of base objects and one for the category of order expansions. These are the three ingredients that are required to infer the combinatorial and dynamical properties of the class of finite powers of a primal algebra and the automorphism group of its Fraïssé limit.

Theorem 3.1. *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be equivalent categories. Then* $\mathbf{C}$ *has the Ramsey property for objects (morphisms) if and only if* $\mathbf{D}$ *does.*

In particular, if $\mathbf{C}$ *and* $\mathbf{D}$ *are dually equivalent and one of them has the Ramsey property for morphisms (objects), the other has the dual Ramsey property for morphisms (objects).*

Proof. Let us prove the statement in case of objects since the proof in case of morphisms is analogous. Let $E\colon \mathbf{C} \to \mathbf{D}$ and $H\colon \mathbf{D} \to \mathbf{C}$ be functors that constitute the equivalence between $\mathbf{C}$ and $\mathbf{D}$, and let $\eta\colon \mathrm{ID}_\mathbf{C} \to HE$ and $\varepsilon\colon \mathrm{ID}_\mathbf{D} \to EH$ be the accompanying natural isomorphisms. Assume that $\mathbf{D}$ has the Ramsey property, and let us show that $\mathbf{C}$ has the Ramsey property (the other direction is analogous). Take any positive integer k and let $\mathcal{A} \to \mathcal{B}$ in $\mathbf{C}$. Then $E(\mathcal{A}) \to E(\mathcal{B})$ in $\mathbf{D}$, so there is a $\mathcal{C} \in \mathrm{Ob}(\mathbf{D})$ such that

$$\mathcal{C} \longrightarrow (E(\mathcal{B}))^{E(\mathcal{A})}_k. \tag{3.1}$$

Let us show that $H(\mathcal{C}) \longrightarrow (\mathcal{B})^{\mathcal{A}}_k$ in $\mathbf{C}$. Note first that $\mathcal{B} \to H(\mathcal{C})$ because $E(\mathcal{B}) \to \mathcal{C}$, whence $\mathcal{B} \cong HE(\mathcal{B}) \to H(\mathcal{C})$. Let

$$\binom{H(\mathcal{C})}{\mathcal{A}} = \mathcal{M}_1 \cup \dots \cup \mathcal{M}_k$$

be an arbitrary k-coloring. Let $\mathcal{M}_i^E = \{E(f)/{\sim_{E(\mathcal{A})}} : f/{\sim_\mathcal{A}} \in \mathcal{M}_i\}$. Then it is easy to show that

$$\binom{EH(\mathcal{C})}{E(\mathcal{A})} = \mathcal{M}_1^E \cup \dots \cup \mathcal{M}_k^E$$

is a k-coloring. Having in mind that $\varepsilon_\mathcal{C} \colon \mathcal{C} \to EH(\mathcal{C})$ is an isomorphism, we have that

$$\binom{\mathcal{C}}{E(\mathcal{A})} = \varepsilon_\mathcal{C}^{-1} \cdot \mathcal{M}_1^E \cup \dots \cup \varepsilon_\mathcal{C}^{-1} \cdot \mathcal{M}_k^E$$

is also a k-coloring. From (3.1), we know that there is a $w \colon E(\mathcal{B}) \to \mathcal{C}$ and a color i such that

$$w \cdot \binom{E(\mathcal{B})}{E(\mathcal{A})} \subseteq \varepsilon_\mathcal{C}^{-1} \cdot \mathcal{M}_i^E. \tag{3.2}$$

Let $w^* = H(w) \cdot \eta_\mathcal{B} \colon \mathcal{B} \to H(\mathcal{C})$ and let us show that

$$w^* \cdot \binom{\mathcal{B}}{\mathcal{A}} \subseteq \mathcal{M}_i. \tag{3.3}$$

Take any $u/{\sim_\mathcal{A}} \in \binom{\mathcal{B}}{\mathcal{A}}$. Then

$$\begin{aligned} w^* \cdot (u/{\sim_\mathcal{A}}) &= (w^* \cdot u)/{\sim_\mathcal{A}} = (H(w) \cdot \eta_\mathcal{B} \cdot u)/{\sim_\mathcal{A}} \\ &= (H(w) \cdot HE(u) \cdot \eta_\mathcal{A})/{\sim_\mathcal{A}} = H(w) \cdot (HE(u)/{\sim_{HE(\mathcal{A})}}) \cdot \eta_\mathcal{A} \end{aligned}$$

because $\eta \colon \mathrm{ID}_\mathbf{C} \to HE$ is natural. On the other hand, (3.2) implies

$$H(w) \cdot \binom{HE(\mathcal{B})}{HE(\mathcal{A})} \subseteq H(\varepsilon_\mathcal{C}^{-1}) \cdot \mathcal{M}_i^{HE},$$

where

$$\begin{aligned} \mathcal{M}_i^{HE} &= \{HE(f)/{\sim_{HE(\mathcal{A})}} : E(f)/{\sim_{E(\mathcal{A})}} \in \mathcal{M}_i^E\} \\ &= \{HE(f)/{\sim_{HE(\mathcal{A})}} : f/{\sim_\mathcal{A}} \in \mathcal{M}_i\}. \end{aligned}$$

So, there is an $m/{\sim_\mathcal{A}} \in \mathcal{M}_i$ such that

$$\begin{aligned} w^* \cdot (u/{\sim_\mathcal{A}}) &= H(w) \cdot (HE(u)/{\sim_{HE(\mathcal{A})}}) \cdot \eta_\mathcal{A} \\ &= H(\varepsilon_\mathcal{C}^{-1}) \cdot (HE(m)/{\sim_{HE(\mathcal{A})}}) \cdot \eta_\mathcal{A} = (H(\varepsilon_\mathcal{C}^{-1}) \cdot HE(m) \cdot \eta_\mathcal{A})/{\sim_\mathcal{A}}. \end{aligned}$$

In order to complete the proof of (3.3), it suffices to note that $H(\varepsilon_\mathcal{C}^{-1}) \cdot HE(m) \cdot \eta_\mathcal{A} = m$ because every dual equivalence is a special dual adjunction. Therefore, $w^* \cdot (u/{\sim_\mathcal{A}}) = m/{\sim_\mathcal{A}} \in \mathcal{M}_i$. □

Example 3.2. The category **FSI** of finite sets and injective maps is dually equivalent to the category **FBAS** of finite boolean algebras and surjective homomorphisms (Stone duality). Since **FSI** has the Ramsey property for objects (Example 2.1), it follows that the category **FBAS** has the dual Ramsey property for objects.

Let us make this statement explicit. For $\mathcal{A}, \mathcal{B} \in \mathrm{Ob}(\mathbf{FBAS})$, let $\mathrm{Surj}(\mathcal{B}, \mathcal{A})$ denote the set of all surjective homomorphisms $\mathcal{B} \twoheadrightarrow \mathcal{A}$. Define $\equiv_\mathcal{A}$ on $\mathrm{Surj}(\mathcal{B}, \mathcal{A})$ as follows: for $f, f' \in \mathrm{Surj}(\mathcal{B}, \mathcal{A})$, we let $f \equiv_\mathcal{A} f'$ if $f' = \alpha \circ f$ for some $\alpha \in \mathrm{Aut}(\mathcal{A})$.

As in the Example 2.2, the fact that $\mathbf{FBAS}^{\mathrm{op}}$ has the Ramsey property for objects takes the following form: for every integer $k \geqslant 2$ and all finite boolean algebras $\mathcal{A}$ and $\mathcal{B}$ such that $\mathrm{Surj}(\mathcal{B}, \mathcal{A}) \neq \varnothing$, there is a finite boolean algebra $\mathcal{C}$ such that for every k-coloring

$$\mathrm{Surj}(\mathcal{C}, \mathcal{A})/{\equiv_\mathcal{A}} = \mathcal{M}_1 \cup \cdots \cup \mathcal{M}_k,$$

there is an $i \in \{1, \ldots, k\}$ and a surjective homomorphism $w \in \mathrm{Surj}(\mathcal{C}, \mathcal{B})$ satisfying $(\mathrm{Surj}(\mathcal{B}, \mathcal{A})/{\equiv_\mathcal{A}}) \circ w \subseteq \mathcal{M}_i$. Since $\mathrm{Surj}(\mathcal{B}, \mathcal{A})/{\equiv_\mathcal{A}}$ corresponds to congruences Φ of $\mathcal{B}$ such that $\mathcal{B}/\Phi \cong \mathcal{A}$, the above statement can be reformulated as follows:

Let $\mathrm{Con}(\mathcal{B})$ denote the set of congruences of an algebra $\mathcal{B}$, and for algebras $\mathcal{A}$ and $\mathcal{B}$ of the same type let

$$\mathrm{Con}(\mathcal{B}, \mathcal{A}) = \{\Phi \in \mathrm{Con}(\mathcal{B}) \colon \mathcal{B}/\Phi \cong \mathcal{A}\}.$$

For every finite boolean algebra $\mathcal{B}$, every $\Phi \in \mathrm{Con}(\mathcal{B})$, and every $k \geqslant 2$, there is a finite boolean algebra $\mathcal{C}$ such that for every k-coloring of $\mathrm{Con}(\mathcal{C}, \mathcal{B}/\Phi)$, there is a congruence $\Psi \in \mathrm{Con}(\mathcal{C}, \mathcal{B})$ such that the set of all the congruences from $\mathrm{Con}(\mathcal{C}, \mathcal{B}/\Phi)$ which contain Ψ is monochromatic.

Example 3.3. By Hu's theorem [7, 8], every variety generated by a primal algebra is categorically equivalent to the variety of boolean algebras. In particular, the category **FBA** whose objects are finite boolean algebras and morphisms are embeddings is equivalent to the category $\mathbf{V}_{\mathit{fin}}(\mathcal{A})$ whose objects are finite algebras in the variety $V(\mathcal{A})$ generated by a primal algebra $\mathcal{A}$ and morphisms are embeddings. Therefore, Theorem 3.1 and Example 3.2 imply that the category $\mathbf{V}_{\mathit{fin}}(\mathcal{A})$ has the Ramsey property for objects for every primal algebra $\mathcal{A}$. In other words, we have the following *Ramsey theorem for finite algebras in the variety generated by a primal algebra*:

For every primal algebra $\mathcal{A}$, for all $\mathcal{S}, \mathcal{T} \in \mathbf{V}_{\mathit{fin}}(\mathcal{A})$ such that $\mathcal{S} \hookrightarrow \mathcal{T}$ and every $k \geqslant 2$, there is a $\mathcal{U} \in \mathbf{V}_{\mathit{fin}}(\mathcal{A})$ such that $\mathcal{U} \longrightarrow (\mathcal{T})^{\mathcal{S}}_k$.

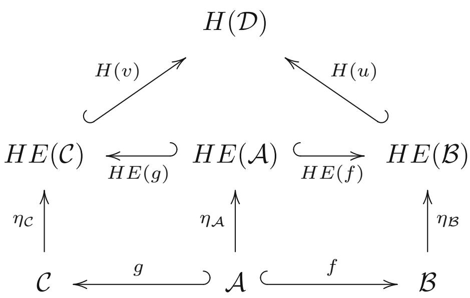
We treat this topic in more detail in Section 4.

Corollary 3.4. *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be equivalent categories whose objects are structures and embeddings are morphisms.*

(*a*) *If one of the two categories has (AP), then so does the other.*
(*b*) *If one of the two categories has (JEP), then so does the other.*
(*c*) *If* $\mathbf{C}$ *is a Ramsey age and* $\mathbf{D}$ *has (HP), then* $\mathbf{D}$ *is also a Ramsey age.*

Proof. (*a*): Let $E \colon \mathbf{C} \to \mathbf{D}$ and $H \colon \mathbf{D} \to \mathbf{C}$ be functors that constitute the equivalence between $\mathbf{C}$ and $\mathbf{D}$, and let $\eta \colon \mathrm{ID}_\mathbf{C} \to HE$ and $\varepsilon \colon \mathrm{ID}_\mathbf{D} \to EH$ be the accompanying natural isomorphisms. Assume that $\mathbf{D}$ has (AP), and let

$f\colon \mathcal{A} \hookrightarrow \mathcal{B}$ and $g\colon \mathcal{A} \hookrightarrow \mathcal{C}$ be two embeddings in $\mathbf{C}$. Then $E(f)\colon E(\mathcal{A}) \hookrightarrow E(\mathcal{B})$ and $E(g)\colon E(\mathcal{A}) \hookrightarrow E(\mathcal{C})$ are embeddings in $\mathbf{D}$, so there is a $\mathcal{D} \in \mathrm{Ob}(\mathbf{D})$ and embeddings $u\colon E(\mathcal{B}) \hookrightarrow \mathcal{D}$ and $v\colon E(\mathcal{C}) \hookrightarrow \mathcal{D}$ such that $u \circ E(f) = v \circ E(g)$. Then the diagram



commutes, because H is a functor and η is natural.

(b): This is similar to (a).

(c): This follows from (a), (b), and Theorem 3.1. □

Unlike (AP) and (JEP), which describe properties of a class of structures with no reference to other classes of structures, (HP) describes a property of a class of structures in relation to another, ambient class of structures. This ambient class of structures is usually implicit in the exposition. For example, the statement that the class $\mathbf{K}_n$ of all finite K_n-free graphs has (HP), where $n \geqslant 3$ is fixed, implicitly assumes that $\mathbf{K}_n$ is considered as a subclass of the class $\mathbf{G}$ of all finite graphs. Although such a requirement can be expressed in terms of category theory, we refrained from doing so in order to keep the exposition concise. For a possible treatment of Fraïssé theory without the (HP), requirement we refer the reader to [20].

Theorem 3.5. *Let $\mathbf{C}^*$ be a reasonable order expansion of $\mathbf{C}$ with the forgetful functor $U\colon \mathbf{C}^* \to \mathbf{C}\colon \mathcal{A}_< \mapsto \mathcal{A}$, where $f \mapsto f$, and let $\mathbf{D}^*$ be a reasonable order expansion of $\mathbf{D}$ with the forgetful functor $V\colon \mathbf{D}^* \to \mathbf{D}\colon \mathcal{A}_< \mapsto \mathcal{A}$, where $f \mapsto f$. Assume that $E^*\colon \mathbf{C}^* \rightleftarrows \mathbf{D}^*\colon H^*$ is a categorical equivalence of $\mathbf{C}^*$ and $\mathbf{D}^*$, that $E\colon \mathbf{C} \rightleftarrows \mathbf{D}\colon H$ is a categorical equivalence of $\mathbf{C}$ and $\mathbf{D}$, and that the following diagrams commute:*

Then $\mathbf{C}^*$ has the ordering property over $\mathbf{C}$ if and only if $\mathbf{D}^*$ has the ordering property over $\mathbf{D}$.

Proof. Assume that $\mathbf{C}^*$ has the ordering property over $\mathbf{C}$ and let us show that $\mathbf{D}^*$ has the ordering property over $\mathbf{D}$. Take any $\mathcal{A} \in \mathrm{Ob}(\mathbf{D})$. Then

$H(\mathcal{A}) \in \mathrm{Ob}(\mathbf{C})$, so by the ordering property, there is a $\mathcal{B} \in \mathrm{Ob}(\mathbf{C})$ which is a witness of the ordering property for $H(\mathcal{A})$. Let us show that $E(\mathcal{B}) \in \mathrm{Ob}(\mathbf{D})$ is a witness of the ordering property for $\mathcal{A}$. Take any $\mathcal{A}_<, \mathcal{B}_\sqsubset \in \mathrm{Ob}(\mathbf{D}^*)$ such that $V(\mathcal{A}_<) = \mathcal{A}$ and $V(\mathcal{B}_\sqsubset) = E(\mathcal{B})$ and let us show that $\mathcal{A}_< \hookrightarrow \mathcal{B}_\sqsubset$.

Let us first show that $H^*(\mathcal{A}_<) \hookrightarrow H^*(\mathcal{B}_\sqsubset)$. Note first that $UH^*(\mathcal{A}_<) = HV(\mathcal{A}_<) = H(\mathcal{A})$ and that $UH^*(\mathcal{B}_\sqsubset) = HV(\mathcal{B}_\sqsubset) = HE(\mathcal{B}) \cong \mathcal{B}$. Since $\mathbf{C}^*$ is a reasonable order expansion of $\mathbf{C}$, there is a $\mathcal{B}_\prec \in \mathrm{Ob}(\mathbf{C}^*)$ such that $\mathcal{B}_\prec \cong H^*(\mathcal{B}_\sqsubset)$ and $U(\mathcal{B}_\prec) = \mathcal{B}$. Since $\mathbf{C}^*$ has the ordering property over $\mathbf{C}$ and $\mathcal{B}$ is a witness of the ordering property for $H(\mathcal{A})$, we have that $H^*(\mathcal{A}_<) \hookrightarrow \mathcal{B}_\prec \cong H^*(\mathcal{B}_\sqsubset)$. Therefore, $\mathcal{A}_< \cong E^*H^*(\mathcal{A}_<) \hookrightarrow E^*H^*(\mathcal{B}_\sqsubset) \cong \mathcal{B}_\sqsubset$. □

4. Primal algebras

Let $\mathcal{B}$ be a finite boolean algebra and let $A = \{a_1, a_2, \dots, a_n\}$ be the set of atoms of $\mathcal{B}$. Every linear order $<$ on A, say $a_{i_1} < a_{i_2} < \dots < a_{i_n}$, induces a linear order on $\mathcal{B}$ as follows. Take $x, y \in B$; let $x = \delta_1 \cdot a_{i_1} \vee \delta_2 \cdot a_{i_2} \vee \dots \vee \delta_n \cdot a_{i_n}$ and $y = \varepsilon_1 \cdot a_{i_1} \vee \varepsilon_2 \cdot a_{i_2} \vee \dots \vee \varepsilon_n \cdot a_{i_n}$ be the representations of x and y, respectively, where $\varepsilon_s, \delta_s \in \{0, 1\}$ and with the convention that $0 \cdot b = 0$ while $1 \cdot b = b$ for $b \in B$. We then say that $x \sqsubset y$ if there is an s such that $\delta_s < \varepsilon_s$, and $\delta_t = \varepsilon_t$ for all $t > s$. In other words, $\sqsubset$ is the *antilexicographic ordering of the elements of* B with respect to $<$. The choice of the antilexicographic ordering induced by $<$ is motivated by the fact that the antilexicographic ordering of a boolean algebra is a linear ordering on the algebra that extends the initial ordering on the atoms (that is, $a_i < a_j$ implies $a_i \sqsubset a_j$). A linear ordering $\sqsubset$ of a finite boolean algebra $\mathcal{B}$ is *natural* [9] if there is a linear ordering $<$ on atoms of the algebra such that $\sqsubset$ is the antilexicographic ordering of the elements of B with respect to $<$. Let **OFBA** denote the category whose objects are finite boolean algebras together with a natural linear order and morphisms are embeddings.

This notion easily generalizes to arbitrary powers of finite algebras. Let $\mathcal{A}$ be a finite algebra and let $<$ be an arbitrary linear order on A. For every $n \in \mathbb{N}$ this linear order induces the antilexicographic order $\sqsubset$ on A^n as follows: $(x_1, \dots, x_n) \sqsubset (y_1, \dots, y_n)$ if there is an s such that $x_i = y_i$ for $i > s$ and $x_s < y_s$. For every permutation π of $\{1, 2, \dots, n\}$ we also have a linear order $\sqsubset_\pi$ defined by $(x_1, \dots, x_n) \sqsubset_\pi (y_1, \dots, y_n)$ if $(x_{\pi(1)}, \dots, x_{\pi(n)}) \sqsubset (y_{\pi(1)}, \dots, y_{\pi(n)})$. Let $\sqsubseteq$ and $\sqsubseteq_\pi$ denote the reflexive versions of $\sqsubset$ and $\sqsubset_\pi$, respectively.

Let $\mathcal{A}$ be a primal algebra. (Recall that every primal algebra is finite and has at least two elements.) It is a well-known fact (see [1] for details on the structure of a variety of algebras generated by a primal algebra) that if $\mathcal{A}$ is a primal algebra and $n, m \in \mathbb{N}$, a mapping $f \colon A^n \to A^m$ is a homomorphism from $\mathcal{A}^n$ to $\mathcal{A}^m$ if and only if there exist $i_1, \dots, i_m \in \{1, \dots, n\}$ such that $f(x_1, \dots, x_n) = (x_{i_1}, \dots, x_{i_m})$. Moreover, f is an embedding if and only if f is injective if and only if $\{i_1, \dots, i_m\} = \{1, \dots, n\}$.

Lemma 4.1. *Let $\mathcal{A}$ be a primal algebra, let $<$ be a linear order on A and let $\sqsubset$ be the induced antilexicographic order. Take any $n, m \in \mathbb{N}$, any permutation π of $\{1, \dots, n\}$ and any permutation σ of $\{1, \dots, m\}$. The mapping $f\colon A^n \to A^m$ is a homomorphism from $\mathcal{A}^n_{\sqsubseteq_\pi}$ to $\mathcal{A}^m_{\sqsubseteq_\sigma}$ if and only if there exist $i_1, \dots, i_m \in \{1, \dots, n\}$ such that $f(x_1, \dots, x_n) = (x_{i_1}, \dots, x_{i_m})$ and the numbers $j_s = \pi^{-1}(i_{\sigma(s)})$, $s \in \{1, \dots, m\}$, have the following properties:*

(1) *$j_m = n$;*
(2) *for all $s < m$, if $j_s = k < n$, then $\{k + 1, \dots, n\} \subseteq \{j_{s+1}, \dots, j_m\}$.*

Proof. Note, first, that (2) is equivalent to the following: if $j_s = k$ is the last appearance of k in the sequence $(j_1, j_2, \dots, j_m)$ then $\{j_{s+1}, \dots, j_m\} = \{k+1, \dots, n\}$. Note that $\{j_1, \dots, j_m\} = \{d, \dots, n\}$ where $d = \min\{j_1, \dots, j_m\}$.

($\Rightarrow$): Since f is a homomorphism from $\mathcal{A}^n_{\sqsubseteq_\pi}$ to $\mathcal{A}^m_{\sqsubseteq_\sigma}$, we know that there exist $i_1, \dots, i_m \in \{1, \dots, n\}$ such that $f(x_1, \dots, x_n) = (x_{i_1}, \dots, x_{i_m})$, and that $(x_{\pi(1)}, \dots, x_{\pi(n)}) \sqsubseteq (y_{\pi(1)}, \dots, y_{\pi(n)})$ implies

$$(x_{i_{\sigma(1)}}, \dots, x_{i_{\sigma(m)}}) \sqsubseteq (y_{i_{\sigma(1)}}, \dots, y_{i_{\sigma(m)}}).$$

Let $j_s = \pi^{-1}(i_{\sigma(s)})$ for $s \in \{1, \dots, m\}$, so that $i_{\sigma(s)} = \pi(j_s)$ for all s.

Let us show that $j_m = n$, that is, $i_{\sigma(m)} = \pi(n)$. Suppose this is not the case and let $i_{\sigma(m)} = \pi(k)$ for some $k < n$. Take any $a, b \in A$ so that $a < b$ and consider the n-tuples

$$\overline{x} = (x_{\pi(1)}, \dots, x_{\pi(k)}, \dots, x_{\pi(n)}) = (a, a, \dots, a, \underset{k\text{th place}\uparrow}{b}, a, \dots, a),$$
$$\overline{y} = (y_{\pi(1)}, \dots, y_{\pi(k)}, \dots, y_{\pi(n)}) = (a, a, \dots, a, \underset{k\text{th place}\uparrow}{a}, a, \dots, b).$$

Then $\overline{x} \sqsubset \overline{y}$ but $(x_{i_{\sigma(1)}}, \dots, x_{i_{\sigma(m)}}) \sqsupset (y_{i_{\sigma(1)}}, \dots, y_{i_{\sigma(m)}})$ as $x_{i_{\sigma(m)}} = x_{\pi(k)} = b > a = y_{\pi(k)} = y_{i_{\sigma(m)}}$, a contradiction.

Let us now show that (2) holds for the sequence $(j_1, \dots, j_m)$. Suppose, to the contrary, that there is an $s < m$ such that $j_s = k < n$ but $\{k+1, \dots, n\} \not\subseteq \{j_{s+1}, \dots, j_m\}$. Take the largest $l \in \{k + 1, \dots, n\} \setminus \{j_{s+1}, \dots, j_m\}$. Note that $l \leqslant n-1$ as $j_m = n$. Take any $a, b \in A$ so that $a < b$, and consider the n-tuples

$$\overline{x} = (x_{\pi(1)}, \dots, x_{\pi(k)}, \dots, x_{\pi(n)}) = (a, \dots, a, \underset{k\text{th place}\uparrow}{b}, a, \dots, a, \underset{l\text{th place}\uparrow}{a}, a, \dots, a),$$
$$\overline{y} = (y_{\pi(1)}, \dots, y_{\pi(k)}, \dots, y_{\pi(n)}) = (a, \dots, a, \underset{k\text{th place}\uparrow}{a}, a, \dots, a, \underset{l\text{th place}\uparrow}{b}, a, \dots, a).$$

Then $\overline{x} \sqsubset \overline{y}$ but, having in mind that $i_{\sigma(s)} = \pi(j_s) = \pi(k)$,

$$\begin{aligned}(x_{i_{\sigma(1)}}, \dots, x_{i_{\sigma(s)}}, \dots, x_{i_{\sigma(m)}}) &= (\dots, \underset{s\text{th place}\uparrow}{b}, \underbrace{a, \dots, a, a, \dots, a}_{\text{no index equals } \pi(l)}) \\ &\sqsupset (\dots, \underset{s\text{th place}\uparrow}{a}, \underbrace{a, \dots, a, a, \dots, a}_{\text{no index equals } \pi(l)}) \\ &= (y_{i_{\sigma(1)}}, \dots, y_{i_{\sigma(s)}}, \dots, y_{i_{\sigma(m)}}),\end{aligned}$$

which is a contradiction.

($\Leftarrow$): Let $f\colon A^n \to A^m$ be a mapping with $f(x_1, \ldots, x_n) = (x_{i_1}, \ldots, x_{i_m})$ for some $i_1, \ldots, i_m \in \{1, \ldots, n\}$, and assume that the numbers $j_s = \pi^{-1}(i_{\sigma(s)})$, for $s \in \{1, \ldots, m\}$, satisfy (1) and (2). Then f is clearly a homomorphism from $\mathcal{A}^n$ to $\mathcal{A}^m$, so let us show that f is monotone. Take $x_1, \ldots, x_n, y_1, \ldots, y_n \in A$ such that $(x_{\pi(1)}, \ldots, x_{\pi(k)}, \ldots, x_{\pi(n)}) \sqsubset (y_{\pi(1)}, \ldots, y_{\pi(k)}, \ldots, y_{\pi(n)})$. Then there is a t such that $x_{\pi(q)} = y_{\pi(q)}$ for all $q > t$ and $x_{\pi(t)} < y_{\pi(t)}$. Let us show that

$$(x_{\pi(j_1)}, \ldots, x_{\pi(j_m)}) \sqsubseteq (y_{\pi(j_1)}, \ldots, y_{\pi(j_m)}). \tag{4.1}$$

If $\min\{j_1, \ldots, j_m\} > t$, then equality holds in (4.1). Suppose, therefore, that $\min\{j_1, \ldots, j_m\} \leqslant t$. Then $t \in \{j_1, \ldots, j_m\}$ because of (2). Let j_s be the last appearance of t in the sequence $(j_1, \ldots, j_m)$. Then we have $\{j_{s+1}, \ldots, j_m\} = \{t+1, \ldots, n\}$, whence follows that strict inequality holds in (4.1).

Therefore, (4.1) holds. The choice of the indices j_s ensures that (4.1) is equivalent to $(x_{i_{\sigma(1)}}, \ldots, x_{i_{\sigma(m)}}) \sqsubseteq (y_{i_{\sigma(1)}}, \ldots, y_{i_{\sigma(m)}})$. □

Let $\mathcal{A}$ be a primal algebra and let $<$ be a linear order on A. Let $\mathbf{OV}_{\mathit{fin}}(\mathcal{A}, <)$ be the category whose objects are isomorphic copies of structures $\mathcal{A}^n_{\sqsubseteq_\pi}$ where $n \in \mathbb{N}$ and π is a permutation of $\{1, \ldots, n\}$, and whose morphisms are embeddings.

Theorem 4.2. *Let $\mathcal{A}$ be a primal algebra and let $<$ be a linear order on A. Then $\mathbf{OV}_{\mathit{fin}}(\mathcal{A}, <)$ is categorically equivalent to* **OFBA**.

Proof. Every finite boolean algebra together with a natural linear ordering is clearly isomorphic to $\mathit{2}^n_{\sqsubseteq_\pi}$ where $\mathit{2}$ is the two-element boolean algebra whose base set is $2 = \{0, 1\}$, $n \in \mathbb{N}$, and π is a permutation of $\{1, \ldots, n\}$ which encodes the initial ordering of the atoms. Hence, $\mathbf{OFBA} = \mathbf{OV}_{\mathit{fin}}(\mathit{2}, \prec)$, where $\prec$ is the usual ordering $0 \prec 1$ of $\mathit{2}$.

Let $\mathbf{B}$ be the full subcategory of **OFBA** spanned by the countable set of objects $\{\mathit{2}^n_{\sqsubseteq_\pi} : n \in \mathbb{N},\ \pi \text{ is a permutation of } \{1, 2, \ldots, n\}\}$, and let $\mathbf{C}$ be the full subcategory of $\mathbf{OV}_{\mathit{fin}}(\mathcal{A}, <)$ spanned by the countable set of objects $\{\mathcal{A}^n_{\sqsubseteq_\pi} : n \in \mathbb{N},\ \pi \text{ is a permutation of } \{1, 2, \ldots, n\}\}$. Clearly, $\mathbf{B}$ and $\mathbf{C}$ are skeletons of **OFBA** and $\mathbf{OV}_{\mathit{fin}}(\mathcal{A}, <)$, respectively, so in order to show that **OFBA** and $\mathbf{OV}_{\mathit{fin}}(\mathcal{A}, <)$ are equivalent it suffices to show that $\mathbf{B}$ and $\mathbf{C}$ are isomorphic. Let $F\colon \mathbf{B} \to \mathbf{C}$ be a functor such that $F(\mathit{2}^n_{\sqsubseteq_\pi}) = \mathcal{A}^n_{\sqsubseteq_\pi}$ and which takes a morphism $f\colon 2^n \to 2^m\colon (x_1, \ldots, x_n) \mapsto (x_{i_1}, \ldots, x_{i_m})$ to $f'\colon A^n \to A^m\colon (x_1, \ldots, x_n) \mapsto (x_{i_1}, \ldots, x_{i_m})$. Lemma 4.1 ensures that F is well defined and bijective on morphisms, so F is clearly an isomorphism of $\mathbf{B}$ and $\mathbf{C}$. Therefore, the categories **OFBA** and $\mathbf{OV}_{\mathit{fin}}(\mathcal{A}, <)$ are equivalent. □

Theorem 4.3. *Let $\mathcal{A}$ be a primal algebra and let $<$ be a linear order on A. Then*

(1) $\mathbf{OV}_{\mathit{fin}}(\mathcal{A}, <)$ *is a Ramsey age and*
(2) *the automorphism group of its Fraïssé limit is extremely amenable.*

Proof. (1): Let us first show that $\mathbf{OV}_{\mathit{fin}}(\mathcal{A},<)$ has (HP). Take any $\mathcal{A}^m_{\sqsubseteq_\sigma}$ and any embedding $f\colon \mathcal{A}^n_{\preccurlyeq} \hookrightarrow \mathcal{A}^m_{\sqsubseteq_\sigma}$. Let $i_1,\dots,i_m$ be indices such that $f(x_1,\dots,x_n)=(x_{i_1},\dots,x_{i_m})$ and $\{i_1,\dots,i_m\}=\{1,\dots,n\}$. Thus, we have that $(x_1,\dots,x_n)\preccurlyeq(y_1,\dots,y_n)$ if and only if $(x_{i_1},\dots,x_{i_m})\sqsubseteq_\sigma(y_{i_1},\dots,y_{i_m})$.

Let us show that there is a permutation π of $\{1,\dots,n\}$ with $\preccurlyeq\,=\,\sqsubseteq_\pi$. For $s\in\{1,\dots,n\}$, let $A_s=\{t\in\{1,\dots,m\}: i_{\sigma(t)}=s\}$. Note that $\{A_1,\dots,A_n\}$ is a partition of $\{1,\dots,m\}$ because $\{i_1,\dots,i_m\}=\{1,\dots,n\}$. Let π be a permutation of $\{1,\dots,n\}$ such that $\max A_{\pi(1)}<\max A_{\pi(2)}<\dots<\max A_{\pi(n)}$ (note that $m\in A_{\pi(n)}$), and define $(j_1,\dots,j_m)$ as follows: $j_s=k$ if and only if $s\in A_{\pi(k)}$. Then it is easy to verify that $\pi(j_s)=i_{\sigma(s)}$ for all s and that $(j_1,\dots,j_m)$ satisfies (1) and (2) of Lemma 4.1. So, Lemma 4.1 ensures that f is a homomorphism, and hence an embedding, of $\mathcal{A}^n_{\sqsubseteq_\pi}$ into $\mathcal{A}^m_{\sqsubseteq_\sigma}$.

Let us show that $\preccurlyeq\,=\,\sqsubseteq_\pi$. On the one hand, $(x_1,\dots,x_n)\sqsubseteq_\pi(y_1,\dots,y_n)$ is equivalent to $(x_{i_1},\dots,x_{i_m})\sqsubseteq_\sigma(y_{i_1},\dots,y_{i_m})$ because $f\colon\mathcal{A}^n_{\sqsubseteq_\pi}\hookrightarrow\mathcal{A}^m_{\sqsubseteq_\sigma}$. On the other hand, we have that $(x_1,\dots,x_n)\preccurlyeq(y_1,\dots,y_n)$ is equivalent to $(x_{i_1},\dots,x_{i_m})\sqsubseteq_\sigma(y_{i_1},\dots,y_{i_m})$ because $f\colon\mathcal{A}^n_{\preccurlyeq}\hookrightarrow\mathcal{A}^m_{\sqsubseteq_\sigma}$. Thus $\preccurlyeq\,=\,\sqsubseteq_\pi$. Therefore, $\mathbf{OV}_{\mathit{fin}}(\mathcal{A},<)$ has (HP).

Now, $\mathbf{OV}_{\mathit{fin}}(\mathcal{A},<)$ is categorically equivalent to **OFBA** (Theorem 4.2), **OFBA** is a Ramsey age [9] and $\mathbf{OV}_{\mathit{fin}}(\mathcal{A},<)$ has (HP); thus, Corollary 3.4 yields that $\mathbf{OV}_{\mathit{fin}}(\mathcal{A},<)$ is a Ramsey age.

(2) This follows from Theorem 2.6. □

We shall now apply Theorem 2.7 to the classes $\mathbf{OV}_{\mathit{fin}}(\mathcal{A},<)$ and $\mathbf{V}_{\mathit{fin}}(\mathcal{A})$, where $\mathcal{A}$ is a primal algebra; let $<$ be a linear order on A. Let us first show that the former is a reasonable order expansion of the latter.

Lemma 4.4. *Let $\mathcal{A}$ be a primal algebra and let $<$ be a linear order on A. Then $\mathbf{OV}_{\mathit{fin}}(\mathcal{A},<)$ is a reasonable order expansion of $\mathbf{V}_{\mathit{fin}}(\mathcal{A})$.*

Proof. Let $f\colon\mathcal{A}^n\hookrightarrow\mathcal{A}^m$ be an embedding, and let $i_1,\dots,i_m$ be indices such that $f(x_1,\dots,x_n)=(x_{i_1},\dots,x_{i_m})$ and $\{i_1,\dots,i_m\}=\{1,\dots,n\}$. Take any permutation π of $\{1,2,\dots,n\}$ and let us find a permutation σ of $\{1,2,\dots,m\}$ such that f is an embedding of $\mathcal{A}^n_{\sqsubseteq_\pi}$ into $\mathcal{A}^m_{\sqsubseteq_\sigma}$. For an arbitrary $s\in\{1,\dots,n\}$, let $A_s=\{t\in\{1,\dots,m\}:\pi(s)=i_t\}$. Note that $\{A_1,\dots,A_n\}$ is a partition of $\{1,\dots,m\}$ because $\{i_1,\dots,i_m\}=\{1,\dots,n\}$.

Let σ be any permutation of $\{1,\dots,m\}$ such that $\sigma^{-1}(A_1)=\{1,\dots,k_1\}$, $\sigma^{-1}(A_2)=\{k_1+1,\dots,k_2\}$, …, $\sigma^{-1}(A_n)=\{k_{n-1}+1,\dots,m\}$, and define $(j_1,\dots,j_m)$ as follows: $j_1=\dots=j_{k_1}=1$, $j_{k_1+1}=\dots=j_{k_2}=2$, …, $j_{k_{n-1}+1}=\dots=j_m=n$. Then it is easy to verify that $\pi(j_s)=i_{\sigma(s)}$ for all s and that $(j_1,\dots,j_m)$ satisfies (1) and (2) of Lemma 4.1. Now, Lemma 4.1 ensures that f is a homomorphism, and so an embedding, of $\mathcal{A}^n_{\sqsubseteq_\pi}$ into $\mathcal{A}^m_{\sqsubseteq_\sigma}$. □

Theorem 4.5. *Let $\mathcal{A}$ be a primal algebra with $<$ a linear order on A. Let $\mathcal{F}$ be the Fraïssé limit of $\mathbf{V}_{\mathit{fin}}(\mathcal{A})$, let $\mathcal{F}_{\sqsubset}$ be the Fraïssé limit of $\mathbf{OV}_{\mathit{fin}}(\mathcal{A},<)$, let $G=\mathrm{Aut}(\mathcal{F})$ and $X^*=\overline{G\cdot\sqsubset}$ (in the logical action of G on $\mathrm{LO}(F)$). Then the logical action of G on X^* is the universal minimal flow of G.*

Proof. The class $\mathbf{OV}_{\mathit{fin}}(\mathcal{A},<)$ is a Ramsey age by Theorem 4.3, while Theorem 3.5 ensures that $\mathbf{OV}_{\mathit{fin}}(\mathcal{A},<)$ has the ordering property over $\mathbf{V}_{\mathit{fin}}(\mathcal{A})$. (The categorical equivalences in question are $\mathbf{FBA} \rightleftarrows \mathbf{V}_{\mathit{fin}}(\mathcal{A})$ and $\mathbf{OFBA} \rightleftarrows \mathbf{OV}_{\mathit{fin}}(\mathcal{A},<)$ established in Example 3.3 and Theorem 4.2.) The statement now follows by Theorem 2.7. □

5. Appendix 1: The Ramsey properties under adjunctions

In this section, we discuss Ramsey properties in adjunctions and prove that right adjoints preserve the Ramsey property for morphisms, while left adjoints preserve the dual of the Ramsey property for morphisms. The status of the Ramsey properties for objects is delicate and is preserved by right, respectively, left adjoints under additional assumptions on the automorphism groups of objects of the form $F(\mathcal{C})$ and $G(\mathcal{D})$.

Theorem 5.1. *Let $F\colon \mathbf{C} \rightleftarrows \mathbf{D}\colon G$ be an adjunction.*

(a) If $\mathbf{D}$ has the Ramsey property for morphisms, then so does $\mathbf{C}$.

(b) If $\mathbf{C}$ has the dual Ramsey property for morphisms, then so does $\mathbf{D}$.

Proof. It suffices to prove (a) as the proof of (b) is dual. Let Φ be the natural isomorphism between the hom-sets.

Take $k \geqslant 2$ and $\mathcal{A},\mathcal{B} \in Ob(\mathbf{C})$ such that $\mathcal{A} \to \mathcal{B}$. Then $F(\mathcal{A}) \to F(\mathcal{B})$. Since $\mathbf{D}$ has the Ramsey property for morphisms, there is a $\mathcal{C} \in \mathrm{Ob}(\mathbf{D})$ such that $\mathcal{C} \overset{hom}{\longrightarrow} (F(\mathcal{B}))^{F(\mathcal{A})}_k$. Let us show that $G(\mathcal{C}) \overset{hom}{\longrightarrow} (GF(\mathcal{B}))^{\mathcal{A}}_k$. Take any k-coloring $\hom(\mathcal{A}, G(\mathcal{C})) = \mathcal{M}_1 \cup \cdots \cup \mathcal{M}_k$. By applying Φ^{-1} and noting that $\Phi^{-1}(\hom(\mathcal{A}, G(\mathcal{C}))) = \hom(F(\mathcal{A}), \mathcal{C})$, we obtain

$$\hom(F(\mathcal{A}), \mathcal{C}) = \Phi^{-1}(\mathcal{M}_1) \cup \cdots \cup \Phi^{-1}(\mathcal{M}_k).$$

Since Φ is bijective, the above is actually a k-coloring of $\hom(F(\mathcal{A}), \mathcal{C})$, so there is an i and a morphism $w\colon F(\mathcal{B}) \to \mathcal{C}$ such that

$$w \cdot \hom(F(\mathcal{A}), F(\mathcal{B})) \subseteq \Phi^{-1}(\mathcal{M}_i).$$

After applying G and multiplying by $\eta_{\mathcal{A}}$ from the right, we have

$$G(w) \cdot G(\hom(F(\mathcal{A}), F(\mathcal{B}))) \cdot \eta_{\mathcal{A}} \subseteq G(\Phi^{-1}(\mathcal{M}_i)) \cdot \eta_{\mathcal{A}}.$$

Loosely speaking, for any set of morphisms $\mathcal{M}$, we have that $G(\mathcal{M}) \cdot \eta = \Phi(\mathcal{M})$, so the above relation transforms to

$$G(w) \cdot \Phi(\hom(F(\mathcal{A}), F(\mathcal{B}))) \subseteq \Phi(\Phi^{-1}(\mathcal{M}_i)),$$

or, equivalently, $G(w) \cdot \hom(\mathcal{A}, GF(\mathcal{B})) \subseteq \mathcal{M}_i$. This completes the proof that $G(\mathcal{C}) \overset{hom}{\longrightarrow} (GF(\mathcal{B}))^{\mathcal{A}}_k$.

Since $\eta_{\mathcal{B}}\colon \mathcal{B} \to GF(\mathcal{B})$, Lemma 2.4 (a) ensures that $G(\mathcal{C}) \overset{hom}{\longrightarrow} (\mathcal{B})^{\mathcal{A}}_k$. Therefore, $\mathbf{C}$ has the Ramsey property for morphisms. □

The analogous statement for objects need not be true in general because, in general, the unit and the counit do not consist of isomorphisms. The following lemma provides a sufficient condition for adjunctions to preserve Ramsey property for objects but the additional condition we impose is rather strong.

Lemma 5.2. *Let* $F\colon \mathbf{C} \rightleftarrows \mathbf{D}\colon G$ *be an adjunction.*

(a) Assume that $\mathrm{Aut}(F(\mathcal{A})) = F(\mathrm{Aut}(\mathcal{A}))$ *for all* $\mathcal{A} \in \mathrm{Ob}(\mathbf{C})$*. If* $\mathbf{D}$ *has the Ramsey property for objects, then so does* $\mathbf{C}$*.*

(b) Assume that $\mathrm{Aut}(G(\mathcal{B})) = G(\mathrm{Aut}(\mathcal{B}))$ *for all* $\mathcal{B} \in \mathrm{Ob}(\mathbf{D})$*. If* $\mathbf{C}$ *has the dual Ramsey property for objects, then so does* $\mathbf{D}$*.*

Proof. Again, we shall focus on (a) because the proof of (b) is dual. The proof of (a), however, is analogous to the proof of (a) in Theorem 5.1 provided we can show that

$$\Phi\left(\binom{\mathcal{B}}{F(\mathcal{A})}\right) = \binom{G(\mathcal{B})}{\mathcal{A}},$$

or, equivalently, $\Phi\big(\hom(F(\mathcal{A}), \mathcal{B})/{\sim_{F(\mathcal{A})}}\big) = \hom(\mathcal{A}, G(B))/{\sim_{\mathcal{A}}}$.

This relation clearly follows from

$$\Phi(f/{\sim_{F(\mathcal{A})}}) = \Phi(f)/{\sim_{\mathcal{A}}} \text{ for all } f \in \hom(F(\mathcal{A}), \mathcal{B}). \tag{5.1}$$

Let us show (5.1).

($\subseteq$): Take $f \in \hom(F(\mathcal{A}), \mathcal{B})$ and $g \in \Phi(f/{\sim_{F(\mathcal{A})}})$. Then $g = \Phi(f \cdot \alpha)$ for some $\alpha \in \mathrm{Aut}(F(\mathcal{A}))$, and then $g = G(f) \cdot G(\alpha) \cdot \eta_{\mathcal{A}}$. Since $\alpha \in \mathrm{Aut}(F(\mathcal{A})) = F(\mathrm{Aut}(\mathcal{A}))$, there is a $\beta \in \mathrm{Aut}(\mathcal{A})$ such that $\alpha = F(\beta)$, so

$$\begin{aligned} g &= G(f) \cdot G(\alpha) \cdot \eta_{\mathcal{A}} = G(f) \cdot GF(\beta) \cdot \eta_{\mathcal{A}} \\ &= G(f) \cdot \eta_{\mathcal{A}} \cdot \beta = \Phi(f) \cdot \beta \in \Phi(f)/{\sim_{\mathcal{A}}}, \end{aligned}$$

with the third equality because η is natural.

($\supseteq$): This is analogous to ($\subseteq$). □

We thank Christian Rosendal for letting us include the following example from his unpublished notes.

Example 5.3. (Homogeneous trees.) Here we present an example that shows the importance of the assumption in Lemma 5.2 (a). Let $\mathbf{C}$ be the category of trees, finite structures in the language $\{f\}$ where $f(a) = b$ if b is the immediate predecessor of a in the partial tree order. Let $\mathbf{D} \subset \mathbf{C}$ be the class of homogeneous trees, trees where the branching number at a node is a function of the level of the node, i.e., for every $n < \omega$, there is some $b(n) < \omega$ such that every node at level n has exactly $b(n)$ immediate successors at level $n+1$. Let $\mathbf{C}_<$, $\mathbf{D}_<$ be the corresponding categories of finite convexly ordered trees (structures in the language $\{f, <\}$ where $<$ gives the linear extension of the partial tree order, and the immediate successors of each vertex form an interval with respect to $<$). It is known that $\mathbf{C}_<$ has the Ramsey property for objects/morphisms, see [2] and see [17] for a discussion. In Rosendal's notes, it is shown that $\mathbf{D}$ has the Ramsey property for objects. This follows by Lemma 2.5 and [9, Proposition 5.6] as $\mathbf{D}_<$ is a cofinal full subcategory of $\mathbf{C}_<$ and it is order forgetful.

This example illustrates a case where we have an adjunction $F\colon \mathbf{C} \rightleftarrows \mathbf{D}\colon G$ but $\Phi\big(\hom(F(\mathcal{A}),\mathcal{B})/{\sim_{F(\mathcal{A})}}\big) = \hom(\mathcal{A}, G(B))/{\sim_{\mathcal{A}}}$ fails. F takes $\mathcal{A} \in \mathrm{Ob}(\mathbf{C})$ to the smallest homogeneous tree containing $\mathcal{A}$ and G gives the inclusion of $\mathbf{D}$ in $\mathbf{C}$. Consider $\mathcal{A}$ as given in Figure 1 and let $\mathcal{B} = F(\mathcal{A})$. Then $|\hom(F(\mathcal{A}),\mathcal{B})/{\sim_{F(\mathcal{A})}}| = 1$ but $|\hom(\mathcal{A}, G(B))/{\sim_{\mathcal{A}}}| = 4$: a map can send (a_1, a_2, a_3) to any of $(c_1, c_2, c_3), (c_1, c_2, c_4), (c_1, c_3, c_4), (c_2, c_3, c_4)$, up to automorphism of $\mathcal{A}$.

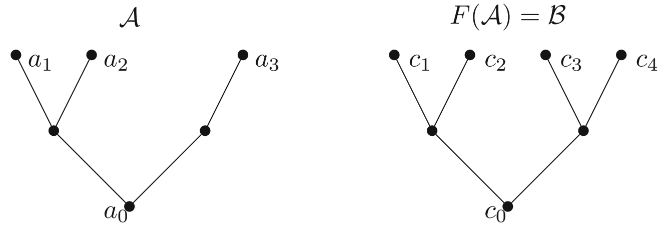


Figure 1. Unordered trees

Notice that this is also a case where $\mathbf{D}$ has the Ramsey property for objects but $\mathbf{C}$ does not. Rosendal provides the example from Figure 1 in his notes. If we impose an ordering $<$ on all members of $\mathbf{C}$ and color copies of $\mathcal{A}$ in $\mathcal{B}$ according to whether the pair (a_1, a_2) occurs $<$ then a_3 or vice versa, then we cannot find a homogeneous copy of $\mathcal{B}$.

6. Appendix 2: Fraïssé limits with identical automorphism group

Here we give categorical language to some results following from [9]. The results in [9] demonstrate in some cases that two categories with different objects have essentially the same Ramsey property. Using results in Appendix 1, we may pin down the underlying functor that is transferring the Ramsey property between these categories.

First, we recap the definition of the canonical language on ultrahomogeneous structures as described in [9]. For a closed subgroup $G < \mathrm{Sym}(\mathbb{N})$, we can construct an ultrahomogeneous structure as follows. Let $\Delta_{h(G)} = \{R_i \mid i \in I\}$ where R_i is an n-ary relation corresponding to the orbit $\mathcal{O}_i$ of an n-tuple $\overline{a} \in \mathbb{N}^n$ under G. Define $\mathcal{M} = (\mathbb{N}, \Delta_{h(G)})$ so that

$$\mathcal{M} \vDash R_i(\overline{a}) \Leftrightarrow \overline{a} \in \mathcal{O}_i$$

Clearly, partial isomorphisms of $\mathcal{M}$ extend to automorphisms of $\mathcal{M}$, and so $\mathcal{M}$ is ultrahomogeneous. Moreover, $\mathrm{Aut}(\mathcal{M}) = G$. We will call $\Delta_{h(G)}$ the *Hodges language corresponding to G* ($(\mathbb{N}, \Delta_{h(G)})$ is called the "induced structure associated to G" in [9]). In the case that $G = \mathrm{Aut}(\mathcal{A})$ for some given countable structure $\mathcal{A}$, we call $\Delta_{h(G)}$ the *Hodges language on* $\mathcal{A}$.

Theorem 2.6 guarantees that for extremely amenable $G \leq \mathrm{Sym}(\mathbb{N})$, $\mathcal{M}_G = \{\mathcal{M} \mid \mathrm{Aut}(\mathcal{M}) = G$ and $\mathcal{M}$ is ordered by $<$ and ultrahomogeneous$\}$ gives a family of structures with age having the Ramsey property for morphisms. The canonical relational structure $(\mathbb{N}, \Delta_{h(G)})$ is in $\mathcal{M}_G$, but so are a variety of structures in functional languages whose description might be more natural. Consider the family of categories

$$\Gamma = \{\mathbf{C} : \mathbf{C} \text{ is the category of structures with } \mathrm{Ob}(\mathbf{C}) = \mathrm{Age}(\mathcal{M}) \text{ for } \mathcal{M} \in \mathcal{M}_G\}$$

We can now make use of our technology of adjunctions to explain in categorical language why all members $\mathbf{C} \in \Gamma_G$ must share the Ramsey property for morphisms.

Theorem 6.1. *Let $\mathbf{D}_1, \mathbf{D}_2$ be two categories of finite structures. Suppose each class of structures is a Fraïssé class and the automorphism groups of the Fraïssé limits are isomorphic. Then there is a third category of finite structures $\mathbf{C}$ and adjunctions*

$$F_1 \colon \mathbf{C} \rightleftarrows \mathbf{D}_1 \colon G_1 \quad \text{and} \quad F_2 \colon \mathbf{C} \rightleftarrows \mathbf{D}_2 \colon G_2$$

such that the G_i are inclusions and $F_2 \circ G_1$, $F_1 \circ G_2$ preserve the Ramsey property for morphisms.

Proof. Let $\mathcal{F}_1, \mathcal{F}_2$ be the Fraïssé limits of the classes $\mathrm{Ob}(\mathbf{D}_1), \mathrm{Ob}(\mathbf{D}_2)$, both with automorphism group G. We may assume that $\mathcal{F}_1, \mathcal{F}_2$ have the same underlying set, $\mathbb{N}$. Now let $\mathrm{Ob}(\mathbf{C})$ be all finite subsets of $\mathbb{N}$ and let the morphisms of $\mathbf{C}$ be embeddings in the Hodges language corresponding to G. Now define $F_i \colon \mathbf{C} \to \mathbf{D}_i$ to take any finite subset $A \subset \mathbb{N}$ and send it to the closure of A under the function symbols in the language of $\mathcal{F}_i$. This map is well-defined and natural by ultrahomogeneity of $\mathcal{F}_i$. Thus, we have an adjunction $F_i \colon \mathbf{C} \rightleftarrows \mathbf{D}_i \colon G_i$, where G_i is the inclusion functor that "forgets" the function symbols. By Proposition 2.5, F_i preserves the Ramsey property for morphisms. By Theorem 5.1, so does G_i. Thus, so do their composites. □

Remark 6.2. It is interesting to note that in the above Theorem, we have $F_i \circ G_i = \mathrm{ID}_{\mathbf{D}_i}$ but $G_i \circ F_i \neq \mathrm{ID}_{\mathbf{D}_i}$.

We thank Christian Rosendal for letting us include the example $\mathbf{K}_r$ studied in his unpublished notes. The example $\mathbf{K}_s$ is from [18].

Definition 6.3. Consider the following classes of trees as categories with embeddings as morphisms.

$$\mathbf{K}_r = \mathrm{Age}({}^{\omega>}\omega, f, <),$$
$$\mathbf{K}_s = \mathrm{Age}({}^{\omega>}\omega, \lhd, \wedge, <, \{P_n\}_n),$$
$$\mathbf{K}_h = \mathrm{Age}({}^{\omega>}\omega, \Delta_{h(\mathrm{Aut}(\mathrm{Flim}\,\mathbf{K}_r))},$$

where we have:

- $\lhd$ is the partial tree order (sequence extension),
- $\wedge$ is the meet in the partial order,
- $<$ is the lexicographic order on sequences (a linear extension of the partial order),
- P_n is a unary predicate picking out the n-th level of the tree,
- f is a unary function symbol giving the immediate $\lhd$-predecessor of any node.

The following is guaranteed by Theorem 2.6.

Corollary 6.4. $\mathbf{K}_r$ *has the Ramsey property for morphisms if and only if* $\mathbf{K}_s$ *has the Ramsey property for morphisms.*

Proof. The classes are Fraïssé classes, so we may assume that $\mathcal{F}_s = \operatorname{Flim} \mathbf{K}_s$ and $\mathcal{F}_r = \operatorname{Flim} \mathbf{K}_r$ share the same underlying set $\mathbb{N}$. Every function and predicate symbol in $\mathbf{K}_r$ is quantifier-free $L_{\omega_1,\omega}$-definable in the language of $\mathbf{K}_s$ and vice versa. Thus, their automorphism groups have the same orbits on n-tuples from $\mathbb{N}$, and thus are the same group. □

Remark 6.5. Substructures of I_r are closed under the function symbols in I_s, so we could set up a direct adjunction $F\colon \mathbf{K}_s \rightleftarrows \mathbf{K}_r\colon G$. In other words, the intermediary category $\mathbf{C}$ from Theorem 6.1 can be jettisoned in favor of the more direct Theorem 5.1.

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